Graph Theory

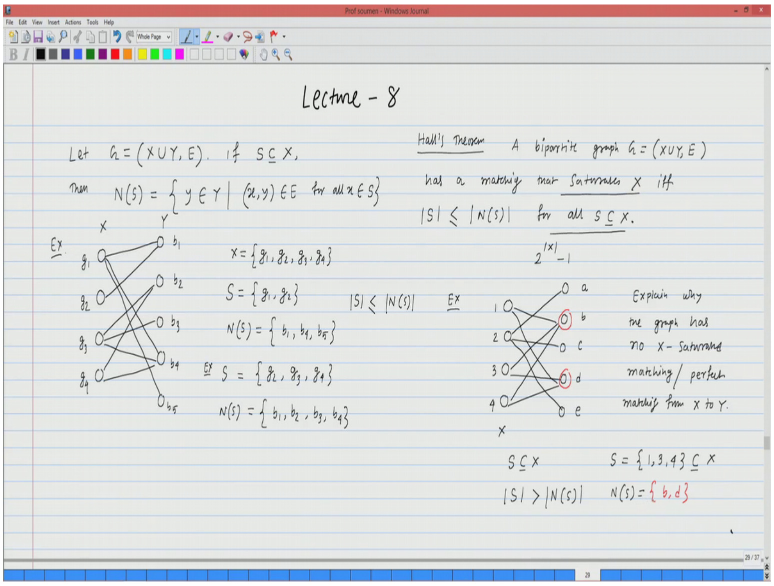
Prof. Soumen Maity
Department of Mathematics
Indian Institute of Science Education and Research, Pune

Lecture - 08
Part 1
Hall's Theorem and Konig's Theorem

Welcome to the 8th lecture on Graph Theory. In the first part of this lecture we learn Halls Theorem; Halls Theorem gives the necessary and sufficient condition for the existence of x saturated matching in bipartite graph and in the second part of this lecture we learn vertex cover and also we will see that the size of maximum matching is equal to the size of minimum vertex cover in bipartite graph.

So, let me start with Halls Theorem first.

(Refer Slide Time: 01:02)



Let G be a bipartite graph X Union Y E now if s is a subset of X then we define the neighbour of S the notation for this is N S. N S is equal to set of some vertices in Y such that xy are adjacent and for all x in S. So, you take 1 vertex from the set S and look at it is neighbour and the neighbour of S is union of all such neighbours of different vertices in x well I will give an example to illustrate this notion of neighbour of set S.

Let me consider the same graph that we consider in the previous class you have a set of 4 girls g 1, g 2, g 3, and g 4 and you have a set of 5 boys: b 1, b 2, b 3, b 4, and b 5 girl 1 or g 1 prefers b 1 b 4 and b 5 g 2 prefers only b 1 and g 3 prefers b 2 b 3 and b 4 g 4 prefers b 2 and b 4 well.

Now, say if I take so here X is the set of girls: g 1, g 2, g 3, g 4, and if I take a subset of X say S is equal to g 1 and g 2. Then the neighbour of S is the union of the neighbour of g 1 and the neighbour of g 2. The neighbour of g 1 consists of b 1 b 4 and b 5 b 1 b 4 b 5 and the neighbour of g 2 is b 1. So, the neighbour of S is this 1. Let me take another S say this is another example say S is equal to g 2 g 3 g 4 and g 2 prefer prefers b 1, g 3 prefers b 2 b 3 b 4, and g 4 prefers b 2 and b 4. The neighbour of S is union of all these 3 things 3 sets that is b 1 b 2 b 3 and b 4; this is what the neighbour of set.

Now, let me state Halls Theorem which gives the necessary and sufficient condition for the existence of X saturated matting matching a bipartite graph G X Union Y, E has a matching that Saturates X so I will explain what is this the meaning of saturates X or X saturated matching.

So, I buy a bipartite graph G has a matching that saturates X if and only if this condition is true that the cardinality of S is less than equal to the cardinality of neighbour of S for all S subset of X . The meaning of this 1 first let me say what is X saturated matching. So, if I say this is my X set X is the set of all girls and Y is the set of all boys, then the meaning of X saturated matching means all the girls are matched and similarly the meaning of Y saturated matching is that all the vertices in Y are matched.

Now the condition say here that you take any subset of girls say if I take g 1 and g 2 this is a 2 subset of the set X and then I can see that the cardinality of S in this case it is less than or equal to it is neighbour cardinality; that means, the meaning of this 1 that this 2 girls collectively they prefer 3 boys. So, that is the meaning of this 1 and this should be true for every subset of curls, this condition is very obvious because if there is a set of say 3 girls they collectively prefer say 2 boys only; that means, there is a set of 3 girls for which the neighbour set is having only 2 boys then you cannot find the matching for these 3 girls 1 of these girls will remain unmatched.

This is very obvious condition that a set of 3 girls or whatever be the number a set of N girls should collectively prefer at least N boys and that is what the condition of the Halls

Theorem what the existence of a X saturated matching. So, here X is the set of all girls. So, all girls will be matched if and only if any subset of the girl, prefer any subset of secure girls, prefer at least K boys. So, that is obvious condition, but the existence of X saturated matching.

Now, you can see that here the number of conditions how many conditions are there it says that for all X for all subset S in X. Basically I have to check 2 to the power of this many conditions. So, you have to consider all singleton sets and then 2 element subsets, 3 element subsets up to this many the whole X. This many conditions you have to check to see whether they exist a matching X saturated matching or not.

Let me give another example to sort of illustrate that just to explain that you know this is an example where you will see that the X saturated matching does not exist, because 1 of the conditions is not satisfied. I have this is my X side and in the Y side again there are 5 vertices; 5 vertices let me call them 1 2 3 4 and here it is a b c d and e. Now 1 is adjacent to b and d 2 is adjacent to or the neighbours of 2 are a c and e, the neighbours of 3 are b and d and the neighbours of 4 are b and d b and d.

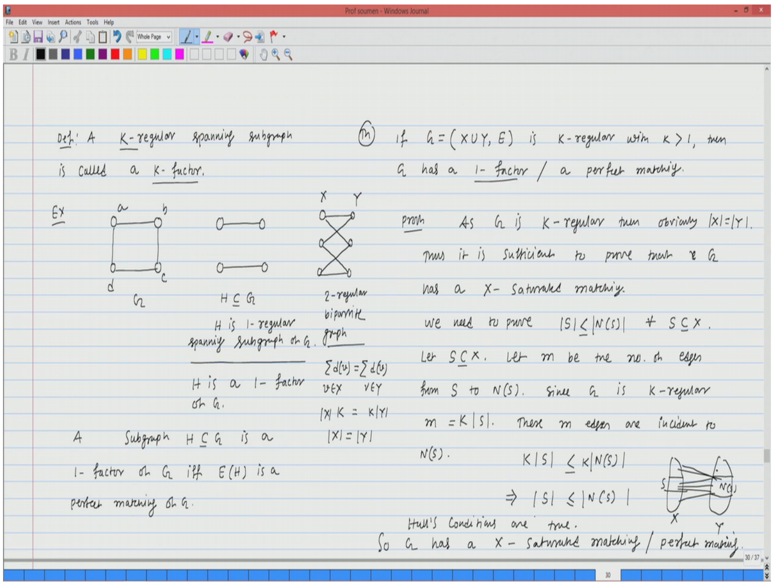
Now, the question you have to explain why the graph has no X Saturated matching, sometime this X saturated matching is also called instead of saying why the graph has no X saturated matching 1 can say why this graph has no perfect matching from X to Y well.

So; that means, you have to find some set of this some subset of this set X for who is the Halls condition is not correct. So, you have to find is a subset of this 4 vertices for which the cardinality of S is greater than N S; that means, a set of say 2 girls or 3 girls a set of 3 girls for example, they prefer collectively 2 boys only. So, you cannot find a perfect matching in that case, and it is not difficult to see in this example that this S could be 1, 3 and 4 you can see that S 1 3 4 if this is the set which is subset of the set X and then that you should look at the neighbour of S the neighbour of S 1 prefer b and d b and d 3 prefers again only b and d and 4 also prefers only b and d.

So, girl 1 3 and 4 they collectively prefer only b and d. Then you cannot find a perfect matching in this case because the Halls condition is sort of violated here. So, we learned the Halls condition for the existence of X saturated matching and I will omit the proof of

Halls theorem, next we will talk about several results and then those results will be proved using the conditions of halls theorem.

(Refer Slide Time: 16:48)



Next we talk about 1 factor of a graph K-regular spanning subgraph is called a K-factor. So, why I am introducing this definition because I want to say that perfect matching is equivalent to 1 factor, let me just give an example to illustrate this definition. So, I have defined what is 1 factor of a graph.

It is a spanning subgraph and it is a K regular spanning subgraph is called K factor of a graph suppose you are given this graph vertices are a b c and d and you know the meaning of spanning subgraph our sub a graph a subgraph is said to be spanning if it contains all the vertices of the graph. So, this is a spanning subgraph of the graph G. So, here H H is a subgraph of the graph G and also it is a spanning subgraph now you can see that every vertex of G is having degree 1.

So, H is H is 1 regular spanning subgraph spanning subgraph of G. So, 1 can also say that. So, telling that H is 1 regular spanning subgraph of G is same as this is same as saying that H is H is a 1-factor of G.

Now, you can see that 1-factor is a spanning 1 regular subgraph and which is same as H is also the edges of the subgraph H this is a perfect matching basically. It matches all the vertices and, 1-factor of a graph is basically a perfect matching of that graph. So, a subgraph sorry a subgraph H is a 1-factor of G if and only, if the edges of the subgraph is a perfect matching of G I hope that you understood this one. So, this is 1 factor of the graph G and this is also a perfect matching of the graph G.

Now we will prove a theorem or a result that if G a bipartite graph of course, X union Y E is K regular with K greater than 1, then G has a 1 factor which is same as telling that G has a perfect matching. So, I believe that you remember what is K-regular graph K-regular graph means every vertex is having degree K. If a bipartite graph is K-regular; that means, the number of vertices in both parts are the same right.

So, here is an example of say a 2 regular bipartite graph. So, every vertex has degree 2. This is degree 2 now yes, this is a 2 regular; 2 regular bipartite graph. And if this is the X side and this is the Y side then we know that the degree sum in the left side d of v v belongs to X is equal to the degree sum will be the same right d d of v v belongs to Y. And since all the degrees are eq are same say k here and here also all the degrees are K then and this sum is cardinality of Y into k and this sum is cardinality of X into K. So, this 2 are equal, X has to be equal to Y. I mean X has to be equal to Y means the cardinality of X has to be equal to the cardinality of Y.

Now let us prove this theorem that a curricular graph bipartite graph has a perfect matching. So, as I stated here as G is K-regular then; obviously, the cardinality of X is equal to the cardinality of Y. Thus it is sufficient to prove that G has a X saturated matching, since the cardinality of X is equal to the cardinality of Y, if the graph is having this is the bipartite graph if it is having X saturated matching, then that is the perfect matching because it will also match all the vertices in the Y side.

Now, a graph will have X saturated matching if and only if the hall condition Halls conditions are true. So, what are to prove is that you take any? So, we need to prove the hall conditions that we need to prove that the cardinality of the neighbour of S is greater than or equal to, the cardinality of S for all s subset of X. Let do it for arbitrary subset say S be a arbitrary subset of X and let M be the number of edges from S 2 NS

Now, what is m since this G is K-regular every vertex in s has degree 3 right. So, the number of edges that is going from S to ns is 3 times the cardinality of S. So, we see the number of vertices in S sorry not 3 times it is K times. It is K times the number of cardinality of the cardinality of S; this is the number of edges that is going from S to N S. So, these N edges are incident to NS.

Now, see that the vertices in NS they are also of degree K now, this NS the cardinality of NS country of less than the cardinality of X, because this many edges are going to NS and suppose NS has cardinality of NS many vertices and so, that in terms of the number of vertices going out of NS, that is this much and this must be greater than or equal to this 1 because suppose here is your X and here is your say S and this is your Y and this is your say NS. All the M edges they are all going to NS only, but some edges of NS are going outside is also right so, the number of edges going out of NS that is greater than or equal to the number of edges this one.

So, the cardinality of NS is this implies that the cardinality of S is less than equal to the cardinality of NS. So, Halls conditions are true, G has a X saturated matching which is also a perfect matching perfect matching. So, perfect matching is also called 1-factor.

This part is slightly tricky, but I hope that you understood because see the every vertex in NS are also of degree K right. So, this and what I explained here some of the edges are going out of not all the edges from NS are going to S they might go outside S. This condition is true.

So, what we prove is that if a graph a K-regular bipartite graph has perfect matching. So, that is all in the first part of eighth lecture.

Thank you very much.