Graph Theory
Prof. Soumen Maity
Department of Mathematics
Indian Institute of Science Education and Research, Pune

Lecture – 07
Part 1
Maximum Matching in Bipartite Graph

Hello, this is my 7th lecture on Graph Theory. In this lecture we will learn augmenting path algorithm to find Maximum Matching in Bipartite Graph. In the previous lecture we have learned: what is augmenting path. Augmenting path is an alternating path such that the path starts and ends with a unmatched vertex. So, augmenting path is with respect to a matching M always.

So, we will talk about augmenting path algorithm and then we will also illustrate the algorithm using an example.

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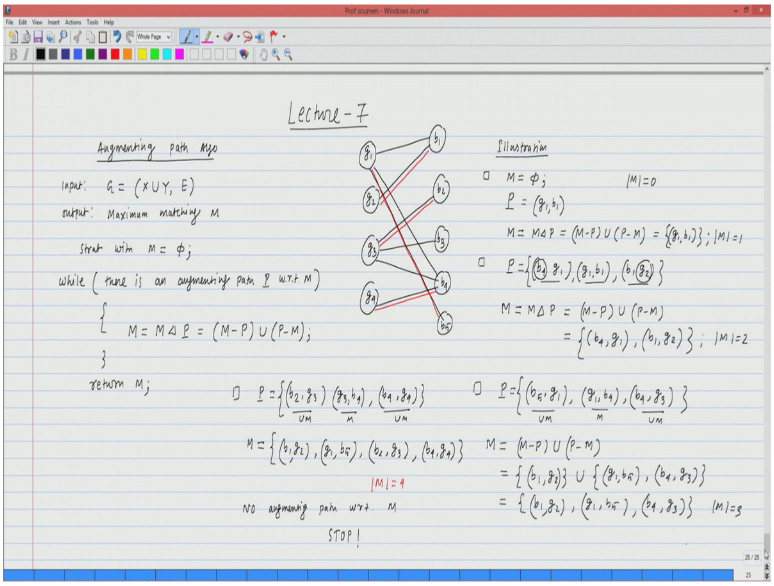

So, here is the Algorithm; augmenting path Algo. So input to this algorithm is a bipartite graph G with X Union Y and set of edges, and output is maximum matching M. This algorithm starts with a null matching it start with M null initially. So, M is the set of matching edges basically. So, initially your matching is null there is no matching edge in the matching M. And then you try to find an augmenting path with respect to the current null matching. So, while there is an augmenting path P with respect to M you do the following you just update the matching. So, you will learn that you know how every time you find an augmenting path the matching size will increase by 1.

This is M is M symmetric difference P I will explain all these things. So, this symmetric difference means this is a set of edges this is also a set up. So, P in terms of set of edges P minus sorry M minus P union P minus M this is called symmetric difference. And you are out of this loop when there is a there is no augmenting path with respect to the current matching and then you returned M, well.

So, this is the same graph that we took we considered in the previous class. Let me illustrate this algorithm with respect to this graph given here. We will try to find maximum matching in this bipartite graph. So, illustration so at the first iteration what we do is that as I said that this algorithm start with the null matching. So, M is equal to phi it is a null matching there is no edge in the matching and then you try to find an augmenting path P. So, here the augmenting path could be just any in all the edges are unmatched. So, you can just pick any one edge arbitrarily let me pick say g 1 b 1 this is an augmenting path, because as I said that an augmenting path is an alternating path, which starts and ends at unmatched vertices.

So, this P the P the pa augmenting path P given here that is g 1 b 1 it is an alternating path it is consists of only 1 edge and it start with an unmatched vertex g 1 and ends at unmatched vertex P1. So, at this moment nothing is matched. So, everything is unmatched. So, this is an augmenting path. Now you update your M. So, M is as I said M is M symmetric difference P; that means, M minus P union P minus M. So, here M minus P is M is null. So, M minus P is also null and P consists of this g 1 b 1 and M is null. So, this is P minus M is consist of this edge g 1 b 1.

So, this is your current matching. So, let me just denote that this is your current matching now at the second iteration you have a matching M and you have to find an augmenting path with respect to this matching. I could not give an example of an augmenting path in my previous lecture now I will give. So, I have to find an augmenting path with respect to this matching now. So, this is an augmenting path say b 4 g 1 b 4 g 1 then g 1 b 1, g 1 b 1 and then b 1 g 2 g 2. So, b 4 g 1 b 1 g 2 this is an augmenting path because this is an unmatched edge, matched edge, unmatched edge. So, it is a alternating path and this path starts at a unmatched vertex b 4.

And ends at another unmatched vertex g 4, this is an augmenting path well. So, you are inside this loop because still with respect to the current matching you have a augmenting path. So, you update your M now as I said every time you update M and the cardinality of M will increase by 1. So, initially the cardinality of this M was 0 and then the cardinality of this match this matching is equal to 1 now we will see that the cardinality of this matching will become 2. So, M is again M symmetric difference P where is M is this 1 and P is this 1. So, M minus P is null M minus P is null because M has this edge and you are subtracting or removing this edge from this set union P minus M.

So, P is this set of 3 edges basically and then you remove this edge from here. So, this will be consists of 2 edges now b 4 g 1 and b 1 and g 2. These are the 2 edges now in your matching. So, that your matching size has become now equal to 2. So, as I as I old before that every time you augment a augmenting path the size of the matching will increase by one at every step. Let me just then see this is no more so g 1 b 1 is no more and edge matching edge. So, what I will do is that I will remove that edge now this is not a not a matched edge let me draw this link. Now, the matched or matching edges are b 1 g 2 b 1 g 2 b 1 g 2 this is a matching edge and the other one is g 1 b 4 g 1 b 4.

So, I had a matching of size 2 at this moment let me see whether there is an augmenting path with respect to this current matching one can try to find an augmenting path with respect to this current matching. So, I have to start with the unmatched there could be several you have to start with an unmatched vertex. So, let me start with let me consider this augmenting path that b5 g 1 and then g 1 b 4 g 1 b 4 and then b 4 g 3 b 4 and g 3 and let me use a bracket here also that is all this is an alternating path.

Because this is unmatched this edge is matched, this is matched edge this is unmatched edge, this is unmatched edge, and this vertex b5 is unmatched vertex and g 3 is also unmatched vertex right b 4 g 3, right. So, g 3 is also unmatched vertex. So, one thing to observe is that when you are finding an augmenting path with respect to a given matching it not necessarily that this augmenting path will include all the matched edges in the matching. So, here you can see that this augmenting path P it includes only one matching edge g 1 b 4, but this path does not include this edge b 1 b 1 g 2 is not included in this augmenting path. So, anyway this is a augmenting path.

Now we augment this one in the current matching. So, you update your M now. So, M is M minus P and union P minus M. M minus P is M minus P you can see that this is b 1 g 2 b 1 g 2 is there and g 1 b 4 is cancelled. So, this one union P minus this again this will be cancelled this 2 will be cancelled and we will be left with g 1 b 5 and b 4 g 3. Now, you can see that your matching is having cardinality this has 3 match matched edges and b 1 g 2 g 1 b5 and b 4 g 3. So, what I will do is that I will update my colours here also.

So, g 1 b 4 is no more matched edge. I will remove this thing I will remove that thing now the new matched edges are g 1 b5 g 1 b5 is the new matched edge and g 3 b 4 is another matched edge right.

Now with respect to this matching now can you find an augmenting path that is the next iteration can you find an augmenting path with respect to the current matching, now the cardinality of this matching is equal to 3 yes. So, you can find this augmenting path you can check that this is an augmenting path b 2 g 3 b 2 g 3 g 3 b 4 g 3 b 4 and b 4 g 4 b 4 g 4 is an augmenting path because this is a matched edge these are unmatched edges and this is an unmatched vertex b 2 is also an unmatched vertex. And if you augment this one you can check that your new matching will be this one b 1 g 2 this will not be effected g 1 b5 will not be affected and then you will have b 2 g 3 and b 4 g 4.

So, this is a new matching with respect to the current augmenting path b 2 g 3 let me just draw it finally. So, this has been removed b 3 g 4 has been removed, now b 2 g 3 is the new matching and b 4 g 4 is another new matching edge. So, the cardinality of M at this moment is equal to 4, but that is not the stopping criteria. Now you try to find an augmenting path with respect to this matching you can check that you will not find an augmenting path with respect to this matching.

So, no augmenting path because the obvious observation that all the girls are matched. So, you will not get an augmenting path augmenting path is always from one unmatched boy to one unmatched girl. Since all the girls are matched no augmenting paths with respect to the current matching M. So, you stop here. This is the end of this algorithm and we have explained augmenting path algorithm with respect to one example here next we talk about why this algorithm is correct why the stopping criteria is that you stop when there is no augmenting path with respect to current matching you stop there and that is the maximum matching and you return that M.

So, why this is true that is a sort of will give a proof for this one that M is a maximum matching if and only if there is no augmenting path with respect to M. If we can prove this theorem that proves the correctness of this algorithm, before proving that theorem I want to talk about another result which is stated here. So, I will talk about this theorem I am considering 2 matching the statement of the theorem.

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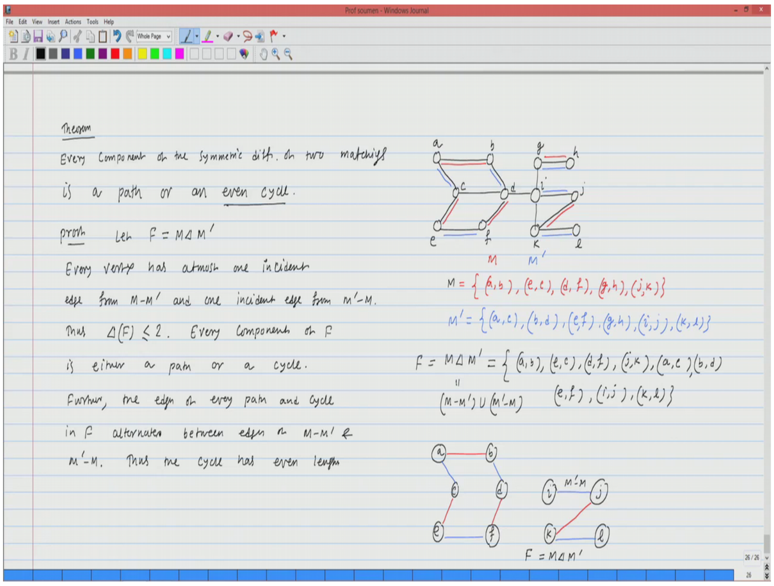

Let me write slightly difficult, but very useful theorem that every component of the symmetric difference of 2 matchings; 2 matchings is a path or an even cycle.

We need to understand this statement of this theorem let me illustrate the statement of this theorem using an example let me consider this graph. So, I am talking about symmetric difference of 2 matchings and what are the properties of that that graph which is just a symmetric difference of 2 matchings. This is the graph we are going to consider now there are 2 matchings I will talk about of this graph let me just denote them by different colours. So, this is one matching of maybe I am this is the red colour matching is matching M and this is a matching and so the M this is the matching M and I will considered another matching, which I will denote by blue colour so, this is part of the other matching M prime note that they satisfy the property of the matching this is M prime.

So, the M prime is the blue colour matching. So, I hope that you know there is no doubt with the statement that M is a matching because you can see that every vertex is adjacent to at most one matched edge in matching M sa sorry M is a matching. So, this vertex is with respect to the red matching this vertex is matched with this matching edge, this vertex is not matched with any, this is unmatched vertex with respect to the red matching right well.

Now if I take the symmetric difference of this 2 matching let me lavel them also. So, this is a b c d e f g h i j k l. So, my M consists of this edges a b e c d f g h j k right and my M prime consist of a c b d e f g h i j g h i j and k l v. Now you consider the symmetric difference so I am concerned about the component of the symmetric difference. I will take the symmetric difference of these 2 matching M symmetric difference M prime you see that the only common thing is g h. So, the symmetric difference will consist of all the edges except g h.

So, M symmetric difference M prime will be consists of a b e c d f not g h because this minus this will cancel g h j k and similarly here also when you compute M prime because this symmetric deference means it is M minus M prime union M prime minus M. So, when you compute M prime minus M only g is g h will cancelled out all the other edges will be there. So, this is a c b d e f not g h i j and k l so, this is the symmetric difference of these 2 matchings. Now let me call this one as f. So, this is a graph right it has some vertices and edges and I draw this graph again. So, this is this is my f graph which is the symmetric difference of these 2 matchings.

So, this is a b c d e f and then you consider that g h is not there. So, I will just remove those 2 vertices also. So, I will be left with i j k l and again I will use 2 different colours for because this is this is due to the matching M, this is due to the matching M, this is due to the matching M, this is due to the matching M and this edge is due to the matching M prime, matching M prime, matching M prime, matching M prime, matching M prime.

This is a beautiful result now what now you can see the claim now you understand that the statement of this theorem that every component of the symmetric difference of 2 matchings here the 2 matchings are M and M prime is a path or an even cycle you can see that.

So, these are this is my this is the graph which is the symmetric difference of 2 matchings M and M prime, you can see that this is one component of this symmetric difference or this is the graph F basically which is the symmetric difference of M and M prime, you can see that one component is a even cycle and the other component is a path this is a path basically right. Now this example will have will help you to understand the proof of this theorem. So, proof of this one let if with a graph which is the symmetric difference of M and M prime. Now it is an easy observation I will just write down the statement and then I will explain that that every vertex; vertex has at most one incident edge from M minus M prime.

That is mostly from M. So, every vertex has at most one incident edge from M minus M prime and one incident edge from M prime minus M. Let me at least explain this line from this example look at a vertex it has at most one edge from this is an edge from M minus M prime and this is an edge from M prime minus M. So, that is what it says that every vertex has at most one incident edge from M minus M prime and one incident edge from M prime minus M look at this vertex I said at most. So, this vertex does not have any incident edge from M minus M prime, but it has a incident edge blue colour is from this is an edge from M prime minus M, but no edge from M minus M prime.

So, this implies that every vertex can have degree maximum 2 thus we write delta F see this delta is different this delta and this delta are different this is the symmetric difference and this is the notation for a capital delta is the notation for the maximum degree of the graph F. The maximum degree of the graph F is now 2 it is at most 2. Since the maximum degree is at most maximum degree is 2 every component of the graph every component of F is either a path or a cycle this is a easy observation. Further you can see that the statement of this theorem that it is not only cycle it is a it is has to be an even cycle.

So, that part we will prove now. Further the edges of every path and cycle in F alternate between edges of M minus M prime and M prime minus M this is true. So, you can see that in a path so if there is a path that path is the alternating path, the edges will be one edge will be from will be blue is the next one will be red again blue red like that and that is true also the cycle is also an alternating cycle.

It cannot happen that you have 2 here also it is a blue edge that is not possible then this is not a matching. So, that is from the definition of the matching. So, the cycle is an alternating cycle the path are also alternating path and since the cycle is an alternating cycle thus the cycle has even length thus the cycle has even length.

So, this is very important result according to me at least that every component of 2 matchings sorry you take 2 matchings M and M prime and the symmetric difference of 2 diff 2 matching will give you a graph and the components of the symmetric difference of 2 matchings are either up either a path or it is a even cycle. We will use this result to prove that that important theorem that a matching is maximum if and only if there is no augmenting path will with respect to that matching M.

Thank you very much.