Numerical Analysis
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Lecture 6
Part 1
Error in Interpolation-2

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In the last class we obtained an expression for error in interpolation as this and bound on error in the interpolation as this, so we shall consider some example and see how we can obtain such an error bound in this case, so let us solve this problem. Supposed say we have given a function f(x) = sin x and it is approximated by a polynomial of degree 9 that interpolates the function at a set of 10 discrete points in this interval 0 to 1.

The question is how large is the error on this interval? Can you provide an estimate on the error bound is the question, so let us work out the details? We are given f(x) to be = sin x and we want to determine the error in interpolation when we represent this function sin x by a ninth degree polynomial which interpolates this sin x at a set of 10 distinct points. So I have to compute on, let us first relate this formula with what we want and what we are given. So we are given that the function is approximated by a polynomial of degree 9, so we would like to find what is f(x) + P n of x?

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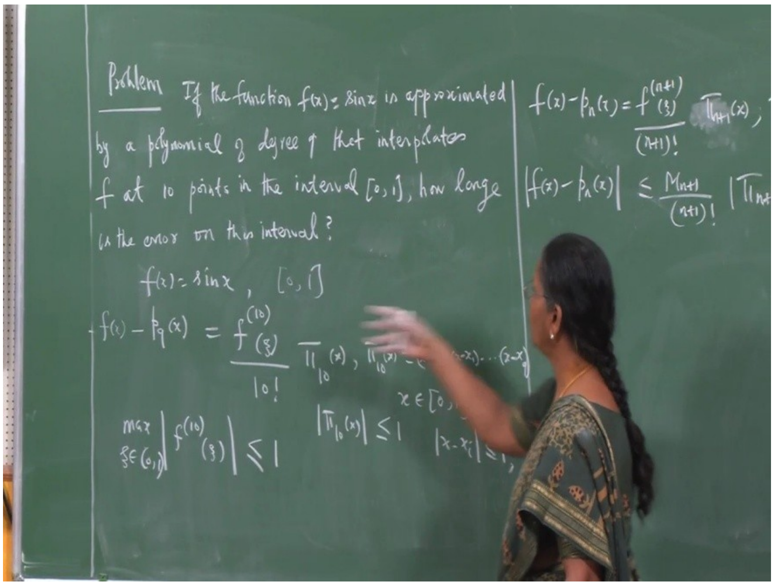

The theorem says it = the 10th derivative this function at some Xi belonging to the interval open interval 0, 1 by 10 factorial multiplied by Pi 10 of x. What is this Pi 10 of x? Pi 10 of x will be x + x 0, x + x 1, etc up to x +. What is n? n is 9, so x + x 9, so it is a polynomial of degree 10. Where do these points like x 0, x 1 etc, x 9, they all lie in the interval 0 1 and x is again a point in the interval 0, 1.

So when I want to determine the error bound I require modulus of Pi 10 of x, so that will be less than or equal to 1 because modulus of x + x i is less than or equal to 1 for x, x i lying in this interval 0 to 1. I require the maximum of the absolute value of the 10th derivative of f, where Xi belongs to the open interval 0 1. What is a function f? f is sin x in the interval 0 to 1, so if you take derivatives the first derivative is cosec, the second derivative is + sin x and so on, so the 10th derivative will be such that the maximum of the absolute value of that will be less than or = 1.

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And therefore I have all the information that I require to compute the error bound namely modulus of f(x) + p 9 of x in this case this modulus of sin x + p 9 of x and that is less than or = M n + 1 and that is 1 by 10 factorial into modulus of Pi 10 of x which is again less than or = 1 and hence the bound on the error in interpolation is such that it is less than or = 1 by 10 factorial.

So how large is the error on this interval requires us to give some information on the error bound and the result says that error in interpolation can0 be greater than 1 by 10 factorial. So the inequality that we have obtained here on the error bound provides us a way of giving the size on error bound for interpolation. So let us now work out the details using this error bound when the points x i are equally spaced and see what happens when I approximate the function f(x) by a constant polynomial or a linear polynomial or a quadratic polynomial. In all these cases we shall take our x i to be equally spaced and determine the error bound in each case.

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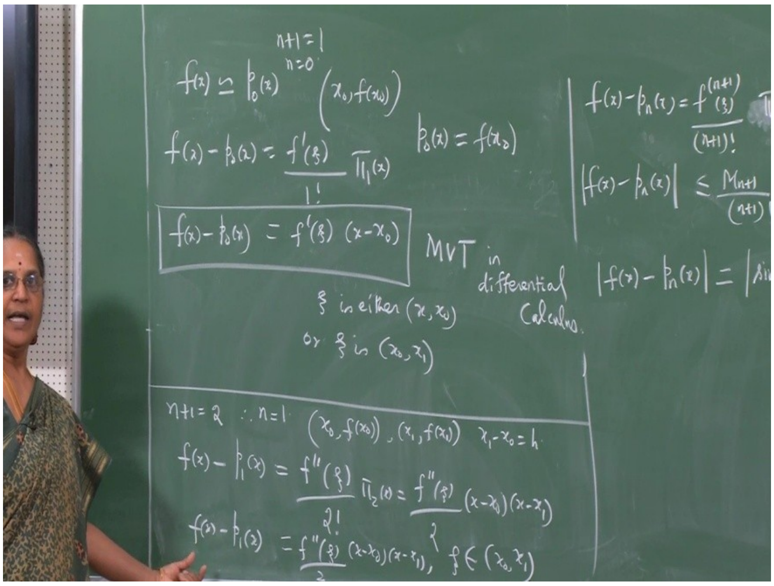

So what is my question? My question is I have a function f(x) I want to approximate this function by a polynomial of degree 0 namely a constant function, what does that mean? I am given information at 1 point namely x 0 comma F of x 0, so my n + 1 is 1 and so n is 0. I write down the error in interpolation in this case. So this tells me f of x + p 0 of x = f dash Xi by 1 factorial into Pi 1 of x, so that is the f dash of Xi into x + x 0. So f of x + p 0 of x is f dash of Xi into x + x 0. What is this P 0 of x? P 0 of x is a polynomial of the degree 0. What is it is property? It interpolates the function of f(x) where at x = x 0. So its property is such that P 0 of x 0 must be = f of x 0, so p 0 of x is a polynomial of degree 0, so it is a constant polynomial which is given by f of x 0.

So where this Xi lies here, Xi lies in the interval x to x 0. Let us now work out the details when n + 1 is 2 and therefore n is 1. So we are given information at x 0, f of x 0 and x 1, f of x 1, so there are 2 points at which the information is given and let x 1 + x 0 be h and hence I can obtain linear interpolating polynomial that interpolates the function f of x. So f of x + a polynomial of degree 1 which I de0e by P 1 of x that = n + 1 derivative of f that is the second derivative of f at some Xi divided by n + 1 factorial, so 2 factorial into Pi 2 of x, so what is it? It is f double prime at Xi by 2 factorial into Pi 2 of x will be x + x 0 into x + x 1. So this gives me the error in interpolation at any x which belongs to the interval x 0 to x 1 and that is f double dash at Xi by 2 into x + x 0 into x + x 1 and where does does xi belongs to? It belongs to the interval x 0 to x 1.

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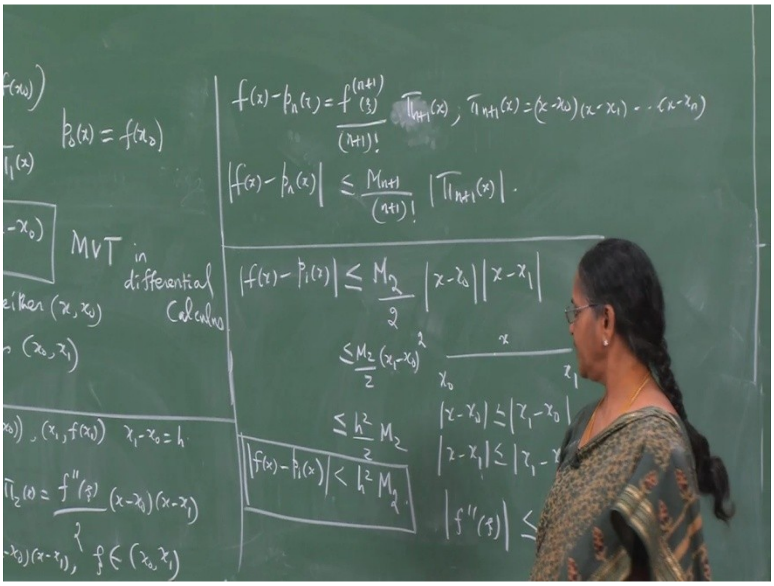

So, can you give the bound on this error, let us work out the details, I want modulus of f of x + P 1 of x, so that will be less than or = M 2 by 2 into modulus of x + x 0 into modulus of x + x 1, so I am given information at x 0 and x 1, x is some point in that interval. So mod x + x 0 is less than or = x 1 + x 0 and mod x + x 1 is less than or = mod x 1 + x 0. So this will be less than or = M 2 by 2 into x 1 + x 0 the whole square that is, it is less than or =, this is h according to our 0ation, so h square by 2 into M 2 which is less than h square into M 2. So modulus of F of x + P 1 of x can be made less than h square times M 2 and what is M 2? Modulus of f double prime of Xi is less than or = M 2. So we have the error bound in case when we approximate F of x by a linear interpolating polynomial.

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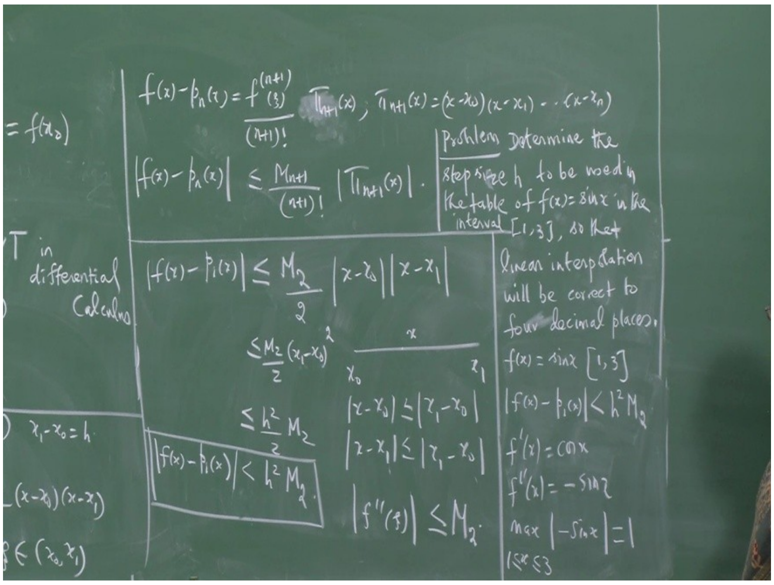

So let us consider an example and understand this result. The problem is to determine the step size h to be used in the table of f of x = sin x the interval 1 to 3, so that the linear interpolation of sine x in this interval will be correct to 4 decimal places. So we are even f of x = sin x and we want to find out the bound on the error in such a way that the linear interpolating polynomial that interpolates the function f of x is such that the result must be correct to 4 decimal places.

We already have computed the bound on linear interpolation error so we shall make use of this, so f of x + P 1 of x is less than h square into M 2. So we compute f dash of x that is cos x, f double dash of x that is + sin x and what is M 2? M 2 is such that modulus of f double dash of Xi is less than or = M 2 for xi in the interval 1, 2, 3. So we can compute the maximum of modulus of + sin x for x in the interval 1, 2, 3 and that is going to be 1.

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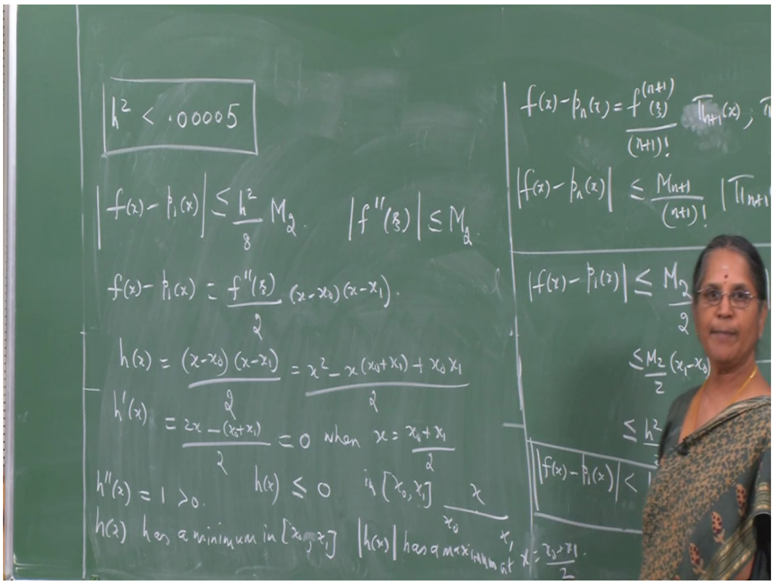

So we have all the information that is required, so that we will be able to answer this question we get h square into M 2 is 1 that must be less than or = 0.00005, from here 1 can compute the value of h. So in the table of values of Sine x you consider points which are spaced at an interval of h units where h square is less than this, then if you approximate sin x by means of a linear interpolation polynomial than your result will be correct to 4 decimal places that is what the conclusion is. When you approximate f of x by a linear interpolating polynomial we have obtained the error bound as this. In fact, one can show the following result namely f of x + P 1 of x in absolute value is less than or = h square by 8 into M 2, where modulus of f double dash of Xi is less than or = M 2 where Xi is in the interval of interest, so let us prove this result in the case of linear interpolation.

So what do we know about f of x + P 1 of x? We already have worked out the details, that is going to be f double dash of Xi by factorial 2 into x + x 0 into x + x 1. So I shall write down h of x to be = x + x 0 into x + x 1 divided by 2 and that is x square + x into x 0 + x 1 + x 0 x 1 by 2. So h dash of x is 2 x + x 0 + x 1 by 2 and it vanishes when x = x 0 + x 1 by 2 and in this interval x 0 to x 1 I see that h of x is, it is x + x 0 into x + x 1.

So I have the interval x 0 to x 1 x is somewhere here, so x + x 0 into x + x 1 is going to be negative in this interval. In addition, I see what is h double dash of x, h double dash of x is going to be 1 so positive and therefore, I have h of x to have a minimum in the interval has a minimum in the interval x 0 to x 1. We have shown that h of x is negative in this interval and h of x has a minimum in the interval x 0 to x 1, so therefore modulus of h of x has a maximum where in the interval at which point at x = x 0 + x 1 by 2. So modulus of h of x has a maximum at the point x = x 0 + x 1 divided by 2. So we can compute the maximum of h of x.

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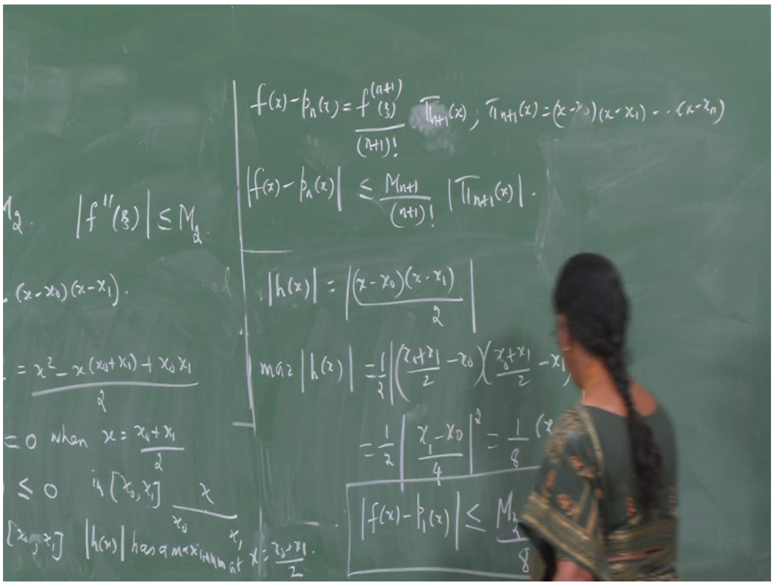

What is the maximum of modulus of h of x? So modulus of h of x is modulus of h x + x 0 into x + x 1 divided by 2 and I want the maximum of modulus of h of x. So maximum of modulus of h of x will be where is it attained? It is attained at x = x 0 + x 1 by 2 + x 0 into x = x 0 + x 1 by 2 + x 1 divided by 2. So it is half of modulus of, what do you get? X 1 by 2 + x 0 by 2, here again x 0 by 2 + x 1 by 2, so it is x 1 + x 0 by 2 into 2, so 4 the whole square, so it is 1 by 2 into x 1 + x 0 the whole square.

But x 1 + x 0 is h, so it h square by 8, so therefore modulus of f of x + P 1 of x is less than or = modulus of f double dash of xi which I de0e by M 2 into maximum of (()) (20:28) into x + x 1 by 2 and that is what is h of x and that has been computed to be h square by 8, so into h square by 8 and this is what we wanted to show. So the error bound in linear interpolation is such that the absolute value of the error in linear interpolation is less than or = h square by 8 into M 2 where modulus of f double of Xi is less than or = M 2 for Xi in the interval x 0 to x 1.

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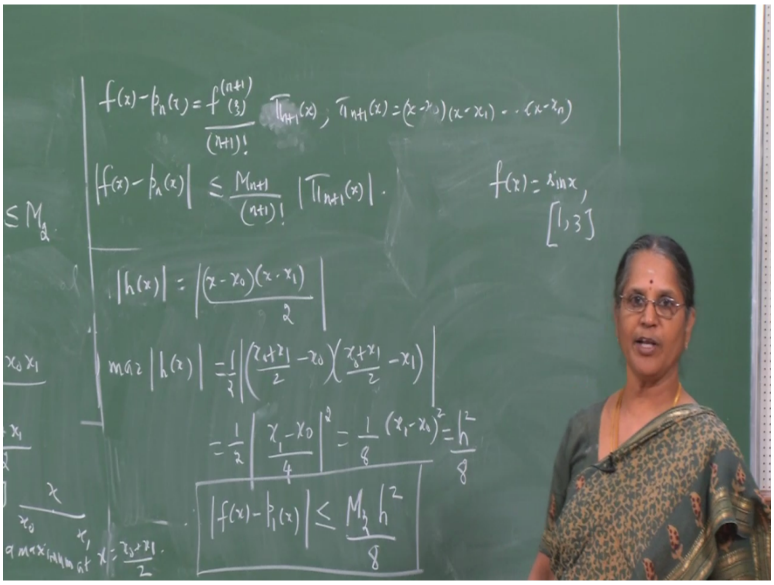

So let us work out the same problem which we had done earlier namely we have the function f of x = sin x in the interval 1 to 3 and we want to approximate this function by a linear interpolating polynomial and we want to use the set of values of x i and f of x i which is sin x i, so that these points x i must be spaced and use these to obtain an approximation to sin x which is a linear first degree polynomial. So what should be the step size and h such that the result is correct to 4 decimal places, that was a problem which we have solved earlier, but we used a different error bound and obtained that my h should be such that h square is less than this.

We shall now make use of this error bound and work out the same problem. So as before we have maximum of modulus of + sin x for x lying between 1 and 3 will be = 1. So I directly make use of this, so modulus of f of x + P 1 of x which is less than or = h square by 8 into 1 and my h should be such that I want h square by 8 into M 2 which is 1 to be less than or = 0.00005 correct to 4 decimal places, so that gives you h square to be less than or = 0.0004, so h is less than or = 0.02. So if you have a table of values of sin x at x i which are equally spaced with step size h to be less than or = 0.2, then your result in approximating sin x by a linear interpolating polynomial will be correct to 4 decimal places.