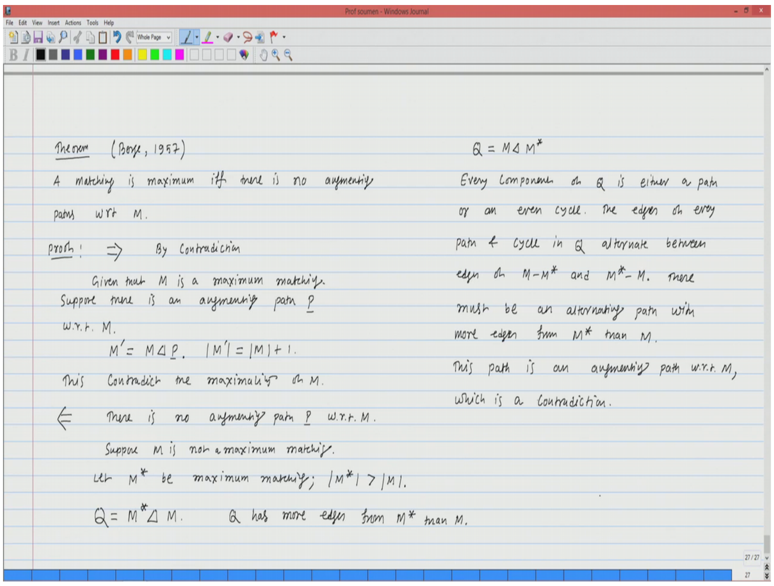
Graph Theory

Prof. Soumen Maity
Department of Mathematics
Indian Institute of Science Education and Research, Pune

Lecture – 07
Part 2
Maximum Matching in Bipartite Graph

Welcome to the second part of 7th lecture. So, we have learnt how to find maximum matching in bipartite graph using augmenting path algorithm. And the stopping criteria for the augmenting path algorithm is that- if there is no augmenting path with respect to the current matching then you stop there, and you return that matching as the maximum matching. We will prove a theorem now which says that a matching is maximum if and only if there is no augmenting path with respect to that matching. This theorem proves the correctness of the augmenting path algorithm.

(Refer Slide Time: 01:21)



Let me the state the theorem now. This is due to merge 1957 a matching is maximum if and only if there is no augmenting paths with respect to M. This proves the correctness of the augmenting path algorithm. So, we will prove this part that you are given that the matching M is maximum so both are by contradiction this part is by contradiction. So, what it says is that given that M is a maximum matching for the sake of contradiction, suppose there is an augmenting path there is an augmenting path P with respect to M.

Since there is an augmenting, but with respect to M you can improved your matching you get another matching M prime which is the symmetric difference of your matching M and the augmenting path P and then we know that .

Once we augment or the once we take symmetric difference of the current matching and augmenting path with respect to that matching; the matching size increases by 1 so the matching size or size of the matching M prime is equal to size of the matching M plus 1 right. These contradict the maximality of M. So, we have we started with the it is given that M is a maximum matching, but you are getting a matching which is of bigger size then the maximum matching. So, that is a contradiction; that means, there is no augmenting path with respect to M.

This part is reasonably and the other part is a given that there is no augmenting path P with respect to M and then you have to prove that the matching M is a maximum matching. So, this part is also using contradiction you have to prove that M is a maximum matching. So, you assume that suppose M is not maximum, M is not a maximum matching.

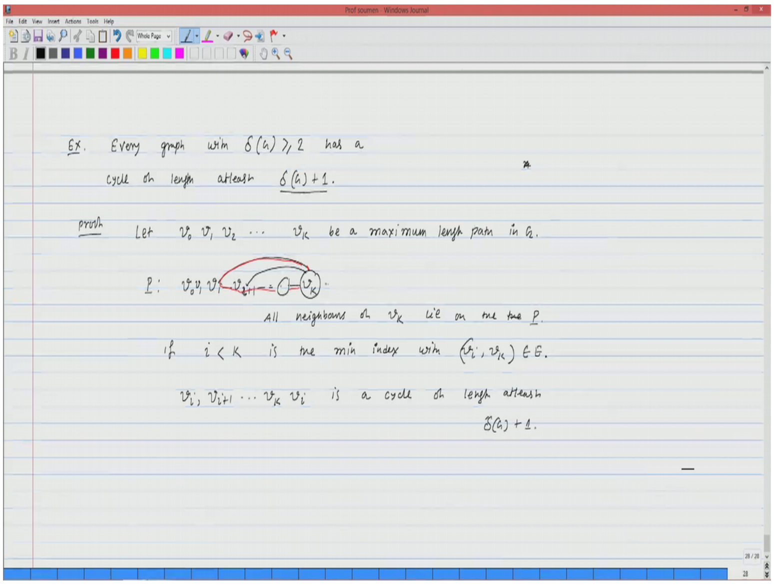
Later there is a bigger matching let M star be maximum matching and of course, M star is maximum matchings for the size of M star is greater than the size of M. Now we use the previous theorem. So, I have 2 matchings M star and M, and M star is a bigger matching now I take the graph like F was the notation for the previous theorem, I take the graph which is the symmetric difference of these 2 matchings. M symmetric difference M star symmetric difference M and we know the property of this graph all right. And since M star is bigger Q has more edges from M star than M this is.

Now from the previous theorem since Q is like symmetric difference of 2 matching, what we know is that that every component of this graph Q is either a path or a cycle right. So, every component of Q is either a path or an even cycle or an even cycle this is the theorem or result that we proved just now and also we know that this is not only their alternating cycle right. So, the edges the edges of every path and cycle in Q alternate between edges of M minus M star and M star minus M. Now, it is easy to understand all this statement because we have just now observed all this thing using an example.

Now since this is a even cycle every component of symmetric difference of M and M star is either an alternating path or an alternating even cycle. So, in the even cycle the number of edges from M minus M star and the number of edges from M minus M star are the same. So, the only difference, but there are more edges from M star then M in the graph Q, then there must be a path that is why there must be an alternating path with more edges from M star, then M and this path is an is an augmenting path is an augmenting path with respect to M, which is a contradiction. Because we have assumed that there is no augmenting path with respect to M and then for the sake of it is given that there is no augmenting path with respect to M and then for the sake of contradiction we considered M still not maximum matching.

Some other matching is maximum matching and then finally, we came to the point which says that there is an augmenting path with respect to M, which is contradiction to the base the first concentration that there is no augmenting path P with respect to M. So, that is the end of this proof. So, we have proved that a matching is maximum if and only if there is no augmenting path with respect to that matching and that that this theorem proves the correctness of the augmenting path algorithm well.

(Refer Slide Time: 13:26)



So, we have some time now so we will prove one problem which is not related to the maximum matching maybe. So, let me consider this problem every graph; every graph with delta G, if you can remember this is the notation for the minimum degree of the graph G minimum degree greater than or equal to 2 has a cycle of length at least delta G plus 1. So, good problem so, the minimum degree is given to be greater than or equal to

2 which is delta G then there is a cycle of length delta G plus 1 in the graph G. So, how do you prove this well, so let v naught v 1 v 2 up to v k be a maximum length path in G . So, here is the maximum length path v naught v 1 v 2 and v k since this is a maximum length path.

This proof is similar to a proof that we did in to prove that to prove a result in trees that every tree with n vertices has at least n greater than equal to 2 has at least 2 leaves well. Since this is a maximum length path I can claim that all neighbours of v k lie on the path, say call let me call this path as P on the path P. This is true because if there is a neighbour of v k which is not here you can list that vertex here and get a larger length path right. So, that is why since this is the maximum length path all the neighbours of v k are here.

Of course, this is this is a neighbour of v k and the other neighbours are also in this path right let me call it this is v 1 this is v i v i plus 1. If i is less than K and i is the minimum index with v i v k belongs to E; that means, this is the minimum index vertex in this arrangement which is a neighbour of v k then what you can do is that you get a cycle this is a cycle you form a cycle right, this is the largest cycle that you can construct here and what is the length of this cycle.

So, v i v i plus 1 v k and then again v i is a cycle of length at least delta G plus 1, this is not difficult to observe if all the neighbours are consecutive you will get a cycle of length exactly delta G plus 1 because delta G is the number of neighbours the minimum number of neighbours that the vertex v k has. So, this is a cycle of length delta G plus 1. So, we have proved that every graph with a with minimum degree delta G has a cycle of length at least delta G plus 1.

So, today we have learned how to find maximum matching in bipartite graph using augmenting path algorithm. And in the next lecture we will learn more properties of related to matching both in bipartite graph and matching in general graphs also.

Thank you very much.