Graph Theory

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Lecture - 01
Part 1
Basic Concepts

Welcome to the first lecture on Graph Theory. My name is Dr. Soumen Maity. I am faculty at Indian Institute of Science Education and Research, Pune. I am grateful to IISER, Pune and NPTEL, IIT Madras for giving me this opportunity to work on this project. This course graph theory is divided into 7 modules. And in the first module we will talk about basic definitions.

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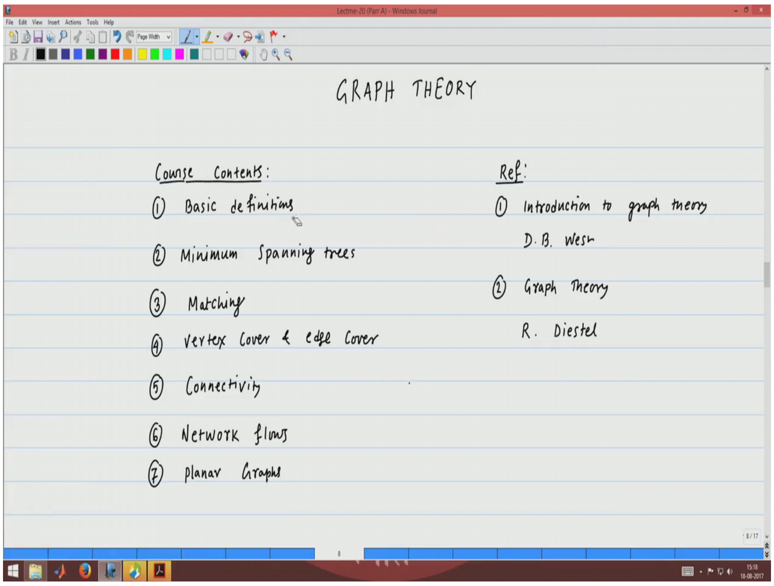

So, these basic definitions will include Eulerian graph, Hamiltonian graph, bipartite graph, complete graph trees and many more. And then next we talk about minimum spanning trees, and here we talk about prims and Kruskal algorithm to find minimum spanning tree in a weighted graph. Third we talk about a matching and we learn how to find maximum matching in bipartite graph. In the fourth module we learn what is vertex cover, and edge cover and also independent set. In the fifth module we learn what is connectivity; we talk about vertex connectivity of a graph and edge connectivity of a graph and how vertex connectivity and edge connectivity are related.

Next we talk about network flows. And in this module we learn how to find maximum flow in a given network using ford Fulkerson algorithm. Also we learn what is minimum cut and we will prove that the maximum flow is equal to the minimum cut. And finally, we talk about planar graphs and colouring. So, we talk about vertex colouring edge colouring and colouring planar graphs.

So, the reference for this course is introduction to graph theory by DB west and the second reference is graph theory by R Diestel. Let me start with the definition of graph.

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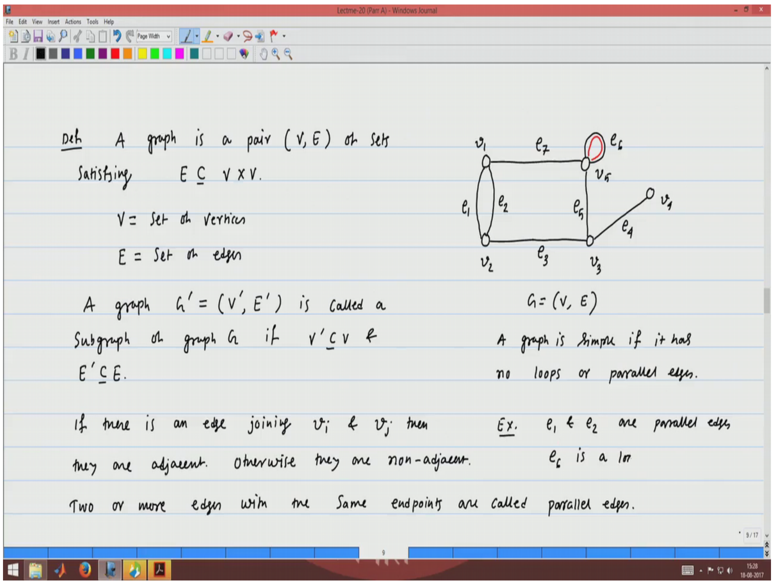

So, a graph is a pair V E of sets satisfying E is a subset of V cross V. Here V is the set of vertices, and E is the set of edges and a graph G prime which is having the vertex set V prime and E prime is called a sub graph or graph G, if V prime is a subset of V and E prime is a subset of E.

So, let me give an example of a graph. So, these are the vertices of the graph, and I call them v 1, v 2, v 3, v 4, v 5 and an edge joints 2 vertices. So, this edge I call e 1 which joints v 1 and v 2. This is another edge which joints e 1 and e 2, I call it e 2. So, the edge e 7 joints v 1 and v 5. E 3 joints v 2 v 3. E 5 joints v 5 and v 3, and e 4 joints with the v 4. And e 6 joints v 5 to v 5 ok.

So, this is a graph G with vertex set v 1 v 2 v 3 v 4 v 5, and the edge set consists of e 1 e 2 up to e 7. If there is an edge joining v i and v j then they are adjacent. So, here you can see that v 2 and v 3 are adjacent because they are joined by an edge e 3, whereas, v 2 and v 5 are not adjacent. Otherwise they are non adjacent. 2 or more edges with the same end point are called parallel edges. And a graph simple is simple if it has no loops or parallel edges.

So, for example, here e 1 and e 2 are parallel edges and e 6. So, e 6 is a e 6 is a loop e 6 is a loop, where as e 1 and e 2 they are parallel edges because they have the same end points.

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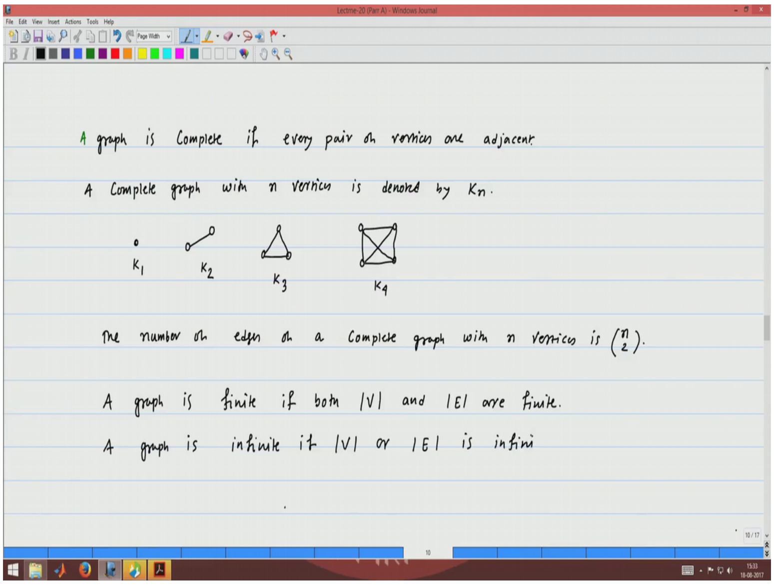

So, next we talk about complete graph a graph is complete if every pair of vertices are adjacent.

So, for example, this is a complete graph with one vertex that is called k 1. So, this is a complete graph with 2 vertices this is denoted by k 2. This is a complete graph with 3 vertices. This is called k 3, this is a complete graph with 4 vertices, it is k 4.

So, a complete graph with n vertices is denoted by k n. So, definitely the number of edges of a complete graph with n vertices is n choose 2. Because if there are n vertices then every pair of vertices are joined by an edge. So, there will be n choose 2 pairs and n choose 2 edges.

Next a graph is finite if both the number of vertices is a cardinality of V, and cardinality of E are finite. And a graph is infinite if cardinality of V or cardinality of E is infinite. So, understand special type of graph now that is called complete graph, and k n is denoted is the notation for complete graph with the n vertices.

Now, next we learn what is degree of a vertex. So, definition the degree of a vertex V in a graph G denoted by d V is the number of edges of G incident to V.

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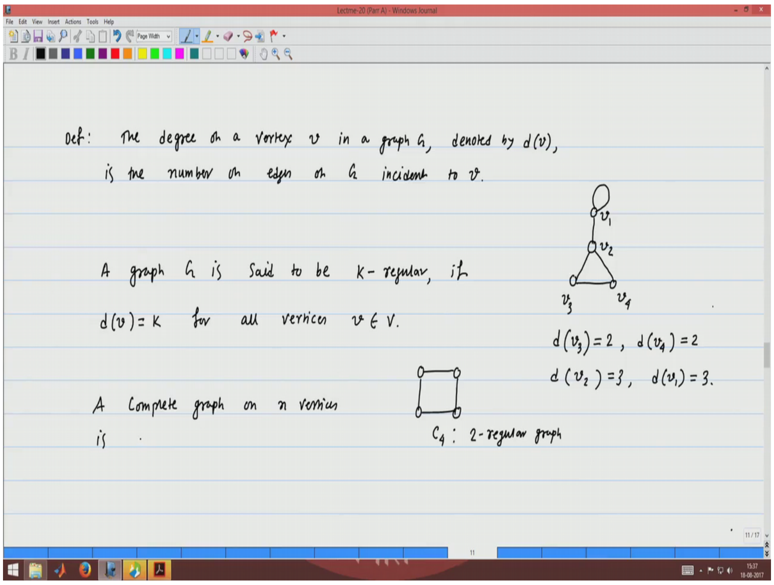

So, let me give an example of a graph. So, I will label this these vertices by v 1 v 2 v 3 v 4 and there is a loop here. So, here you can see that the degree of degree of v 3 is equal to 2 because 2 edges are incident to v 3 degree of v 4 is also 2 because there are 2 edges incident to v 4. Degree of v 2 is equal to 3, because there are 3 edges incident to v 2 and the degree of v 1 is also 3 because this will be counted 2 degree plus 1 degree here.

A graph G is said to be k regular if the degree of V is equal to k for all vertices V belongs to k. So, an example of a regular graph this is called c 4 cycle of length 4 you will learn soon. So, here you can see that the degree of every vertex is 2. So, this is 2 regular graph. And one observation that a complete graph, graph on n vertices is n minus regular.

So, this is because a complete graph with n vertices. Every vertex is adjacent to all the remaining n minus vertices. So, every vertex is having degree n minus 1. So, a complete graph with n vertices is n minus 1 regular. Next we talk about a theorem.

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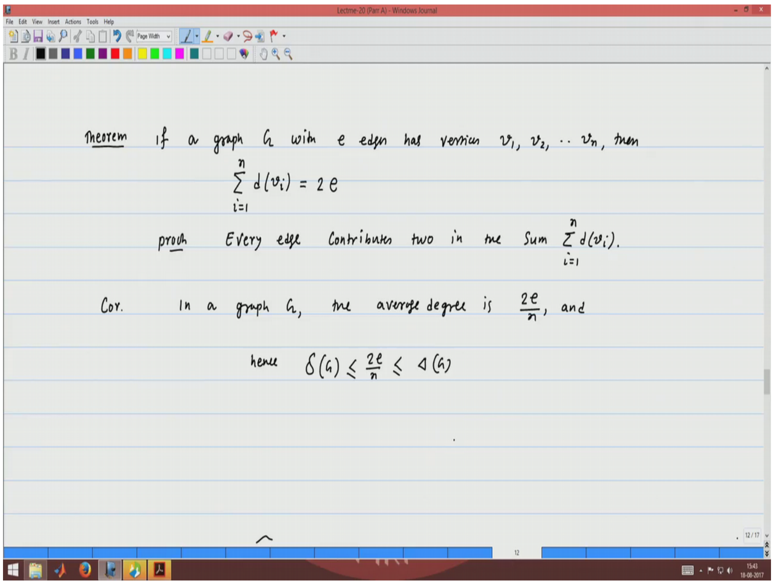

If a graph G with E edges has vertices v 1 v 2 V n. Then the degree sum degree of the vertex v i, i equal 1 to n is equal to twice the number of edges.

So, E is stands for the number of edges and n stands for the number of vertices. So, in our previous example if I see that here the degree sum there are 4 vertices. So, the degree sum is summation of d i, sorry d of v i i equal from 1 to 4 is equal to 10. And this graph has 5 edges. So, E equal to 5. So, this degree sum is equal to twice of E right. So, it is easy observation because the proof is that every edge contributes 2 in the degree sum right. That is why this is the degree sum is equal to n.

So, we can see here that this edge for example, this is contributing 2 degree in the degree sum. So, one degree is counted as degree of v 4 degree of v 4 is 2 because of this edge and this edge. And the degree of v 3 is also 2 because of this edge and this edge. So, this edge is contributing 2 in the degree sum that is why the degree sum is twice the number of edges. Now we denote something called maximum degree; so that this notation that delta of G is equal to the maximum degree. So, we will use this notation several times and the maximum degree here is 3. So, this is 3 here and small delta G is the notation for minimum degree, and the minimum degree here is 2, right.

Now, corollary of this theorem in a graph G, the average degree is twice e by n, because this is the degree sum and there are n vertices. So, that is average degree of every vertex is twice e by n. Ana obviously, this average degree twice e by n. It will be smaller than the maximum degree and this will be the average degree will be greater than the minimum degree of the graph g. So, these are all fine.

Next we talk about another theorem.

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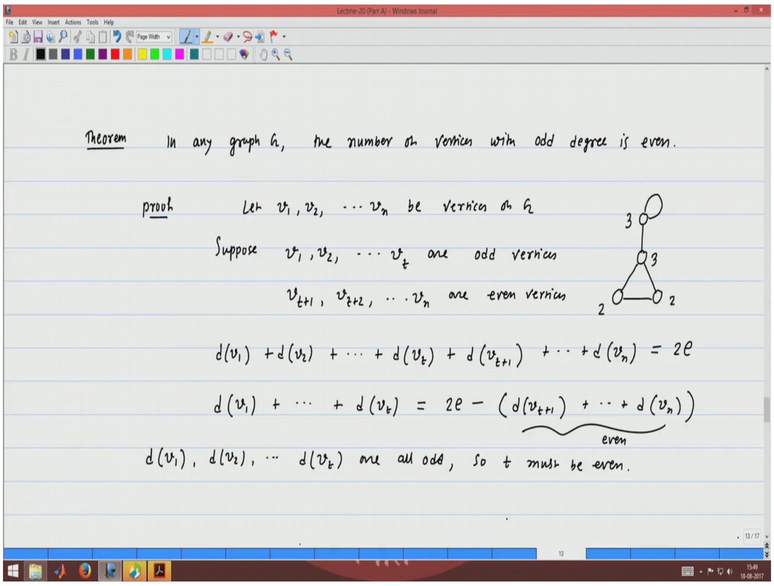

This says that in any graph G the number of vertices with odd degree is even. So, if I draw this graph again, now these vertex has degree 2 this vertex has degree 2 this vertex has degree 3 this vertex has degree 3. Now you can see that the number of vertices having odd degree is 2. And that is always true, why this is true? Here is the proof of this. Let v 1 v 2 and V n be vertices of G. And for simplicity suppose v 1 v 2 up to v t they are odd vertices. And v t plus 1 v t plus 2 up to V n are even vertices. The meaning of this one is that odd vertices means the vertices with odd degree even vertices means the vertices with even degrees. So, these are the even vertices these are the odd vertices.

Now, what we know is that we know that the degree of v 1 plus the degree of v 2 plus the degree of v t plus the degree of v t plus 1 plus the degree of v n that is the degree sum is equal to twice e right. Now what we can do is that I can write degree of v 1 plus degree of v t these are the odd vertices equal to twice E minus degree of v t plus 1 plus degree of v n. Now this is 2 E is even and these are all even vertices.

So, this is also even. So, this is even. So, the whole thing the right hand side is even right. The whole thing is even, even minus even. So, it is even now we know that this each term here they are odd because this is the odd degree. So, the left hand head left hand side this will be even only if the number of terms are the number of terms is even. So, here degree of v 1 degree of v 2 degree of v t are all odd, but their sum is even the sum is even. So, t must be even and the t is the number of odd vertices. So, what you have proved is that if in any graph the number of vertices with odd degrees is even.

So, in the first part of first lecture we have learnt some basic definitions like degree of vertex and some related results. In the second part we learn what is walk trail connected graph complement of a graph, and many more.

Thank you very much.