Graph Theory
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Lecture – 10
Part – 2
Proof of Halls Theorem

Welcome to the second part of lecture 10 on Graph Theory. So, in this lecture we talk about halls theorem. So, halls theorems talks about the necessary and sufficient condition for the existence of A saturated matching in bipartite graph. So, we have not proved this theorem before. So, today we will talk will prove this theorem, and also in this part we solve some problems.

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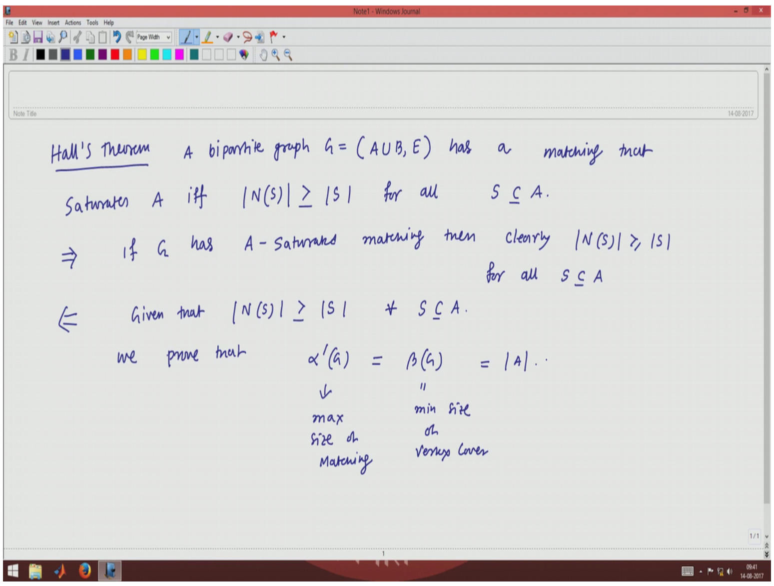

So, Hall's theorem, a bipartite graph G which is A union B E has a matching that that saturates A that saturates A means all the vertices of A are matched if and only if the neighbor of S the cardinality of neighbor of S is greater than or equal to the cardinality of S for all for all S subset of A. So, this is the necessary and sufficient condition for the existence of A saturated matching.

So, first we prove that if what if G has A saturated matching, then clearly N S cardinality of the neighbor of S is greater than equal to the cardinality of S for all S subset of A. So, this is quite trivial now the only e part that if this conditions are satisfied then graph G has A saturated matching. So, here you are given that you given that N S is greater than equal to the cardinality of S for all S subset of A. And what we need to prove that we prove that alpha primes A; that means, this is the size of maximum matching this is the maximum size of a matching. This is equal to cardinality of A this is equal to B G, which is equal to the minimum size of vertex cover, which is equal to the cardinality of A.

So, we prove this theorem with the help of Konig's theorem, because we know that for bipartite graph the size of maximum matching is equal to the size of minimum vertex cover. So, we will prove that the size of minimum vertex cover is equal to the cardinality of A. Because we say that graph has if those conditions are satisfied then the graph has a A saturated matching; that means, all the vertices of a are matched ok.

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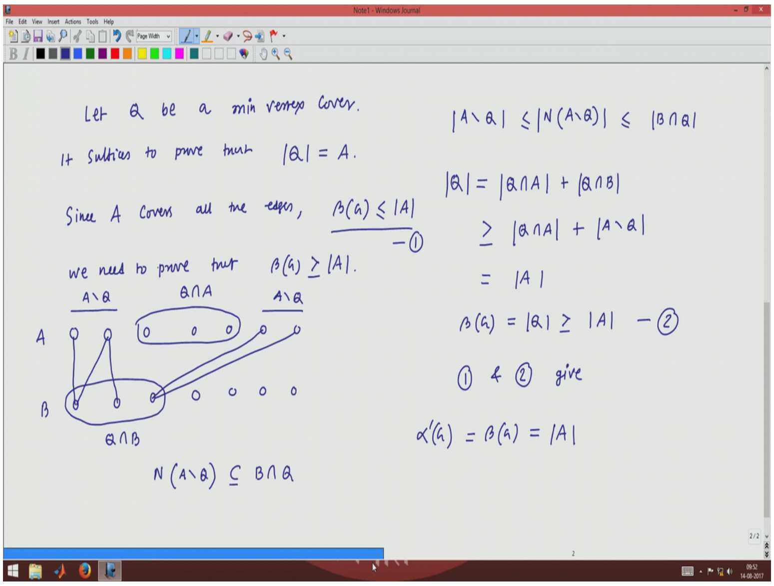

So, let Q be a minimum vertex cover. So, it surfaces to prove that Q is equal to A. So, first since A this is the one side of the bipartite graph A since A covers all the edges, then definitely this beta G is less than equal to the cardinality of A. So, this is a obvious vertex cover if you in a bipartite graph if you take one side of the bipartite graph all the vertices in one side of a bipartite graph then it covers all the edges of the bipartite graph. So, that is a trivial vertex cover. So, the min size of minimum vertex cover must be less than or equal to the cardinality of A.

So, what we need to prove now? We need to prove that this beta G is greater than equal to A. So, start with a minimum vertex cover let Q be a minimum vertex cover. So, this Q is not is necessarily that one side of the bipartite graph. So, it contained vertices from both the sides of the graph g. So, let me just draw a bipartite graph. So, Q is the minimum vertex cover. So, Q consists of. So, this is the part A and this is the part B. So, Q contains some vertex from part A suppose these are the vertices from part A in Q. So, this is Q intersection A and also Q contains some vertex form part B say these 3 vertices from part B. So, this is Q intersection B.

Now, think about the edges. So, this is edges from A minus Q. So, this is my A minus Q and also this 2 edges are also in A minus Q. So, all the edges out of A minus Q for example, this is one edge going out of A minus Q, this is another edge, this is another edge. So, this is an edge which is going out of A minus Q, this is another edge.

Now, all these edges will be covered by B intersection Q. So, that is why the neighbor of A minus Q. So, the neighbor of A minus Q must be a subset of B minus B intersection Q. So, this is subset of B intersection Q right. Because see all these edges in A minus Q they are not covered by Q intersection A. So, they must be covered by Q intersection B, because A Q is a vertex cover and it is a minimum vertex cover for the graph G ok.

Now, by halls theorem A minus Q the cardinality of this one is less than equal to N A minus Q. And just now we prove that this is by halls theorem because A minus Q this is also A minus Q A minus Q is a subset of A. And this one is less than equal to B intersection Q cardinality. So, this one comes for from the condition that that the neighbor of A minus Q is a subset of B intersection Q ok.

So now what you can write is that we can write the cardinality of Q is cardinality of Q intersection A, because the vertex cover has some vertex in the part A and some vertex in part B, this is quite obvious. Now this one is greater than A minus Q. So that means, that cardinality of Q is greater than or equal to Q intersection A cardinality, plus A minus Q into 6, sorry A minus Q cardinality right. So, what we have proved is that. So, this is equal to this is nothing but the cardinality of A.

So, what we have proved here is that the size of minimum vertex cover which is the cardinality of Q is greater than or equal to the cardinality of A. So, this is my equation say this is my equation 1 and this is my equation 2, and then equation 1 and equation 2, give that beta G is equal to the cardinality of A. So, we have proof that the minimum vertex cover size is equal to the cardinality of A, and by Konig's theorem this is equal to the maximum size of matching.

So, what we have proved is that if the necessary if the condition that that N S the cardinality of N S is greater than equal to the cardinality of s for all is subset of A, then we prove that the size of the maximum matching is equal to A; that means, there is a A saturated matching in the graph G. This is the proof of halls theorem using the Konig's theorem because we proved actually you prove that the size of the minimum vertex cover is equal to the cardinality of A. And by Konig's theorem we know that the size of the minimum vertex cover is equal to the size of the maximum matching. So, he proved actually prove that the size of the maximum matching is equal to the cardinality of A ok.

So, next we prove will solve some problems.

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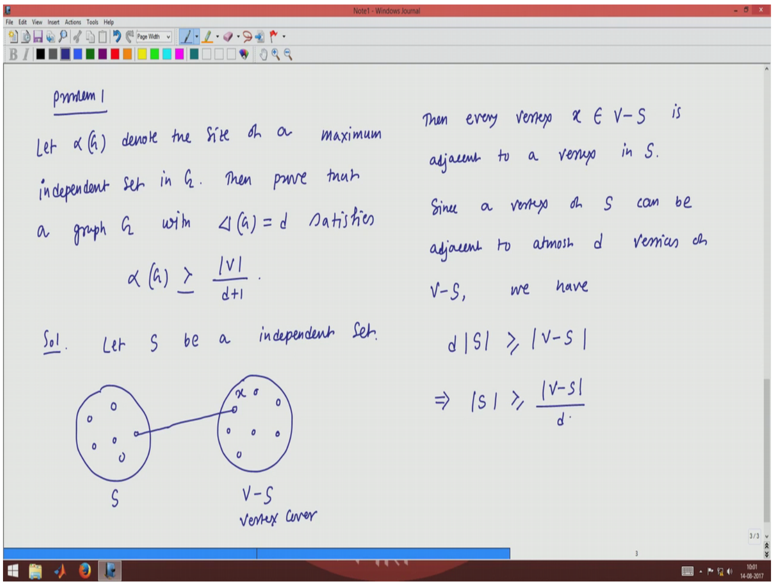

So, problem 1 related to matching vertex cover and all these things. Let alpha G denote the size of a maximum independent set in G, then prove that a graph in G with delta G, delta G is the maximum degree delta G equal to d; that means, the maximum degree the graph G has maximum degree d the graph satisfies, alpha G better than equal to cardinality of v by d plus 1. So, this is what you have to prove. So, l denote the size of the maximum independent set, then in G and G is a graph such that the maximum degree is equal to d then the cardinality of the alpha cardinality of the maximum independent set is greater than or equal to cardinality of v by d plus 1.

So, you can try proving this independently and then you can look at the solution here, let s be a independent set. So, you know what is independent set it is a set of vertices in the graph G such that no 2 of them are adjacent. So, they are all there is no edge between any 2 vertices in the set s. And then this one is v minus s. So, I am partitioning the vertices of the graph G into 2 parts s and v minus s. And then it is not difficult to observe that v minus a is the minimum if this is the maximum independent set then this is the minimum vertex cover because every edge or all the edges will be covered by the vertices in v minus s. Then every vertex x belongs to v minus s is adjacent to a vertex in s.

So, every vertex x in v minus s is adjacent to a vertex in s, suppose there is a vertex x which is vertex x in v minus s, which is not adjacent to any of these vertices here, then this s can be in the independent set right. And you will get a bigger independent set, but since this is a bigger independent set the large largest or maximum independent set, that is why this is true that every vertex x is adjacent to a vertex in s.

Since a vertex of s can be adjacent to at most d vertices of v minus s because this can be adjacent at most d vertices because the maximum degree is d. So, a vertex of s can be adjacent to at most d vertices of B minus s. Thus we have cardinality of s in 2 d because there are s many vertices here. And from for each vertex in s there could be at most d vertices adjacent to that x in v minus s. So, that is why this cardinality of v minus cardinal sorry cardinality of s into d is greater than or equal to v minus s. And also because of the fact that you say every vertex here is adjacent to a vertex in s.

So, there will not be any vertex left out without a neighbor N s. So, that is why this condition is true, and this implies that the cardinality of s is greater than equal to v minus s by d. So, we prove this problem or and next we talk about another problem.

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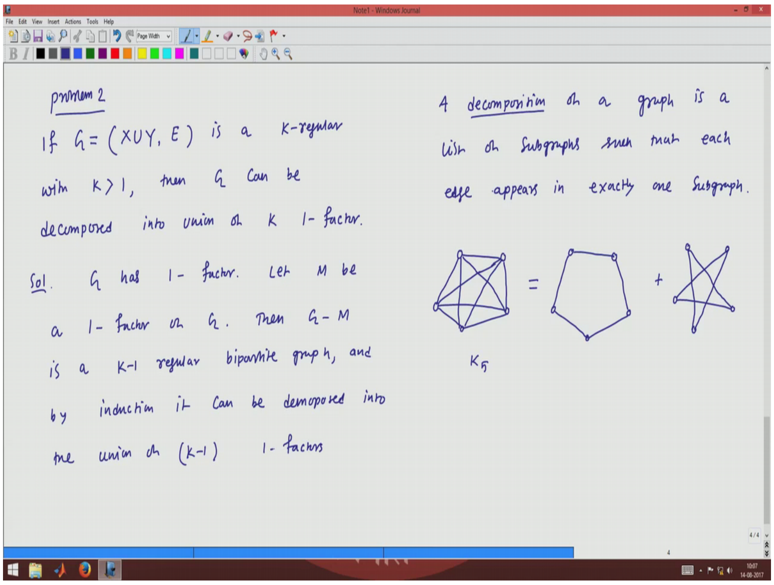

Say problem 2. So, before I talk about this problem. Let me talk about a term called graph decomposition, decomposition of a graph is a list of subgraphs, such that such that each edge appears in exactly one subgraph. So, it is the decomposition of the edges mostly.

Let me just give an example to illustrate what I mean by decomposition of a graph. So, consider this graph. So, every vertex here is having degree 4. So, it is basically k 5. And now I can decompose this into these 2 graphs these 2 sub graphs and you can see that. So, this graph can be written as union of these 2 graphs these 2 sub graph, this is one and the other one is right. So, k 5 can be decomposed into this plus this graph. So, this is what the decomposition of a graph means.

Now, let me state the problem if G is a bipartite graph x union sometimes we write A union B sometimes we write x union y is a k regular with k greater than 1, then G can be decomposed into union of k 1 factory, I hope that you know, what is one factor one factor is a perfect matching basically.

Since G is a k, k regular graph. So, it has a it has one factor. So, G has one factor one factor is one regular spanning sub graph or it is the perfect matching basically, let m be one factor of G. Then what you can say is that then G minus m is a k minus 1 regular bipartite graph right G minus m means you remove all the edges in m from G then the k regular graph will become k minus 1 regular bipartite graph. And by induction it can be decomposed into the union of k minus 1 one factors. So, this proves that that the graph G which is a k regular bipartite graph can be decomposed into k 1 factors that is all for today.

Thank you very much.