Graph Theory
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Lecture - 08
Part 2
Hall's Theorem and Konig's Theorem

Welcome to the second part of lecture 8. Now, we will talk about vertex cover. Informally speaking, a vertex cover is a set of vertices; which cover all the edges in the graph. We will see the formal definition now. And it is minimization problem given a graph you have to find vertex cover of minimum size, because you know if you include all the vertices of the graph anyway it will cover all the edges. So, maximizing the vertex cover is not a problem at all.

And then after knowing the definition of vertex cover, we will prove that the size of minimum vertex cover is equal to the size of maximum matching in bipartite graph.

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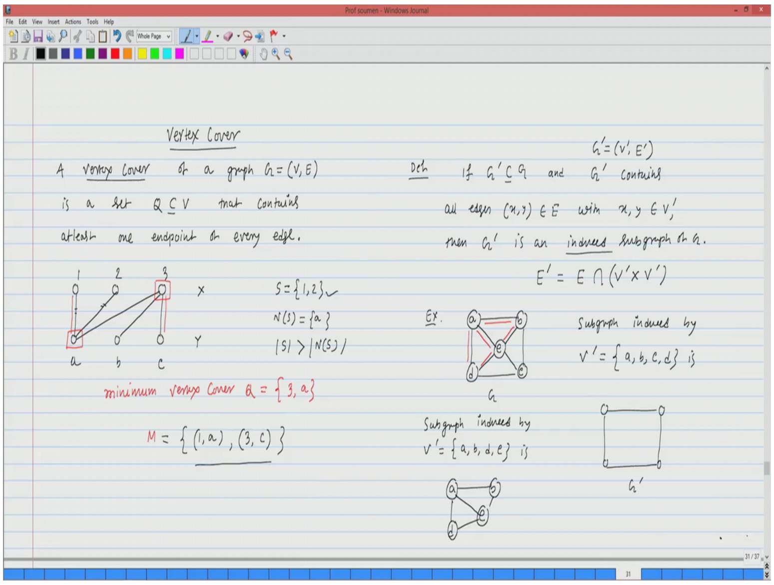

So, we learn formally, what is vertex cover? So, a vertex covers of a graph G, so it has the vertex set V, and the E set G is a set Q of the vertex set v; that contains at least one endpoint of every edge. So, once a vertex of Q contains an endpoint of an edge we say that the edge is covered by that vertex. Anyway let me just given example to explain the definition. Let me consider this graph, this is a bipartite graph. So, this side is one part say x part this is y part, but of course, this definition is for any graph. So, the vertices are 1, 2, 3 and a, b, c and these are the edges of this graph.

Now, a vertex cover is a subset of vertices that contains at least one endpoint of every edge. So, let me just choose these two vertices. Now you can see that this is a vertex cover. So, this is a minimum vertex cover minimum vertex cover Q is consist of 3 and a, why this is true? Because, you see that you take any edge here, so this edge is covered by vertex 3, this edge is covered by vertex 3, that is what the; because it contains one endpoint. So, this set contains one endpoint of this vertex, this edge. This edge is covered by both the vertices, this edge is covered by vertex a, this edge is covered by vertex a again. So, this is a vertex cover and this is a minimum vertex cover of course, if I include more vertex here say if I also include 2, then also it is a vertex cover, but you can it is not difficult to see that this q which consists of vertex 3 and a is the minimum vertex cover.

Now, as I said that we there will be theorem which sort of finds the relationship between the size of minimum vertex cover and matching maximum matching. So, what is the size of maximum matching in this bipartite graph? So, can you get a matching of size 3 here? You apply say for example, augmenting path algorithm in this graph. So, you will see that you can find matching of size maximum 2. So, this is this edge. So, let me consider the matching M sorry. So, the matching M is consists of on the edges 1, a, and 3, c and in this example for the bipartite graph of course, I can see that the size of minimum vertex cover is equal to the size of maximum matching.

Let me also say that since we have just learned Hall's theorem. You see that the Hall's theorem conditions are not satisfied for this graph. If you say; this is my x side. Now if I consider my one subset S to be say 1, 2, 3 and what is the neighbor of S here? The neighbor of S is you know; let me take the subset S equal to only 1 and 2, and then the neighbor of 1 and 2 is.

So, let me take S equal to 1 and 2 only. Then the neighbor of S is equal to a only a. So, a 1 and 2 collectively prefer a only. So, here you can see that the cardinality of s is greater than the cardinality of N S for this S. So, the condition of Hall's theorem is violated. So, you cannot get a perfect matching or x saturated matching here. So, that is also say that there cannot be matching up size 3. So, this is the size of the maximum matching and this is a small example you can it is not difficult to observe that. There is, this is the maximum this is the size of the maximum matching.

Before, I moves to Konig's theorem; let me just give one more definition which will be sort of used in the theorem. We know what is sub graph if G prime is a sub graph of G and G prime contains all edges x, y in the a set E with x and y belongs to V prime, then G prime is an induced subgraph of G and it is induced by V prime.

So, this V prime is the vertex set for G prime. So, we know short of what is subgraph? And this is what I am trying to define is that the induced subgraph, induced by the vertex induced by a set vertex set V prime. And this definition actually says that this the edge set of G prime is equal to E; can I write in this way? E intersection V prime cross V prime, I think this is also true. And now let me give example to illustrate this definition. Suppose the graph is this one: a, b, c, d and e. This is the original graph G.

Now, what is the subgraph? Subgraph induced by say V prime equal to a, b, c and d, so all the edges in the graph G involving a, b, c, d. So, these are the edges which involves a, b, c and d. So, these are the edges in the graph G, which involves a, b, c and d. So, those edges will be there in G prime. So, this G prime here the subgraph induced by this is the following. This is the subgraph induced by a, b, c, d. I hope that this definition is clear. Maybe one more example say subgraph induced by V prime equal to say a, b, d and e, a, b, d and e. Now a, b, d and e, will involve all this edges. So, all these edges will be in the subgraph, but not this one. So the subgraph induced by this is this one, a, b, d, e, I hope the definition is now clear.

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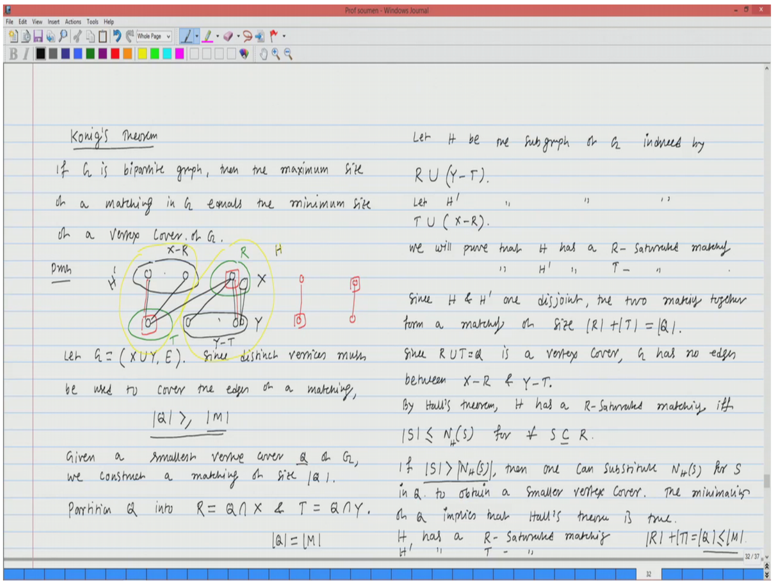

Next, we move to the very important theorem called Konig's theorem. If G is a bipartite graph, then the maximum size of a matching in G equals the minimum size of a vertex cover of G. Let me site the example that, we considered during the definition of vertex cover. So, it is bipartite graph. So, this vertex is adjacent to these vertices, and this vertex is adjacent to these vertices. And we know that this is a vertex cover and we also know that this has a matching of size maximum 2. So, this is true this theorem is true for this example and it is also in general to, let G be this graph X union Y, E.

Now, the proof of this theorem goes in this way. You start with a minimum vertex cover, and then you construct a maximum matching from that and then you will get one relation between their sizes. And the other way is that you start from matching and then you try to construct a vertex cover. So, let me start with; so since this is this is a matching you know that matching is a collection of disjoint edges, and to cover only the edges of matching you need two vertices right forget about the other edges, because matching is a collection of disjoint edges. So, you need does that many vertices to cover the edges of the matching.

Since, distinct vertices must be used to cover the edges of a matching, the cardinality of Q this is the Q is the minimum vertex cover, that cardinality of Q must be greater than equal to the cardinality of M, because this many vertices you need to cover. So, the matching here is this one right; this one and this one. So, this to cover this matching you need two vertices and to cover the remaining edges you need more vertices, that is why the cardinality of the minimum vertex cover is more than the cardinality of the matching.

Now, given a smallest vertex cover Q of G, we construct matching of size Q. Suppose this is the X part, this is the Y part and Q is a smallest vertex cover. Now we define, we partition this Q, so some vertices of Q could be from the X parts, some vertices of Q could be from the Y part. So, we partition Q into two parts: the one part is R which is Q intersection X, and the other part is T. So, I am introducing two new notations: R and T, this is Q intersection Y. So, in this example this is my R. So, in this example maybe this is just it consists of only one set, but in general this is the Q, this is a vertex in Q. So, Q intersection X that is R, and this is part of Q there could be more vertices also in general and this is T right; this is T, Q intersection Y.

Now we define two subgraph. A let H be the subgraph of G induced by R union Y minus T. So, what is y minus t? This part is Y minus T because the whole thing is Y, so this is Y minus T and this is X minus R, this is X minus R right. So, H 1, H be the subgraph G induced by these vertices. You understand the definition of subgraph induced by some set of vertices. And similarly let H prime be the sub graph of G induced by T union X minus R.

So, this is your sub graph is your H. This is the H subgraph, and this subgraph is H prime, this subgraph is H prime and this is H. Now what will do is that? We will prove that, we will prove that: H has a R - saturated matching; that means, H has a matching of size R, and also will prove that H prime has a T - saturated matching. And since now we can see that this graph these two graphs.

Since, H and H prime are disjoint, the two matching together form a matching of size R plus T, which is equal to the cardinality of Q. See we are given; we started with the vertex cover of size cardinality of Q, and then given that we are producing a matching of size cardinality of Q. So, that is what this and then once this is true then you can say that the matching the cardinality of M is greater than equal to cardinality of Q.

So, since R union T which is equal to the Q is a vertex cover, G has no edges between X minus R and Y minus T. This is just an object based in see there cannot be an edge from here to here because if there is an edge from here to here suppose there is an edge from here to here, then that is that edge is not covered by the vertex cover. So, vertex cover all the vertices are here and all the vertices are here. So, there cannot be an edge between X minus R and T minus Y minus T, and in that case if it is there it is not covered by the vertex cover. So, this edge there is no edge between this and this.

Now, to prove that: H has a R saturated matching. We have to prove that the Hall's conditions are true. So, by Hall's theorem, H has a R saturated matching if and only if S is less than equal to the neighbor of S in edge of course, so the neighbor of S in the graph H for all S subset of R. So, this is this must be true. Now if this is not true, if say S is greater than N S, then will come to a contradiction; if this is true that N S is the neighbor is smaller than this one.

So, S is a subset of R. So, S is somewhere here, somewhere here and it is neighbor is further small. Then one can substitute N H S for S in Q right. And since N H S smaller than so since this is smaller than this one, you will get a smaller vertex cover. To obtain a smaller vertex cover, think about this part. This is slightly tricky, but not so difficult. But Q is what we assume that Q is the minimum vertex cover.

So, the minimality of Q implies that Hall's theorem is true; that means, this cannot be true this has to be true. And so H has a R - saturated matching. And similarly H prime has a T - saturated matching. So, you get a matching finally, of size R plus T which is equal to the cardinality of Q. So, this is by this construction, but one can construct a matching of bigger size the matching maximum matching could be bigger than this one right. So, R plus T is equal to Q.

So, we got a matching of size cardinality of Q and the maximum matching size is greater than this one. So, what we got is that we got from this part we got Q is greater than equal to M and here we got Q is less than equal to M. So, finally, then one can say that Q is actually from this two inequality combining these two I get the cardinality of Q is the cardinality of M. So, that is the proof of the theorem, that the size of the maximum matching is equal to the size of minimum vertex cover in bipartite graph.

Thank you very much.