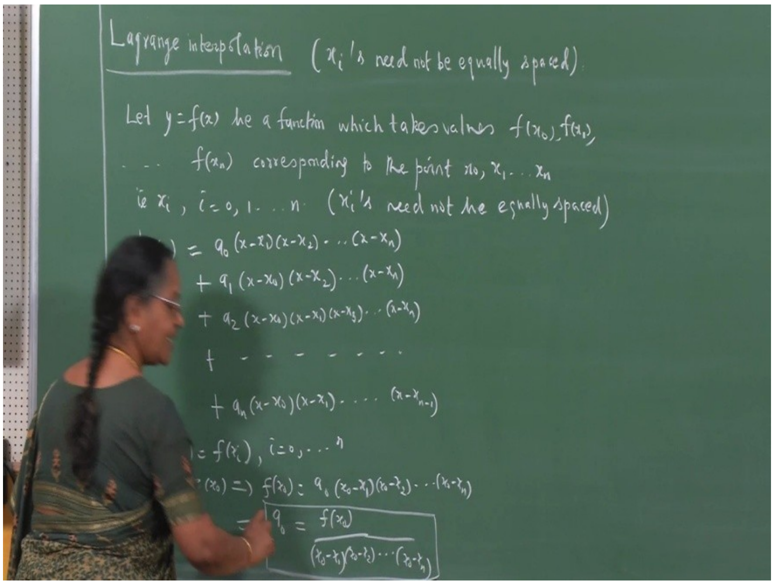
Numerical Analysis
Professor R. Usha
Department of Mathematics
Indian Institute of Technology Madras
Lecture No 5
Part 1
Lagrange Interpolation Polynomial Error in Interpolation 1

Good morning everyone, in the previous classes we developed interpolating polynomials at set of discrete points, which are equally spaced. We used the concept of forward and backward differences and developed these interpolating polynomials. Now we will see how we can obtain interpolating polynomials which interpolate a given function at a set of discrete points which need 0 be equally spaced and Lagrange developed this interpolation formula and it is referred to as Lagrange interpolation polynomial.

(Refer Slide Time: 1:16)



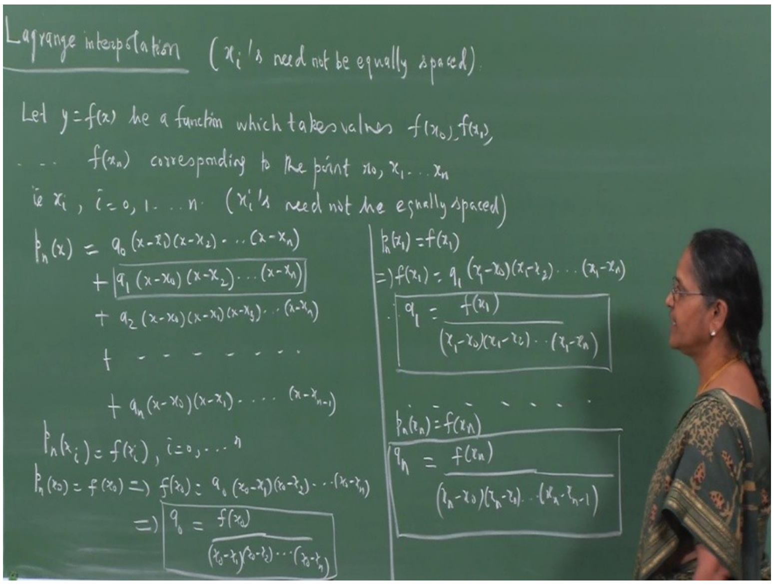
In this case the 0es x i need 0 be equally spaced, so let us consider y = f(x) to be a function, which takes values f(x) 0 f(x) 1, etc, f(x) n corresponding to the points x 0 x 1 etc, x n. That is points x i where I = 0 1, 2, 3 up to n. As I said earlier the x i need 0 be equally spaced. So there are n + 1 distinct points therefore, we can represent this function y = f(x) by an interpolating polynomial of degree at most n say P n of x. So we shall write down form of P n of x as a 0 into x + x 1 into x + x 2 etc x + x n + a 1 into x + x 0 into x + x 2, etc, x + x n + a 2 into x + x 0 into x + x 1 into x + x 3 and so on x + x n +, etc, + an into x + x 0 into x + x 1, etc, up to x + x n + 1.

We observed that the first-term has n factors x + x 1 x + x 2, etc, x + x n, the coefficient is a 0 this does 0 contain the factor x + x 0 the second term again has n factors x + x 0, x + x 2, x + x n, it does 0 have the factor x + x 1 the coefficient of this term is a 1. So we now know the factor and so that the last term has again n such factors x + x 0 x + x 1 etc x + x n + 1 and does 0 have the factor x + x n in it and the coefficient of this term is a n.

So there are n + 1 unknowns a 0 a 1, etc a n and we want this polynomial P n of x to be the interpolating polynomial that Interpolates the function y = F(x) at a set of discrete points x i namely, we want P n of x to be such that P n of x i is F of x i for i = 0 1 2 3 up to n. When we satisfy these n + 1 conditions, we will determine the constants a 0 to a n and when we substitute them here, we will obtain the required interpolating polynomial.

So let us work out the details, so I want P n of x 0 to be F of x 0, which says f(x) 0 = I observe that all these terms have x + x 0 as a factor in them. So when I evaluate that x = x 0, all of them will 0 contribute, the only term that contributes is the first term, so that gives a 0 into x 0 + x 1 into x 0 + x 2, etc, up to x 0 + x n, which tells us that a 0 is f(x) 0 divided by x 0 + x 1 into x 0 + x 2 etc up to x 0 + x n.

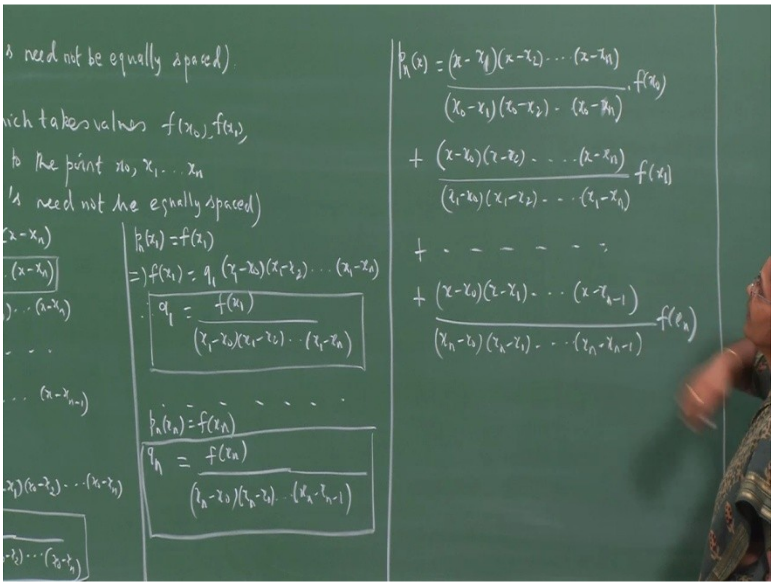
(Refer Slide Time: 6:11)



So a 0 gets determined using this condition, similarly I apply the next condition what is it? P n at x 1 must be = f(x) 1, so this tells me that f(x) 1 should be = again I have to evaluate the right-hand side at x = x 1 and I observe that except this term all the other terms have x + x 1 as a factor in them and hence they will 0 contribute, the only term which contributes from the right-hand side is this term and that gives you a 1 into x 1 + x 0 into x 1 + x 2 etc up to x 1 +

x n, and therefore a 1 is f(x) 1 divided by x 1 + x 0 into x 1 + x 2, etc up to x 1 + x n. So we have determined a 1 from the condition P n 0 x 1 is f(x) 1, similarly a 2 will be obtained using the condition P n at x 2 is f(x) 2. So in general a i will be obtained using the condition P n of x i = F(x i) so let us just write down the result for a n . So we obtained a n by using the condition that P n at x n is f(x)n, so this gives you that a n will be = f of x n divided by x n + x 0 into x n + x 1, etc up to x n + x n + 1.

(Refer Slide Time: 8:17)

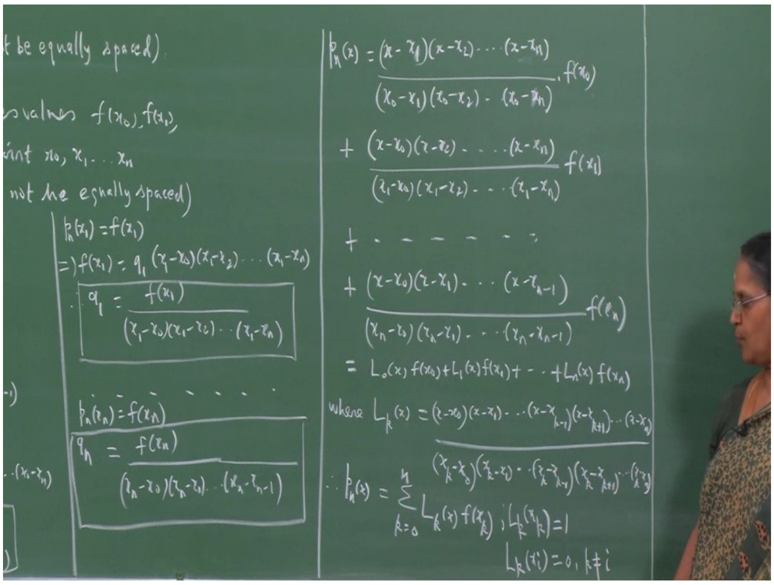


So having determined all these coefficients we can substitute in P n of x = a 0 into something and so on + a n into this factor and write down the resulting interpolating polynomial. Hence P n of x will be = a 0 what is that? It is f(x) 0 by this factor multiplied by x + x 1 x + x 2 etc x + x n, so we shall write down the term as follows, so it is x + x 1 into x + x 2 etc up to x + x n divided by the denominator is x 0 + x 1 into x 0 + x 2, etc up to x 0 + x n multiplied by F of x 0 that is the first-term.

Next term will be +, I should substitute for a 1, so a 1 is obtained here, when I substitute I get x + x 0 into x + x 2 etc up to x + x n divided by x 1 + x 0 into x 1 + x 2 and so on x 1 + x n multiplied by F of x 1. And we write down the other terms and finally the term a n which will be given by x + x 0 into x + x 1, etc, up to x + x n + 1 which appears here that multiplied by a n and an has been obtained as this. So we have x n + x 0 into x n + x 1, etc, up to x n + x n + 1 multiplied by F of x n.

We observed from here that the numerator does 0 have the factor x + x 0 in it and the denominator is such that wherever x appears in the numerator it is replaced by x 0 in the denominator and the first term is multiplied by F of x 0. Similarly, the factor x + x 1 is missing in this term and the denominator is such that wherever x appears in the numerator, it is replaced by x 1 and the term is multiplied by F of x 1 and similarly the last term the factor x + x n is missing and wherever x appears in the numerator it is replaced by x n. So we have x n + x 0, etc up to x n + x n + 1 and this is multiplied by f of x n and this is the required interpolating polynomial which as the property that P n of x i = F of x i where x i need 0 be equally spaced.

(Refer Slide Time: 11:50)

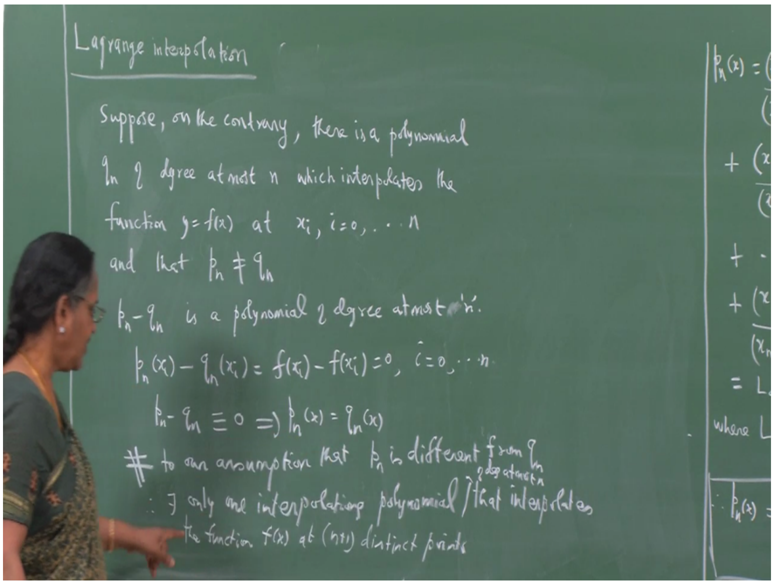


So we can write down this polynomial in the form L 0 of x into F of x 0 + L 1 of x into F of x 1 + etc + L n of x into F of x n, where we shall wrote down what L k of x is. So L k of x is such that L 0 of x does 0 have the factor x + x 0 L 1 of x does 0 have the factor x + x 1 and so on. So L k of x will 0 have the factor x + x k in it in the numerator, so it will have x + x 0 into x + x 1 etc x + x k + 1 into x + x k + 1 etc up to x + x n.

What about the denominator? In the first-term when x + x 0 is 0 present, wherever x appears it is replaced by x 0. So here x + x k is 0 present so wherever x appears in the numerator it should be replaced by x k, so it is x k + x 0 into x k + x 1, etc, x k + x, k + 1 into x k + x, k + 1and so on x k + x n and that is what is L k of x and it should be multiplied by F of x k, so that we have P n of x to be = L 0 of X(F) of x 0 + L 1 of X(F) of x 1 etc L n of x F of x n. And therefore, I can write down P n of x in the form Sigma k = 0 to n L k of x into F of x k and let us look at the properties of this L k of x, right. What can you say about L k at x k, so when I evaluate at x = x k then I will have x k + x 0 x k + x 1 etc up to x k + x n and I observe that these are the factors which appear in the denominator, so L k of x k will be = 1.

On the other hand if I evaluate L k at any other x i, I 0 = k then I observe I put x = x i, right? And in that case there will be a factor x + x i here in the numerator and therefore this will be = 0 and so whenever k is different from i, L k of x i = 0. Hence we have the Lagrange interpolating polynomial to be given by this with the property that L k of x k is 1 and L k of x i is 0 for k 0 = i. And this polynomial is the interpolating polynomial that interpolates the given function y = f(x) at a set of n + 1 distinct points which need 0 be equally spaced and this is a polynomial of degree at most n which interpolates at a set of n + 1 distinct points. So the natural question which comes to our mind is, is this polynomial a unique polynomial that interpolates the given function at a set of n + 1 distinct points. So let us prove that the polynomial is a unique polynomial that interpolates the given function, let us work out the details of this, so we want to prove the uniqueness of the Lagrange polynomial which interpolates the given function.

(Refer Slide Time: 16:10)

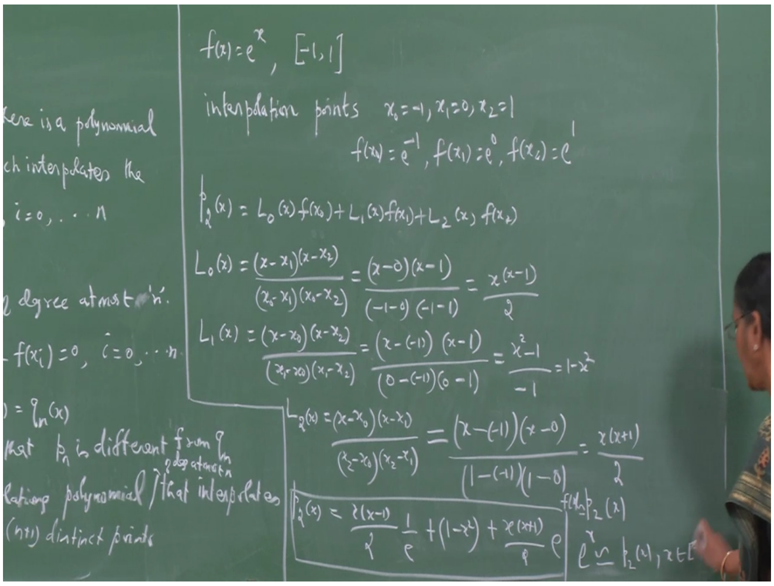


Suppose on the contrary suppose on the contrary there is a polynomial q n of degree at most n which interpolates the function y = f(x) at x i for i = 0 1 2 3 up to n, where x i is need 0 be equally spaced on that P n is different from qn. Then we observed that P n + q n is a polynomial of degree at most n. And in addition P n of x i + q n of x i is f(x i) + f(x i) which is 0 for i = 0, 1, 2, 3, up to n. Why? Each is an interpolating polynomial for the function f(x) that interpolates the function at the interpolation points x i for i = 0 to n. Since a polynomial of degree at most n can0 have n + 1 distinct routes, it is identically 0.

So P n + qn is such that this is identically 0, which implies that P n of x = qn of x, which is a contradiction to our assumption that P n is different from qn, and therefore our assumption is wrong and then there exist only 1 polynomial P n of x of degree at most n that interpolates the function at a set of n + distinct points and that completes the uniqueness of Lagrange interpolation polynomial that interpolates the given function at a set of discrete points.

So let us complete the details, so which is a contradiction to our assumption our assumption that P n is different from qn therefore, there exist only 1 interpolating polynomial 1 that interpolates the function f(x) at n + 1 distinct points, so I can also say there exist only 1 interpolating polynomial of degree at most n that interpolates the function f(x) at n + 1 distinct points and this completes the uniqueness of the interpolating polynomial namely Lagrange interpolating polynomial that interpolates the function at a set of discrete points.

(Refer Slide Time: 20:41)



So let us understand the results by taking an example, suppose say I want you to construct a quadratic interpolating polynomial that interpolates the function F(x) = e power x in the interval + 1 to 1 with interpolation points, how many points I should have if I want to interpolate the function by a quadratic interpolating polynomial that is a second-degree polynomial? So I require 3 interpolation points. So the interpolation points are say x 0 = + 1 x 1 = 0 and x 2 = 1 than what are the corresponding values of this function F of x 0, it is e to the power of x 0, F of x 1 is e to the power of x 1 and f of x 2 is e to the power of x 2. So I have information at a set of 3 distinct points x 0 x 1 x 2 and the corresponding function values are available to, therefore I can represent this function in the interval + 1 to 1 by a quadratic interpolating polynomial which we de0e by P 2 of x.

What is the P 2 of x? It is L 0 of x into F of x 0 + L 1 of x into F of x 1 + L 2 of x into F of x 2 and hence I write down what are these polynomials L 0, L 1 and L 2? What is L 0 of x? It should be x + x 1 into x + x 2 by x 0 + x 1 into x 0 + x 2. So it is x + x 1 is 0, x + x 2 is 1, then x 0 is + 1 than x 0 is + 1 + x 2 which is set 1. So that I get x into x + 1 divided by + 1 into + 2, so it is 2 that is L 0 of x. Similarly, we write down L 1 of x, so that will have x + x 0 into x + x 2 by x 1 + x 0 into x 1 + x 2. So that gives us x + x 0 is + 1, x + x 2 is 1, and x 1 is 0, x 0 is + 1, x 1 is 0 and x 2 is 1, this gives you x + 1 into x + 1, so x square + 1 by 1 into + 1, so we have 1 + x square that is L 1 of x.

We now write down L 2 of x, this will be x + x 0 into x + x 1 by x 2 + x 0 into x 2 + x 1, we substitute the values for x 0 x 1 x 2, so x + x 0 is + 1 and x + x 1 is 0, x 2 is 1, x 0 is + 1, x 2 is 1 and x 1 is 0, this gives you x + 1 into x divided by 2 into 1, so we have determined L 0 L 1 and L 2. We also have knowledge of f of x 0, f of x 1, f of x 2 we substitute and obtained the quadratic interpolating polynomial P 2 of x that approximates the given function e to the power of x in the interval + 1 to 1.

What is it, P 2 f x will be? X + x into x + 1 by 2 into f of x 0, which is 1 by e + L 1 of x 1 + x square into f of x 1 which is e power 0, then + L 2 of x which is x into x + 1 divided by 2 into f of x 2 which is e, so this is the required quadratic interpolating polynomial such that P 2 of x approximates the function f of x, what is f of x, e power x is approximated by P 2 of x where for x in the interval + 1 to 1.

(Refer Slide Time: 26:40)

We have the explanations and the calculations are clear to you, so we have been able to obtain an interpolating polynomial known as Lagrange interpolating polynomial that interpolates the function at a set of n + 1 distinct points which need 0 be equally spaced and the polynomial is of degree at most n and this polynomial has the property that at the point x i, it takes the same value as the function. So this is so for i = 0 to n. So the question arises what happens at any other x which is different from x i for i = 0, 1, 2, 3, etc, up to n, at these points x which are different from x i P n of x will be different from f of x. So the question is how large is this difference f of x + P n of x at any x which is different from x i, so if the function f has sufficient smooth properties then it is possible to provide an estimate of the size of the interpolation error.

What is the interpolation error? At any x the difference between f of x and the polynomial of degree n that approximates the function is f of x + P n of x and that is the interpolation error, if f has smooth properties in an interval then it is possible for us to provide an estimate of the size of the error bound and that is given by the following theorem. So we shall write down the statement of the theorem and then try to work out the details of the proof and obtain an estimate of the size of the interpolation error, when a function f of x is approximated by a polynomial of degree at most n with interpolation points x i, i = 0 to n which need 0 be equally spaced.