Graph Theory
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Lecture - 10
Part 1
Matching in General Graphs

Welcome to the tenth lecture on Graph Theory. So, in the first part of this lecture we learn matching in general graphs. More specifically we learn Tutte's theorem, which characterizes graphs with perfect matching. And it is a generalization of halls theorem from bipartite graph to general graphs. So, a perfect matching is a matching in which all the vertices are matched. So, let me start with Tutte's theorem; so matching in general graphs.

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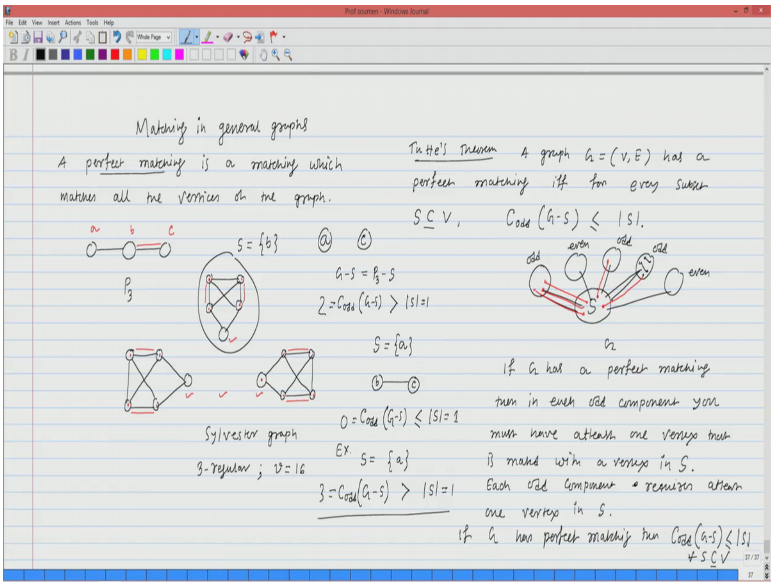

So, please recall that a perfect matching is a matching which matches all the vertices of the graph. So, let me start with one small graph. This is called p 3 path graph. Now for this graph you see you try to find maximum matching in this graph.

So, either you can includes include this as matched edge, but the other one. So, you can see that you. So, here with respect to this matching, only b and c are matched. So, a is not matched. And there is no way you can find a perfect matching for this graph of course, the obvious reason is that here you can see that the graph is having p 3 is having only 3 vertices. And you must have noticed that you need even number of vertices for a perfect matching.

Let me give an example of another graph which has even number of vertices, but still you cannot find a perfect matching. So, here is the graph. This is a 3 regular graph; that means, every vertex is having degree 3. And another copy of the same thing of now you can notice that the graph is 3 regular graph every vertex has degree 3 and it has 16 vertices. And this graph is called Sylvester graph.

Now, we will try to find matching or say maximum matching in this graph for this part of the graph if I just I may include this edge and this edge in my matching. So, these 4 vertices are matched, and similarly here also I can include this edge and this edge in my matching. And here also I include this 2 edges in my matching. So, all these vertices are now matched.

So, you are left with 4 vertices now. So, this 4 vertices no you are left with yeah you are left with 4 vertices; this one, this one, this one and this one. Now what maximum you can do is that you can include this in the matching then these 2 vertices are matched, but you cannot find a matching for these 2 vertices ok.

So, this is a 3 regular graph, and it has number of vertices is equal to 16. Then let me state Tutte's theorem which characterizes graph with perfect matchings. So, here is the theorem a graph G V E has a perfect matching. If and only if for every subset is of v, c odd this is the number of odd components of s minus, sorry of G minus s; if this is less than or equal to the cardinality of s.

Let me explain this one and then you will realize that this is an obvious necessary condition that, number of odd components of G minus s must be less than or equal to the cardinality of s. And this has to be true for every subset s. Now this graph say p 3 does not have a perfect matching because this condition is not satisfied. For p 3 let me consider my s to be equal to b, then if I remove s from the graph G I will be left with a and c.

So, this is So, G minus s G minus s this is my G minus s which is nothing but p 3 minus s. So, it has 2 components and both are odd components odd components means the a component having odd number of vertices is called an odd components. And a component having even number of components is called an even component.

So, here G minus s has 2 add components a and c. So, c odd G minus s is equal to 2 here which is greater than the cardinality of s which is equal to 1. So, this graph does not satisfy the necessary and sufficient condition of Tutte's theorem. So, that is why this graph does not have a perfect matching. Let me explain the same thing of course, you know if you take different subsets for example, for p 3 if you now take say s equal to a, then your G minus s is this graph. You are removing vertex a when you remove vertex a you remove the vertex a and it is adjacent edges also. So, you will be left with b c. So, here you can see that c odd G minus s G minus s, this is equal to 0 because G minus s s is having only one even component.

So, the number of odd components is equal to 1 component and that is a even component because it has 2 vertices. So, the number of odd components is 0, which is smaller than the cardinality of s which is equal to 1. So, for this is of course, the task condition is true, but this condition should be true for every subset. So, since this condition is not true for s equal to a for one subset, that is why this graph does not have a perfect matching.

Let me explain the same thing with respect to this graph bigger graph. If I call this vertex as vertex a and if I consider my s to be equal to the set a only. Then you can see that G minus s will be having. So, once you remove you remove all the adjacent edges. So, you will be left with 3 components. So, once you remove you remove all these edges.

So, this is G minus s. And you can see that it has 3 odd components, these are all odd components because it has 5 vertices. So, G minus s c odd is equal to 3 here, which is greater than the number of vertices in s which is equal to 1. So, here the Tutte's condition is violated. So, that is why this graph does not have a perfect matching.

Now, in general let me explain just give an outline why these are this is the condition required for existence of perfect matching. Let me draw this figure. I am not proving this theorem I am just giving an outline what why this is a necessary condition. So, this is my s and once I remove the vertices in s in general for a general graph I will be left with some components right. Like here I have 3 components one is remove s from the graph I have 3 components I can have any number of components.

So, some of these components are odd some maybe even also we do not know odd. So, this is even. Now if I look at the whole complete graph G then the edges are there could be one edge more than one edges anything. This is the complete graph not G minus s this is the graph G. And once I remove this s from g you will be left with these components.

Now, imagine that G has a perfect matching, if g has a perfect matching, then in each odd component you must have at least one vertex that is matched with a vertex in s, in s. This is this has to be true because this component has odd number of vertices.

Now, these 4 vertices can be matched suppose it has 5 vertices, these 4 vertices can be matched using the edges inside this component. Now one vertex will be left out and that vertex must be matched with the vertex in s. Similarly here also at least one vertex of this component will be matched with the vertex in s, it can be more than one also. So, it can be 3 also, 3 or 5 any odd number right not necessarily 1.

So, that is why it says that if G has a perfect matching then each odd components then in each odd component. You must have at least one vertex that is that is matched with a vertex in s. So, each odd component in other words each odd component requires at least one vertex in s. That is why if g has a perfect matching then the number of odd components c odd G minus G minus s this must be less than equal to the cardinality of s for all s in v ok.

So, this is what we sort of explained why the number of odd components in a graph, when you take any subset of the given a graph G you take any subset of any subset s of the vertex set. And if you remove those vertices from the graph, then you will be left with some number of components now number of odd components must be less than or equal to the number of vertices in s. So, this is a necessary and of course, sufficient condition for the existence of perfect matching in general graph.

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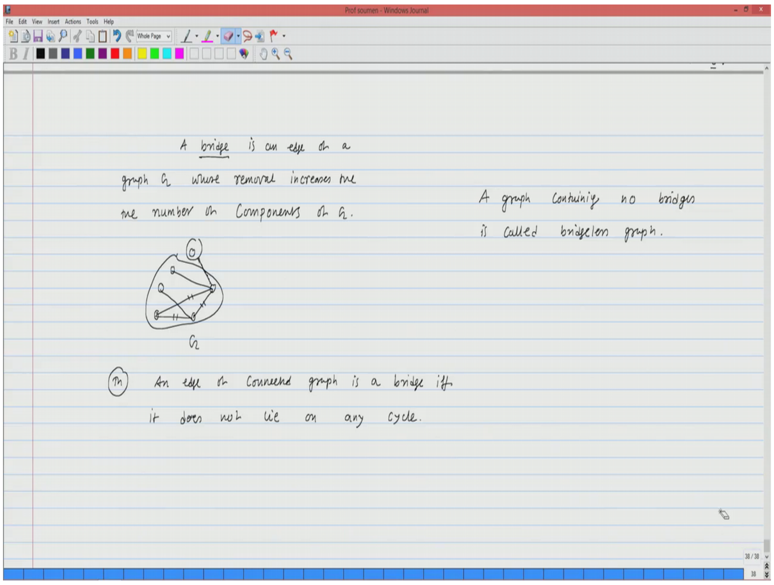

So, next we talk about one theorem related to perfect matching. So, before that I want to introduce a few definitions. First one is what is a bridge. So, a bridge is an edge of a graph G whose removal increases the number of components of G ok.

So, let me give an example. Consider this graph. Now in this graph you can see that if you. So, this edge is in bridge, because if you remove this edge you will have a component here. You will have it will increase the number of components at this moment the graph has only one component. Now if you remove this edge it will have 2 components now.

Now, you can see that the graph is having 2 components. This is one component and this is another component. So, that edge is a bridge. And similarly this edge is also a bridge this edge is also a bridge, but this edge is not a bridge. Even if you remove this one now go back to the original graph, if you remove this edge it will not increase the number of components. So, I hope that the definition of bridge is clear.

Now, you can see that this edge is also not an component not a bridge this edge is also not a bridge. Now you can see that there is a cycle here, and it can be proved easily that an edge of a connected graph is a bridge, is a bridge if and only if it does not lie on any cycle ok.

So, if an edge is on a cycle it is not a bridge. And finally, a graph containing no bridges is called bridgeless graph. So now, we are in a position to state our theorem.

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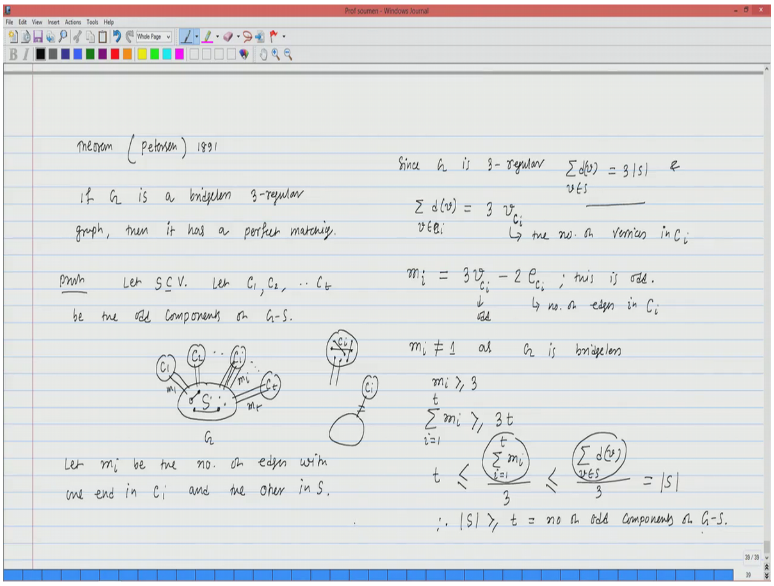

So, theorem is this theorem is due to Peterson. We have learned what is Peterson graph before and it is in 1891.

If G is a bridgeless 3 regular graph then it has a perfect matching. It is a nice result just you can see that we have seen an example here before. This is a 3 regular graph Sylvester graph, this is the 3 regular graph, but it is not a bridgeless graph these are. So, this is a bridge, this is another bridge, this is another bridge.

So, so it does not have if the graph is 3 regular and bridgeless it has a perfect matching that is what the theorem says. Now let me proof this theorem. So, to prove this theorem what will do is that will. So, that Tutte's conditions are true for this graph, and that will once we prove that the Tutte's conditions are true for this graph then we are done then it has a perfect matching ok.

So, let consider a subset of vertex v, and let c 1 c 2 c t be the odd components of G minus s. So, here is this. So, this is s, and if I remove s from the graph I will have several components say c 1 c 2 c i and ct. And in the original graph of course, they are all is in edge at least one edge is there from here to here.

We do not know how many edges are there between s and c 1 in the graph G. Let m i be the number of edges with one end in c i and the other in s. So, the number of edges from s to c i is m i this is mt this is m one we do not know that number at this moment.

Now, since g is 3 regular what we know is that the degree sum degree of v for v belongs to s. This is equal to 3 into cardinality of s, this is easy because every vertex in s has degree 3 it is a 3 regular graph. And summation of this degree sum for v belongs to G i is equal to what is 3 times, the number of vertices in G i.

So, this v G i v G i is the number of vertices in the component sorry in the component not G i it is a c i in the component in the component c i. So, this is v sorry about this v c i. So, v c i is the number of vertices in c i. Now this is a very important step that I am trying to compute the number of edges m i from c i to si you can see that this number is 3 times v c i that is the degree sum for the vertices in c i. So, 3 v c i minus 2 e c i ok.

So, see inside one component c i there are some vertices there are this many c v i vertices and c e c i edges. And this is the total degree for all the vertices in this component. And out of this total degree this much degree is consumed by the edges inside the component. And the remaining part will come out of this component.

So, that is why this is the number of edges between s and c i, and this is so this is the number of edges in c i. And you can check that this number this is this is odd right. Because odd this is this is odd right. Well, so the number of vertices in the- oh sorry this is this is odd because see this number this is odd component. So, v c i is odd this is odd this number is odd. So, that is why the whole thing is odd ok.

Now, this m i can not be equal to 1, because this is a bridgeless graph if there is an only one edge from c i to s then that is a bridge. So, m i can not be equal to 1 as G is bridgeless. So, m i is greater than equal to 3, right. Now from here what I get is that summation of m i I equal to 1 to t this is greater than equal to 3 t. And I do it for all components and that is why this is greater than equal to 3 t.

So, what I can write is that t is less than or equal to summation of m i, i is from one to t by 3 this is less than equal to summation of d V degree sum in s for v belongs to s. So, this component is smaller than this component, because you know this is the total degree sum this one is the degree sum for all the vertices in s. And this degree sum includes these edges also right there might be some edges inside this edge also.

So, that is why this sum plus all this m i is the degree sum for s. So, this component this term is bigger than this term this is by 3. So, that is why summation d v i v belongs to s by 3 is greater than this quantity. And from here I know that this is equal to cardinality of s. So, what we have proved is that the cardinality of s is greater than or equal to t, where the t is the number of odd components of G minus s. So, so by Tutte's theorem G has a perfect matching ok.

So, what we have proved just now is that if G is a bridgeless thee regular graph, then it has a perfect matching and we proved that the Tutte's condition conditions are true for this graph. So, it has a perfect matching.

Thank you very much.