Graph Theory
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Lecture - 01
Part 2
Basic Concepts

Welcome to the second part of first lecture. In this lecture we learn what is walk, what is trail, path cycles in a graph, we learn when a graph is said to be connected graph and also we learn what is a complement of a graph and some related results. A let me start with walk in a graph so the definition here.

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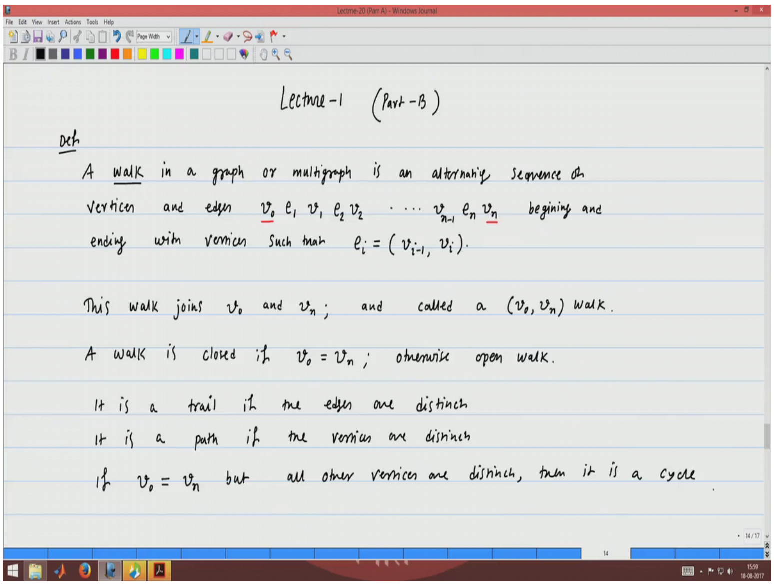

So, a walk in a graph or multi graph, so multi graph means you, here multiple parallel edges and loops are allowed and simple graph parallel edges and simple edges are not allowed.

So, walk in a graph or multi graph is an alternating sequence of vertices and edges v naught e 1 v 1, e 2, v 2 like this, and then v n minus 1, e n v n beginning and ending with vertices. So, the sequence begins with a vertex and end with the vertex, and such that this e i is basically an edge joining v i minus 1 v i.

So, this walk joins v naught and v n and it is called v naught v n walk and a walk is closed if the first vertex is equal to the last vertex otherwise open walk. So, this walk it is a trail if all this edges are distinct if the edges are distinct. I will explain using an example and this walk it is a path this walk is a path if the vertices are distinct and hence the edges are also distinct. If v naught these are really important because you know this definitions will be using throughout this course v naught equal to v n, but all other vertices are distinct then it is a cycle.

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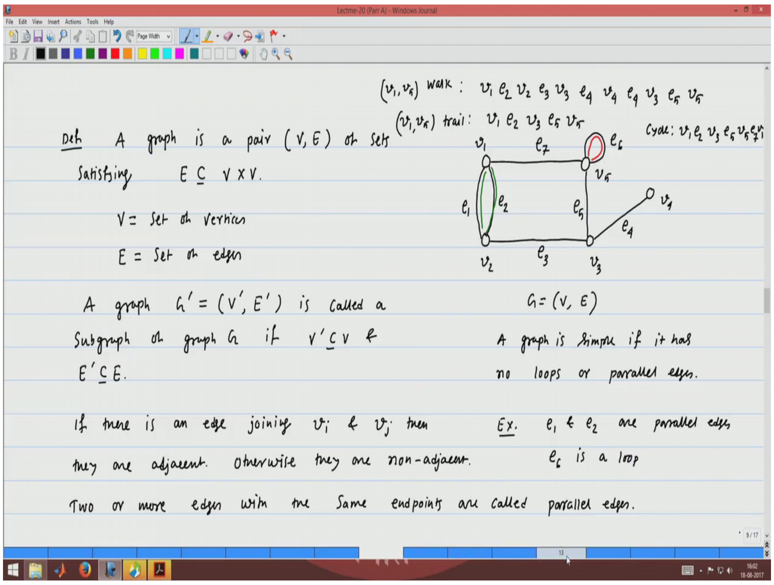

I will explain this definitions using this example maybe. Let me just give an example of a walk say v 1, v 5 walk. So, you start with v 1 and then go to v 2 with say anyone edge say e 2 and then you go to v 2 and then from v 2 you go to v 3 using the edge e 3 v 3, please note that you know we are using this edge also because there could be multiple edges between 2 vertices like here the 2 edges between v 1 and v 2. So, which edge you use here to denote that we write the edge levels in this sequence also. So, v 3 then e 4 you go to v 4 and then again use e 4 to come to v 3 and then you go to v 5 using e 5 v 5. So, this is a walk and this is a walk because of course, you know you can see that the same edge appearing twice. So, this is not a trail.

Now v 1 v 5 trail could be like this v 1, e 2, v 3, e 5, v 5. So, this is a trail because all the edges are a distinct here and this is also a path because all the vertices are also distinct and let me just give one more example of a cycle maybe. So, this is a cycle if you the in the in the case of cycle this first vertex will be equal to the last vertex so, you v 1, e 2, v 3, e 5, v 5 and then sorry e e 7, v 1 so this is a cycle right.

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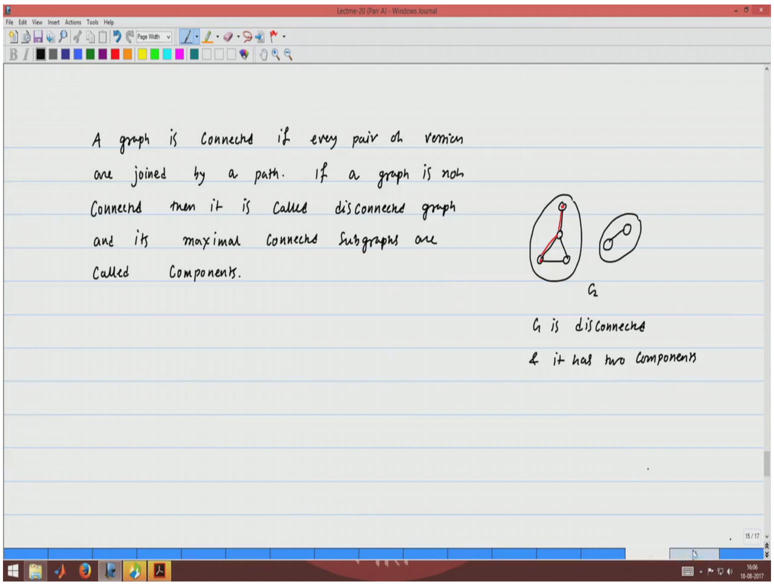

Now, we talk about when a graph is connected a graph is connected if every pair of vertices are joined by a path. If a graph is not connected then it is called a disconnected graph and it is maximal connected sub graphs are called components. Of course, the graph that we had before say for example, this graph this is a connected graph because there exist a path between every pair of vertices. Say if I take this vertex and this vertex there is a path they are not adjacent, but there is a path between these 2 vertices.

Now and this is true for any pair of vertices and now, if I add another edge another edge here or 2 vertices here and this is my graph now, this is my graph G, now you can see that there is no path from this vertex to this vertex. So, this graph is this graph is disconnected graph. So, this G is disconnected and it has 2 components. So, this is one component, this is the maximal connected sub graph of this graph and the other component is this one. If a graph is disconnected then it has at least 2 components.

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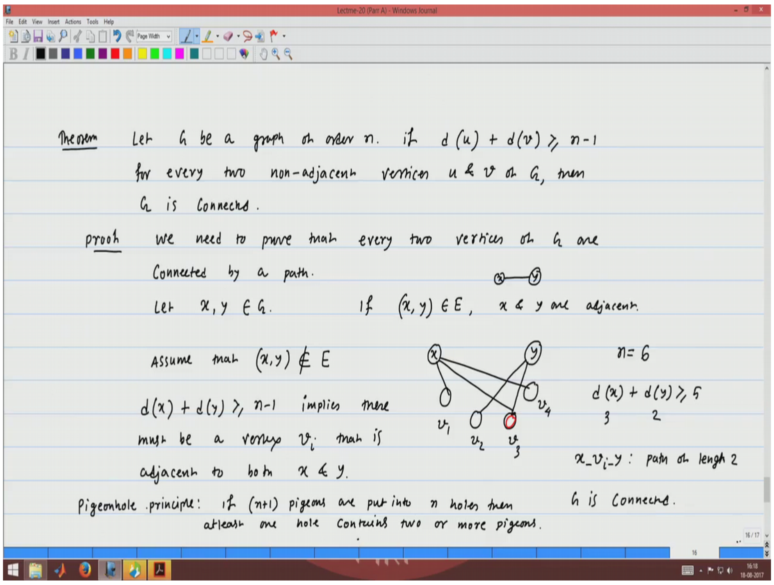

Now, we talk about a result theorem let G be a graph of order n graph of order n means it has n vertices, if degree of u, plus the degree of v is greater than equal to n minus 1 for every 2 non-adjacent vertices u and v of G, then G is connected. So, this says that you know G is a graph of order n it has n vertices, if the degree sum of 2 vertices, 2 non-adjacent vertices u and v if the degree sum is greater than or equal to n minus 1 then the graph is connected graph.

Let us prove this theorem. So, we need to prove that every 2 vertices of G are connected by a path. In order to prove a graph is connected what you have to do is that you have to prove that every pair of vertices in the graph are joint by a path so, that is what we have to do here. Now first let x and y be 2 vertices of G, the first case is that if x and y are adjacent; that means, they are is an edge between x and y like this is x this is y, then there is a path so this is a path of length 1.

So, you are done basically; that means, x and y are adjacent. So, assume that x and y assume that x and y are non-adjacent that mean there is no path between sorry there is no edge between x and y then. Let me draw the figure this is x this is y and they are not adjacent for the sake of simplicity or for to understand suppose n is equal to say 6 there are 6 vertices in the graph. So, these are the other 4 vertices in the graph.

Now, the degree sum is of according to this theorem the degree of x plus, the degree of y is equal to is greater than or equal to greater than or equal to 5. Now suppose the degree of x is 3 and degree of y is 2, see x and y are not adjacent then x degree 3 x suppose x is adjacent to this one this one and this one fine. So, the degree of x is 3 now the degree of y is 2. So, then y is again cannot be adjacent to x because x and y is not in edge. So, y can be adjacent to this, but y has to be adjacent to another vertex suppose this one. Now what we claim is that the degree of x plus the degree of y greater than equal to n minus 1 implies there must be a vertex.

In generally v I that is that is adjacent to both x and y and that v k is this one this is what let me call them v 1, v 2, v 3 v 4. So, in this example this v I is v 3. Basically and this has to be true because the degree sum is greater than or equal to n minus 1 and another way of understanding this fact is this is called pigeon hole principle, it says that if n plus 1 Pigeons are put in into n holes, then at least one hole contains 2 or more Pigeons. So, this is a very well known fact, here in terms of pigeonhole principle if you want to understand this one.

So, you have you have 4 pigeons sorry you have 4 holes. So, every vertex is like a hole and one edge is corresponds to one pigeon coming here. So, since there are 5 pigeons because 5 edges are there and 4 holes. So, at least 1 hole contains 2 pigeons, that is what it will be true in general.

So, what we understood is that this implies there must be a vertex v I that is adjacent to both x and y. So, x v I y is a path of length 2 I did not say what is path of length what is the length of a path. So, it has 2 edges in this path. So, I emitted the edge now because anyway the sequence of vertices are enough. So, there are 2 edges. So, that is why the length of this path is 2. So, we proved that when x and y are not an edge, there is a path of length 2 or there is a path between x and y always. So, we have finally, proved that G is connected. Now this is another result or theorem we can say this is this can be proved similar way.

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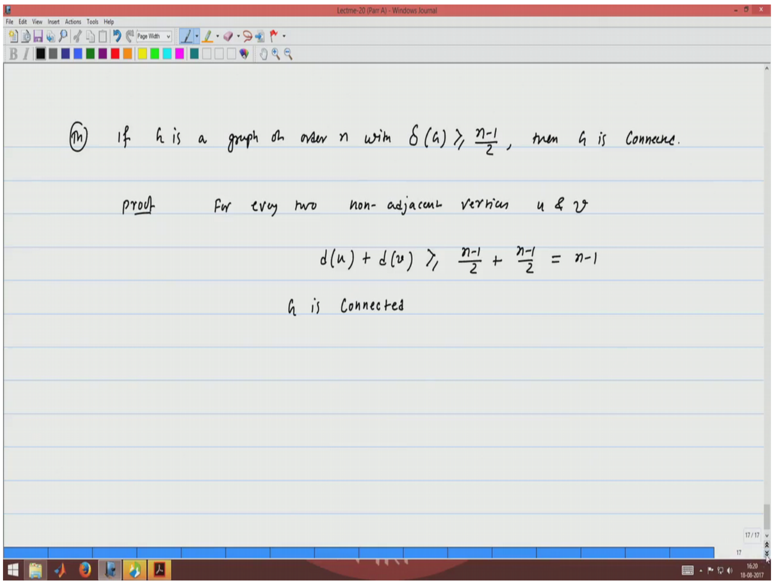

If G is a graph of order n with minimum degree greater than equal to n minus 1 by 2 then G is connected. So, the proof is very obvious from the previous result here for every 2 non adjacent vertices say u and v what we have that the degree of u, plus the degree of v is greater than equal to n minus 1, by 2 plus n minus 1 by 2 because the minimum degree is greater than or equal to n minus 1 by 2. So, sum of these 2 degrees is greater than equal to n minus 1.

So, using the previous theorem G is connected well.

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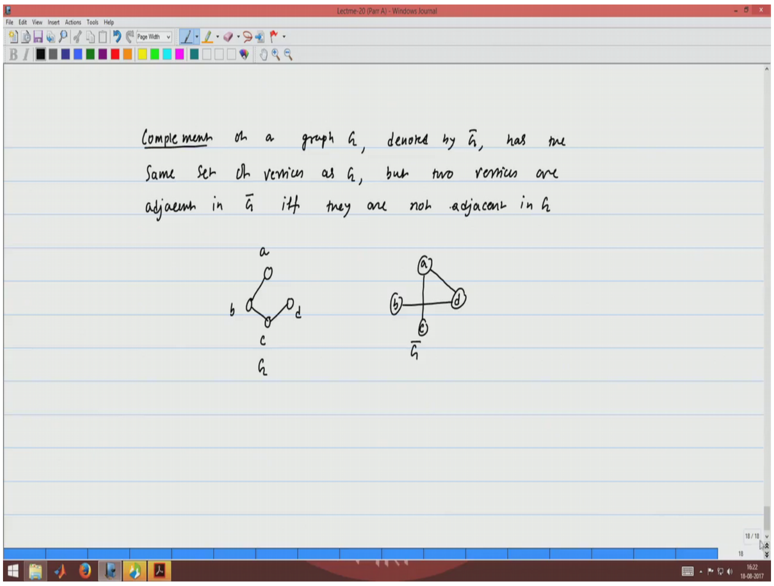

Next we talk about complement of a graph complement of a graph G denoted by G complement has the same set of vertices as G, but 2 vertices are adjacent in G compliment if and only if they are not adjacent in G.

So, let me just give a small example suppose this is the graph G let me label them a b c d this is my graph G and then what is G complement is it has the same set of vertices a b c and d. Now a and d will be adjacent because they are not adjacent in G a and c will be adjacent G.

Because they are not adjacent in g similarly d and b will be adjacent because they are not adjacent in g. So, this is the g compliment in g.

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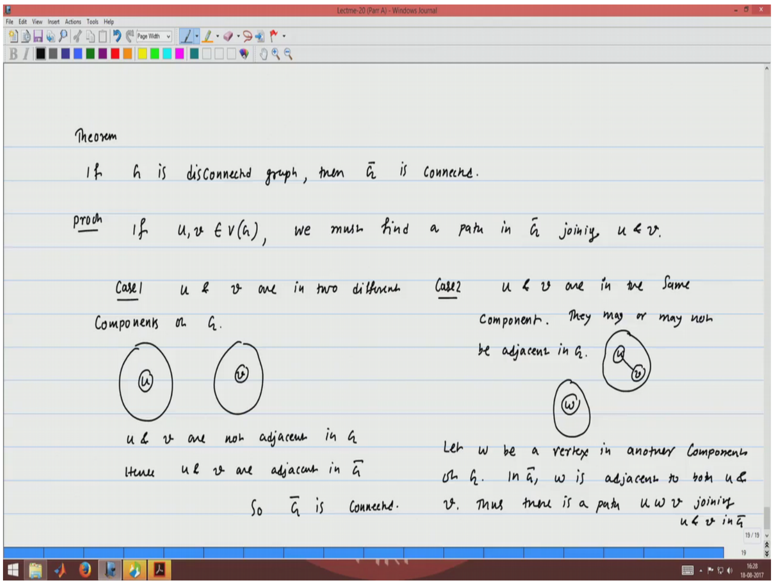

Now, let me prove a theorem if G is disconnected graph, then G complement is connected. So, I will just keep the outline of prove. If u and v are 2 vertices in the graph G, this is the set of vertices in the graph G notation we must find a path in G compliment joining u and v right.

Because as I said to prove that any graph is connected you have to prove that they are exist a path between every pair of vertices. So, we took a pair of vertices u and v and then we have to prove that there is a path between u and v in g compliment. So, the case one is that u and v are in v 2 different components of G; G is disconnected graph. So, the first case is that u is a vertex, which is in one component of G and the other vertex v is in another component of G. Since there in 2 different component u and v are not adjacent in G.

Hence u and v are adjacent in G compliment by the definition of g compliment. So, there is there is a path of length one between u and v in g compliment. The case 2 is that u and v are in the same component. So, in G they are in the same component. So, u is here and v is here now there could be 2 cases if there in the same component, but if u and v are not adjacent then they will be adjacent in G compliment. So, it is done suppose they are adjacent in G right. So, they may be they may or may not be adjacent in G.

Now, suppose w is another vertex in the other component, let w be a vertex in another component of G. Then w is neither adjacent to u nor adjacent to v. So, w will be in G compliment w is adjacent to both u and v because they are not adjacent in G. So, they will be adjacent in G compliment. Thus there is a path u w v joining u and v in G complement. So, we took a arbitrary pair of vertices and then we prove that in both the cases there is a path between them so G complement is; so G complement is connected.

In this lecture we have learnt what is a connected graph and what is the complement of a graph. And then finally we proved that if g is disconnected graph then G complement is a connected graph.

Thank you very much.