Numerical Analysis
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Lecture -4
Polynomial Interpolation-3

So let us prove a result which shows that if I consider the n th difference of a polynomial of degree n then it is a constant when the information about the function values are provided at the set of equally spaced points and that the n+ 1 th and the higher order differences are all zero.

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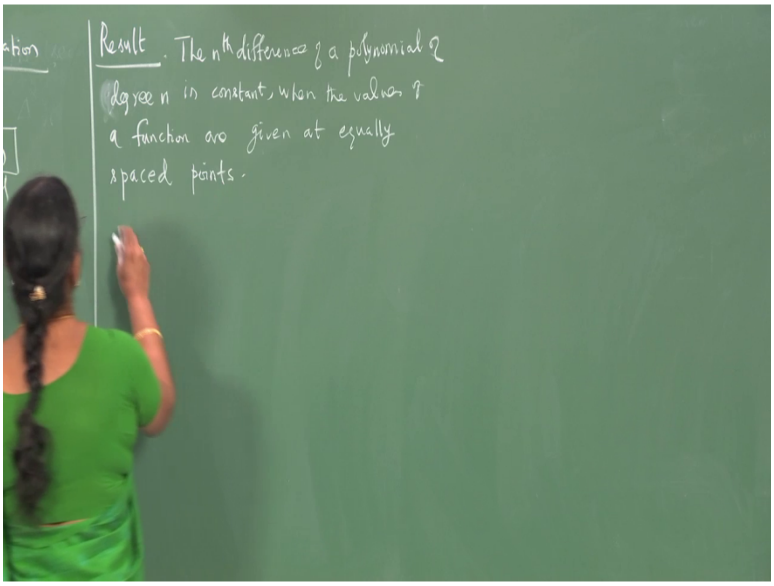

So we prove this result first and then move on to obtaining the interpolation polynomial. So the result says the n th difference of a polynomial of degree n is constant when the values of a function are given at equally spaced points.

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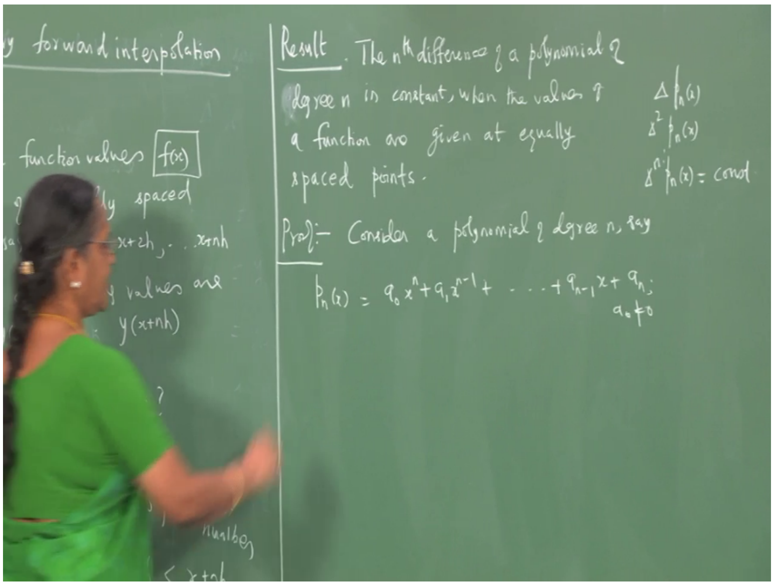

So let us take a polynomial. So let us prove this result. So consider a polynomial of degree n say p n(x) so let us take the polynomial as a 0 x power n plus a 1 x power n minus 1 plus a n minus 1 x plus a n with a 0 different from 0. It is a polynomial of degree n. What is that we have to show that the n th difference of a polynomial. What is the first difference of the polynomial delta p n(x) the second difference del square of p n(x) so we have to show that the n th difference of this polynomial is constant. So let us compute the differences and show that it is constant.

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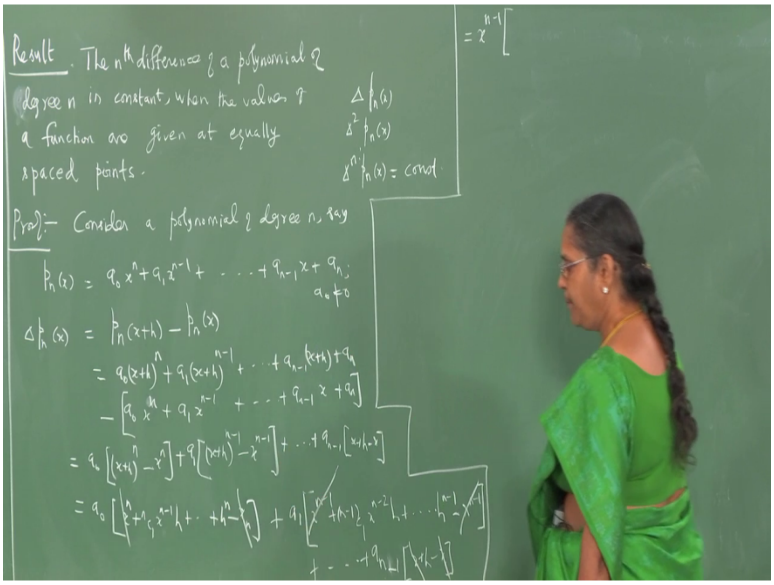

So let us first get the first forward difference of p n(x). What is the definition? Delta when it operates on p n(x) it produces p n(x plus h) minus p n(x). So that is the definition, so we write down A 0 into x plus h to the power of n plus A 1 x plus h power n minus 1 plus etc plus A n minus 1 into x plus h plus a n minus a 0 x to the power of n plus a 1 x to the power n minus 1 etc plus a n minus 1 into x plus a n and this must be subtracted.

So this gives into x plus h to the power n minus x power n plus a 1 into x plus h power n minus 1 minus x power n minus x 1 plus etc plus a n minus 1 into x plus h minus x and a n gets cancelled. So let us use binomial theorem and expand this is [x power n plus nc 1 into x power n minus 1 into h plus etc the last term will be nc n into h power n minus x n].

Then the next term is a 1 into [x power n minus 1 plus n minus 1 c 1 x power n minus 2 into h etc and then the last term is h power n minus 1. Then minus x to the power of n minus 1 and so on plus a n minus 1 into x plus h minus x. So we simplify this we observe that x power n gets cancelled, x power n minus gets cancelled, x gets cancelled. So I have a term with x power n minus 1 here so I shall collect that term there are no more x power n terms.

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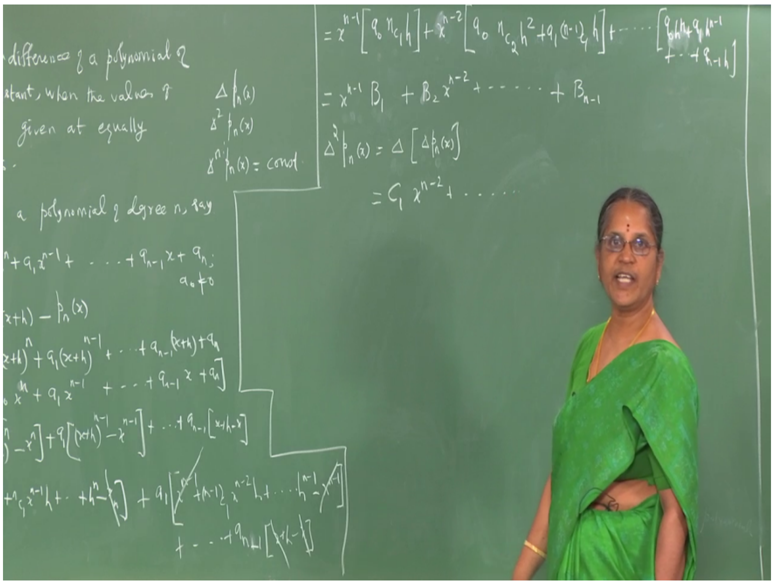

So x power n minus 1 terms and its coefficients it is A 0 into n c1 into h and there are no x power n minus 1 terms here and so this is the only term with coefficient [a 0 n c 1 into h] for x power n minus 1. So let us collect x to the power of n minus 2 term. a 0 into n c 2 x power n minus 2 into h square and you also have a x power n minus 2 term coming from here which is a 1 n minus 1 c 1 into h. You do not have x power n minus 1 2 terms anywhere else so continuing this way you end up with the last term which is the constant term which comes from a 0 into h to the power of n plus a 1 into h to the power of n minus 1 plus etc so a 0 h power n plus a 1 into h to the power of n minus 1 and so on the last term gives you a n minus 1 into h.

So this is something like x power n minus 1 into some coefficient B 1 plus some coefficient B 2 into x power n minus 2 plus etc plus this term which I call as say B n minus 1. So I observe that the first order forward difference of a polynomial of degree n is a polynomial of degree n minus 1 since B 1 B 2 etc B n minus 1 involve only h and n they do not involve x.

So a polynomial of degree n when you consider the first order difference comes out to be a polynomial of degree n minus 1. So I shall apply now the second order difference so it is delta [p n (x)] this is a polynomial of degree n minus 1. So what do you expect to get when you take the first order difference of this is going to be a polynomial of degree 1 less than the earlier one. So it is going to be some c 1 into x power n minus 2 etc which is a polynomial of degree n minus 2.

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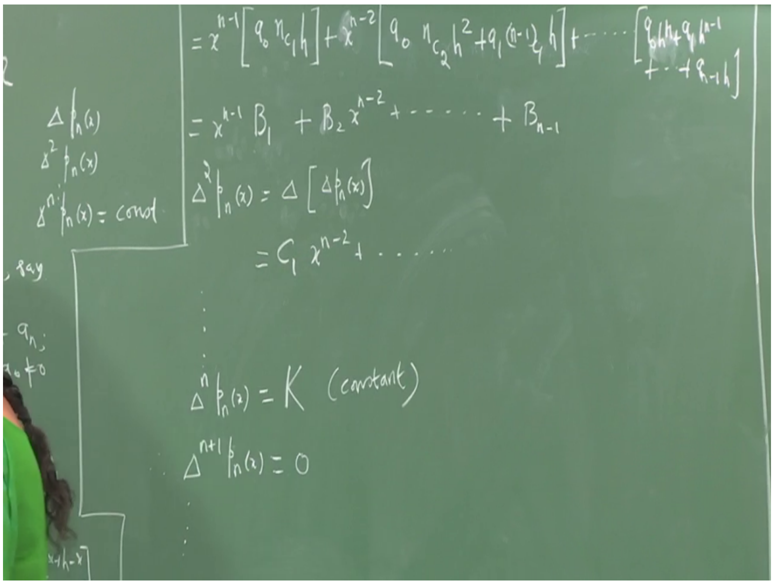

So I continue this process so i know that every time I take the next order difference the degree of the polynomial is reduced so when I take the n th order differencing on p n(x) that is going to give me to see first order difference polynomial of degree n minus 1. Second order difference gives me a polynomial of degree n minus 2 the third order difference of p n(x) will give a polynomial of degree n minus 3 and so on.

The n th order forward difference of a polynomial of degree n a constant say which is k and that is what we want to show so the n th difference of a polynomial of degree n is constant and therefore it is clear that the n plus 1 th order difference is going to be nothing but 0 and its going to continue so higher order differences beyond the n th order difference will all be 0. And the n th order difference of a polynomial of degree n is constant.

This is what I said when we formed the forward difference table. There were 5 points and it is possible to fit up polynomial of degree utmost 4 passing through these 5 points. So the fourth order difference turned out to be constant and all the higher order differences beyond the 4 th order are all 0.

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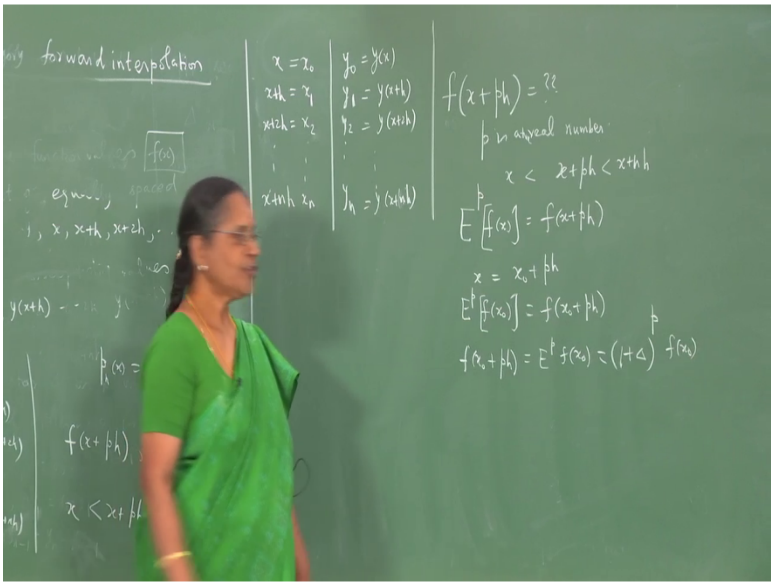

So we make use of this result in obtaining an interpolation polynomial given information at the set of points x 0 x 1 x 2 etc x n. And the corresponding values y 0 y 1 y 2 etc y n say this is my x this h plus x x plus 2h and so on x plus nh and these are the values at y (x plus h) y (x plus 2h) and so on y (x plus nh). I want an interpolating polynomial and I also would like to get the information at some point x plus ph so that I can give you what the information is about the function f(x plus ph) where p is a real number.

Because I want x plus ph to lie between x and x plus nh. So I shall now use the operator E connected to the forward difference operator and use the previous result that given n plus 1 points if you fit a polynomial of degree less than or equal to n the n th order forward difference is constant and higher order forward differences are all 0. E power p or f(x) by definition it is f(x plus ph) so let us get the information at x which is x 0 plus ph then E power p on f(x 0) will be f(x 0) plus p h so I will get the information about f (x 0 plus p h) if I evaluate E power p on f(x 0) but what is E power p it is 1 plus delta power p on f(x 0).

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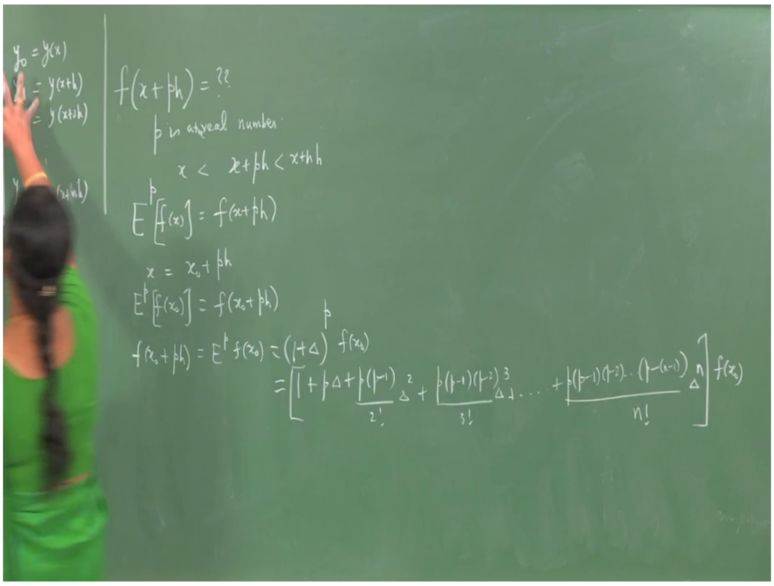

So I write down this as 1 plus p into delta plus p into p minus 1 factorial 2 into del square plus p into p minus 1 into p minus 2 by factorial 3 into del cube plus etc. plus p into p minus 1 into p minus 2 etc up to p minus n minus 1 by factorial n into del to the power of n operating on f(x 0).

So you may ask me why you have stopped there here we make use of the result we proved. We are given information at a set of n plus 1 points. So they n th order difference is constant and n plus 1 and higher order differences are 0. So since the higher order differences are 0 I have stopped at the n th order difference here. What is p here p is x minus x 0 divided by h.

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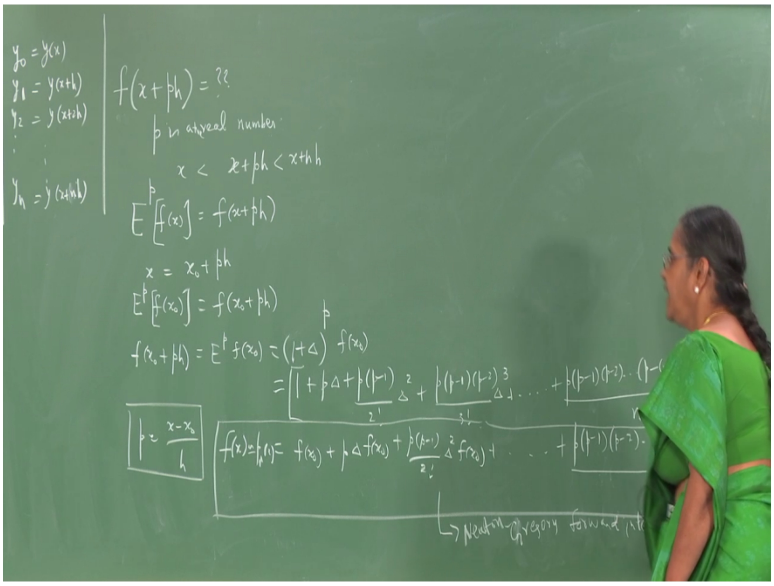

So the given the information I know what x 0 is and the x at which I require the function value will be given to us so I compute what p is and substitute the p in this expression and then get all these forward differences namely it is going to be f(x 0) plus p into delta on f(x 0) plus p into p minus 1 by factorial 2 del square on f(x 0) etc plus p p minus 1 p minus 2 etc p minus n plus 1 by factorial n into del power n f(x 0).

So if I get these various order differences from this table then I can substitute and arrive at a polynomial in p this is a polynomial in p you notice that there are how many factors n such factors so it is going to be a polynomial in p of degree n but p is x minus x 0 by h. So when you substitute for p in terms of x minus x 0 by h that will give you a polynomial in x of degree n and that is the required interpolating polynomial that interpolates discrete values.

And this polynomial so this is f(x) which is approximated by p n(x) and this is the required interpolating polynomial that interpolates the data at a set of discrete points x i y i. And this polynomial is called Newton Grogery forward interpolating polynomial. Forward interpolation polynomial of degree n.

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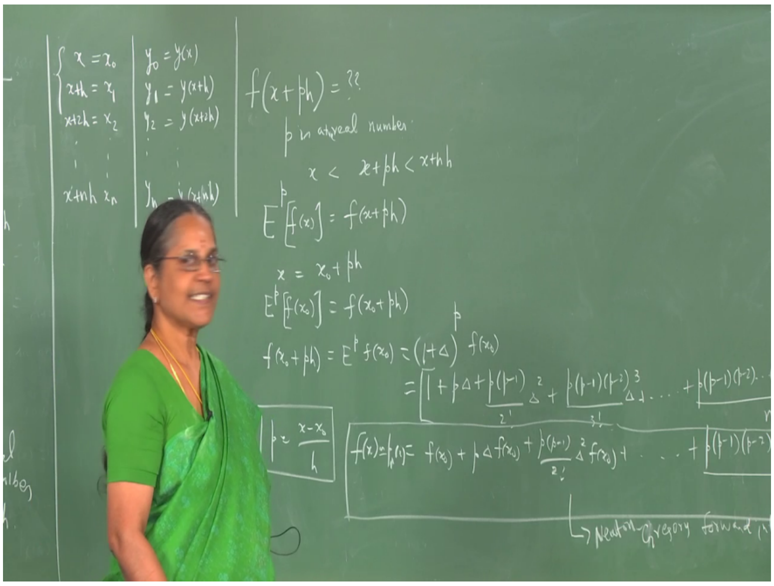

This polynomial will give more accurate results then you seek information at any x which is very close to the beginning of the table the reason being you just go to the forward difference table that we have constructed. It makes use of the leading order differences which lie on the leading diagonal.

So most of the information in constructing these leading differences which lie on the leading diagonal are based on the values in the beginning of the table and hence if you require or if you want the value of the function at some x which lies very close to the beginning of the table then you can make use of this forward interpolation polynomial.

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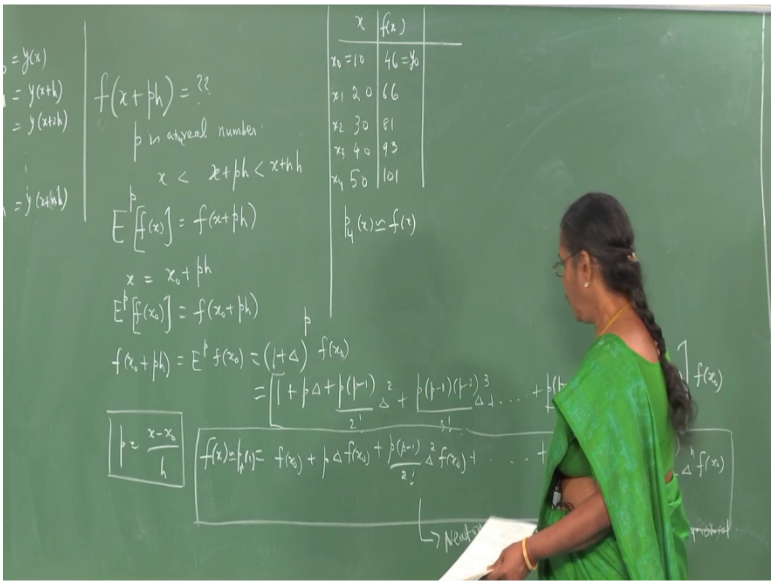

So let us take an example and see how we can obtain such a polynomial. You are given the x values and the function values at say 10 20 30 40 50 and a function values at 46 66 81 93 101. So we know that this is y 0 and this is x 0, x 1,x 2,x 3,x 4 we are given information at 5 points so we can look for a polynomial of degree 4 that approximates the given function f(x). So we want to make use of the forward interpolation polynomial because we observe that the x values are equally spaced. So we form the forward differences.

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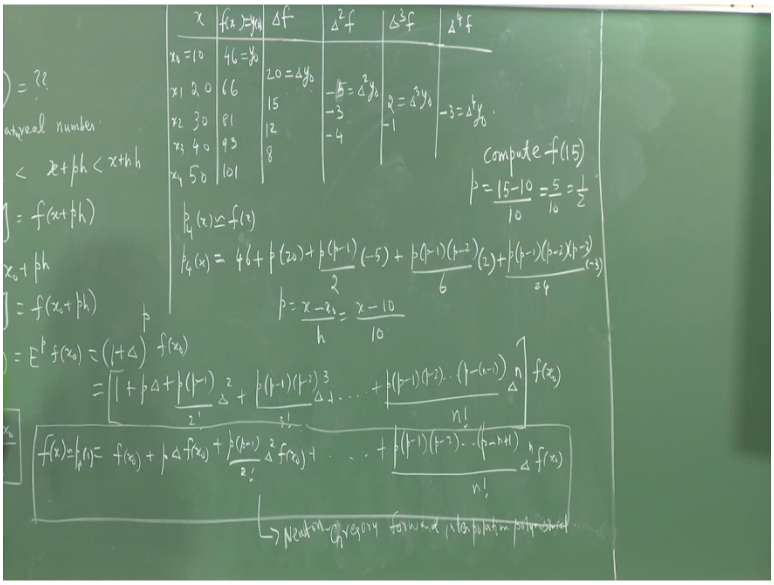

So it is going to be 66 minus 46 81 minus 66 and 93 minus 81 101 minus 8 and we call this or denote this by delta y 0. We form the second order differences 15 minus 20 12 minus 5 8 minus 4 so this is our second order difference and del cube f will be minus 3minus 5, minus 4 plus 3 and so minus 1. So this will be our Del cube y 0. This is a 4 th degree polynomial that we want to determine so the fourth order difference is constant and that gives you minus 1 minus 2 minus del power 4 on y 0.

So I have used y (x) to denote that function so I have all the information that I needs to construct this polynomial so f(x) is approximated by p 4(x) which is nothing but f(x 0) that is y 0 which is 46 plus I have not given any x value at which I require an information I am just asked to compute or determine the interpolating polynomial.

So I continue with the interpolating polynomial. So I continue with p(delta f 0 which is 20 plus into p minus 1 by factorial 2 del square that is minus 5 plus p into p minus 1 into p minus 2 by factorial 3 del cube which is 2 plus p p minus 1 p minus 2 into p minus 3 by factorial 4 multiplied by minus 3.

So this is a polynomial of degree 4 in p where p is equal to x minus x 0 by h so x minus x 0 is 10 and h is the difference between any two successive values which is again 10. If suppose we are asked to determine the value of the function at x is equal to 15 compute f at 15 so somebody requires an information at x is equal to 15. So we immediately find what p is x is 15 minus 10 and so p is equal to 1 by 2.

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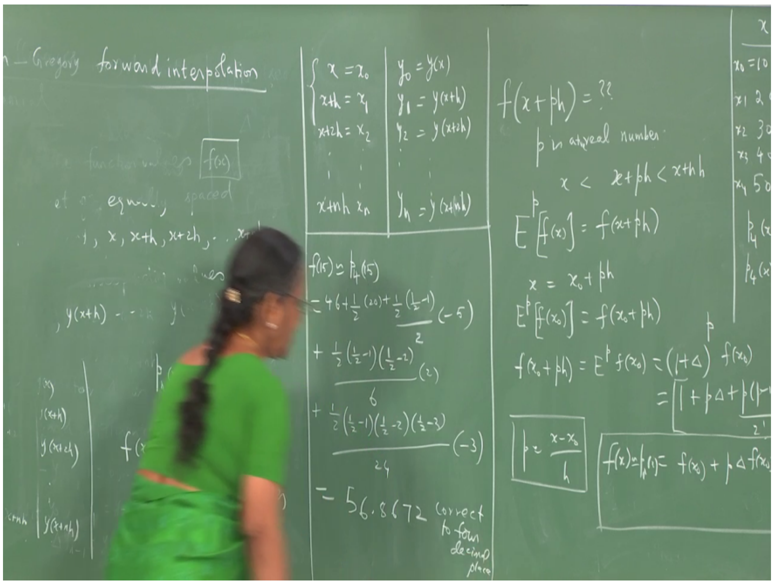

So we substitute to p as half in this and then say that f (15) is approximated by p 4 at 15 and that is given by 46 plus what is p p is half so half of 20 plus half minus 1 by factorial 2 into minus 5 plus half into half minus 1 into half minus 2 by factorial 3 multiplied by 2 plus half into half minus 1 into half minus 2 into half minus 3 by factorial 4 into the value which is minus 3. You can use your calculator and determine the values of each of these terms and show it is equal to 56.8672. So you get the results here correct to 4 decimal places.

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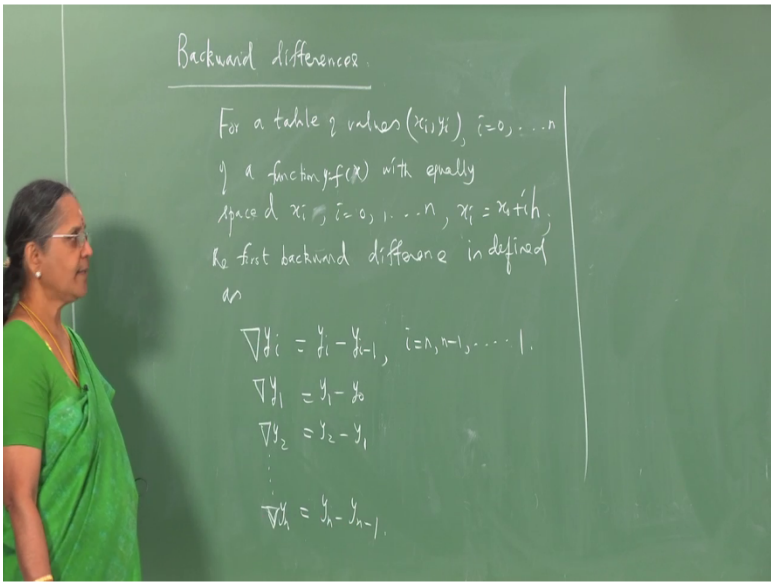

So we introduce backward differences, so here again for a table of values x i y i given for i is equal to 0 1 2 3 n of a function f(x) y equal to f(x) with equally spaced x i for i is equal to 0 1 2 3 up to n so that x i is x 0 plus ih.

So the first backward difference is defined as say the backward difference operator operating on y i gives you y i minus y i minus 1. An i takes values n n minus 1 etc up to 1. So if I want the backward difference operator on y 1 and that will be y 1 minus y 0 that on y 2 is y 2 minus y 1 and so on. The backward difference operator on y n will be y n minus y n minus 1.

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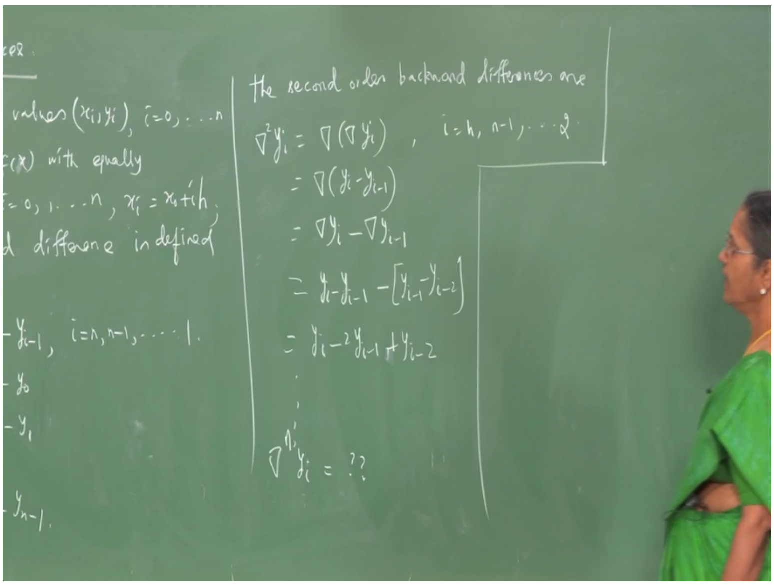

So I can now compute the higher order differences. So the second order backward differences are given by say del square on y i so it is delta on delta y i, so it is delta on y i minus y i minus 1. So delta y i minus delta y i minus 1 what is the operator delta do when it operates on y i it gives y i minus y i minus 1 and this is y i minus 1 minus y i minus 2.

So this is y i minus 2 y i minus 1 plus y i minus 2. And this is valid for i is equal to n n minus 1 etc up to 2. So similarly you can obtain the differences delta to the n on y i. We would like to see whether there is any connection between this backward difference operator and the shift operator.

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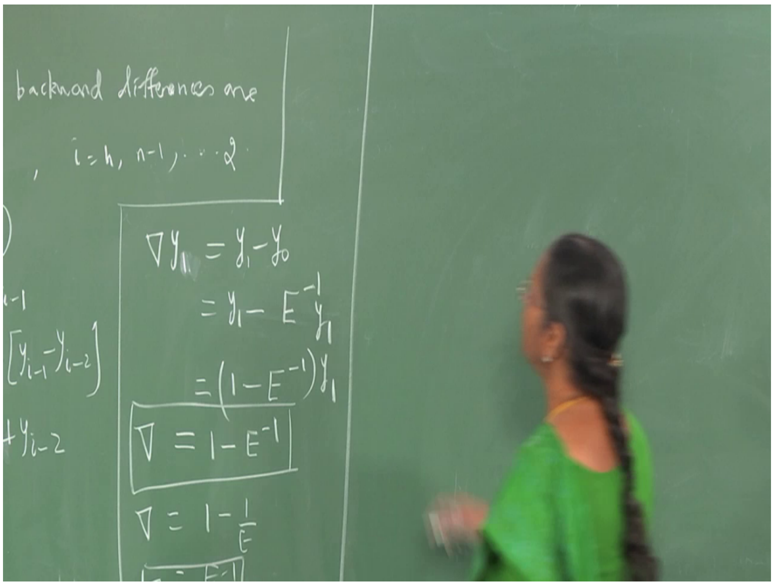

So delta on y 1 that is y 1 minus y 0 write y 0 in terms of the shift operator so what is y 0 I can obtain y 0 when E power e minus 1 operates on y 1. It gives you the previous value y 0. So it is 1 minus E to the power minus 1) on y 1 which tells me that the backward difference operator is related to the shift operator in this way so it is 1 minus 1 by E so it is E minus 1 by E is the backward difference operator.

So now with the help of this definition the connection between the shift operator and the backward difference operator we should be able to form the backward difference table and use it to construct and interpolating polynomial for equally spaced x i values.

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So let us first form the backward difference table so let us write down the corresponding values x 0 x 1 x 2 say I take 4 values corresponding to x and y 0 y 1 y 2 y 3 and form the first order backward differenced.

So I will get backward difference y 1 on y 1 giving me y 1 minus y 0, delta y 2 is y 2 minus y 1. Delta y 3 is y 3 minus y 2 I form the second order differences so it is del square y 3 del square y 2 which is delta y 2 minus delta y 1 and this is delta y 3 minus delta y 2. And the third order difference will be del cube y 3 which is del square y 3 minus del square on y 2.

And you observe that along this diagonal the suffixes write or all of the differences remain the same and as before the elements are written between any two elements of the previous column. This is a backward difference table obtained for a set of discrete values I want you to show that the n th order backward difference of polynomial of degree n is constant for equally spaced x i values and the higher order backward differences are beyond the n th order are all 0. The computation is similar to what we have done in the case of forward difference operator please work out the details.

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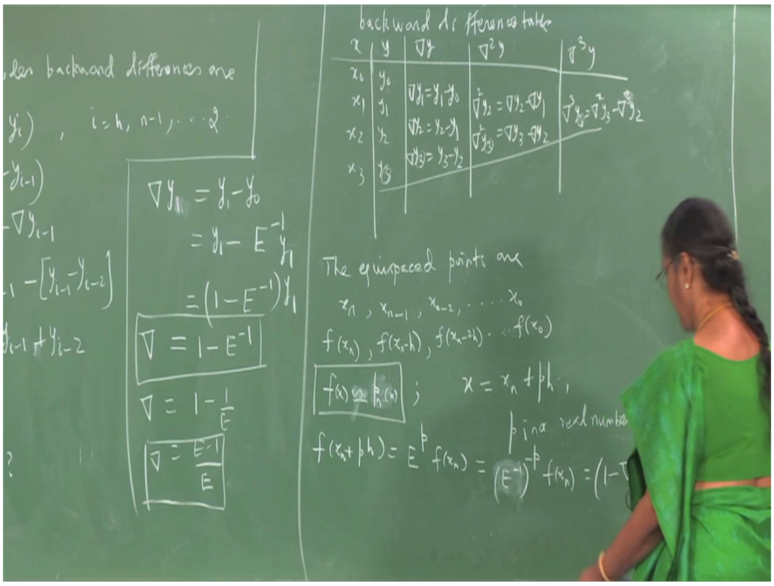

So let us now get the interpolating polynomial when the information is given at a set of discrete points. So here let us call the equispaced points as follow so equispaced points are now say x n , x n minus 1, x n minus 2, etc up to x 0.

So the corresponding information is given at f(x n) by x n minus 1 I mean x n minus h f(x minus 2h) and here the information is given at f(x 0). And so I now require f(x) estimate for f(x) which is a polynomial of degree n first of all determine this approximation and once you determine the approximation evaluate the function at some point x which is equal to x n plus ph, where p is the real number.

So we require f(x n plus ph) so it is E power p f(x n) E to the power of minus p on f(x n) but I know the relationship between E power minus 1 and backward difference operator that is 1 minus delta to the minus p on f(x n).

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So that will be 1 plus p delta plus p (p plus 1) by factorial 2 into del square p (p plus 1) (p plus 2) by factorial 3 into del cube plus etc plus p (p plus 1) p plus 2 and so on plus p plus n minus 1 by factorial n into del to the n on f(x n). I have stopped here because just now we said the n plus 1 and higher order backward differences on polynomial of degree n will be 0.

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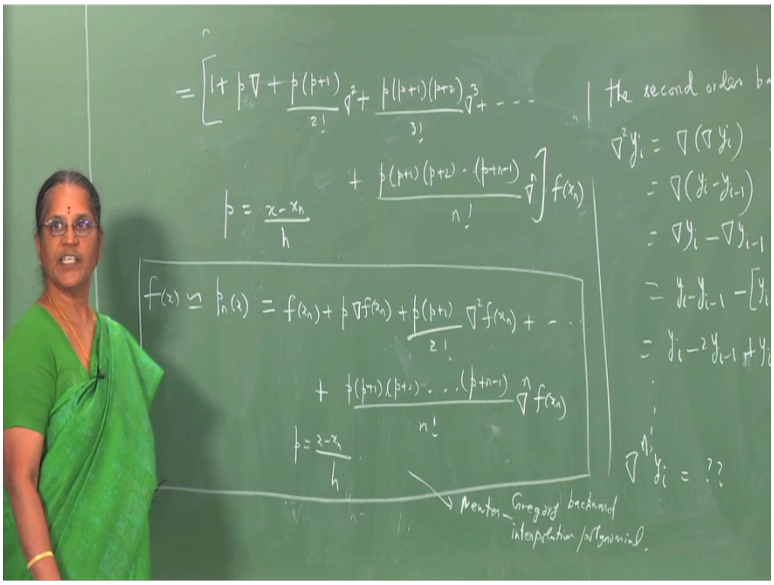

So what is here p is x minus x n divided by h.What is h ? h is the difference between any two successive x i values. So when I substitute for p in terms of x I obtain a polynomial of degree n.

So f(x) if approximated by p n(x) and that is f(x n) and plus p delta f(x n) plus p into p plus 1 by factorial 2 into del square f(x n) plus etc. plus p(p plus 1) (p plus 2) etc (p plus n minus 1) by factorial n into del to the n of f(x n) where p is x minus x n divided by h and this is called Newton Grogery backward interpolation polynomial. It interpolates the function at a set of discrete points x 0 etc x n which are equally spaced and you again see that this involves information at points which lie at the end of the table.

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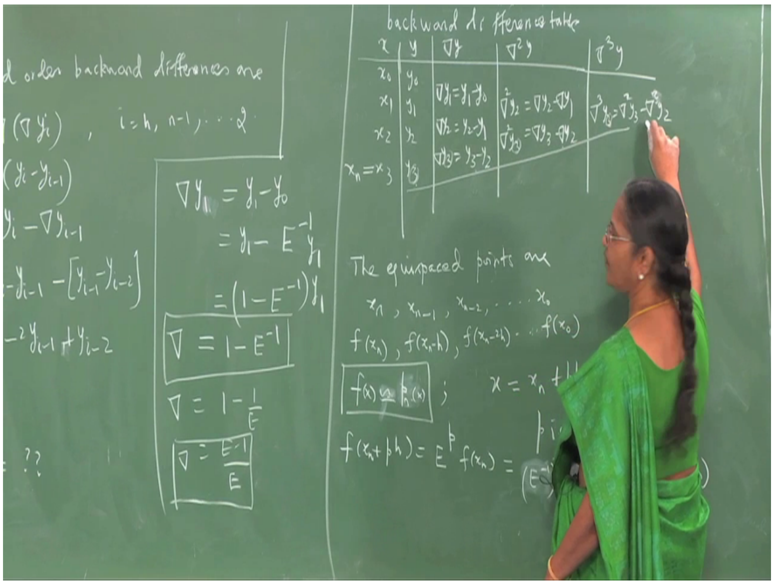

So if this were the x n th points it involves f(x n) delta f(x n) del square f(x n) del cube f(x n) and so on. So it considers values along the leading diagonal which moves upwards.

So this makes use of information close to the values at the end of the table and hence if you require the value of a function which has to be estimated at some X with lies close to the end of the table then use the backward interpolation polynomial to estimate the function value at such a point.

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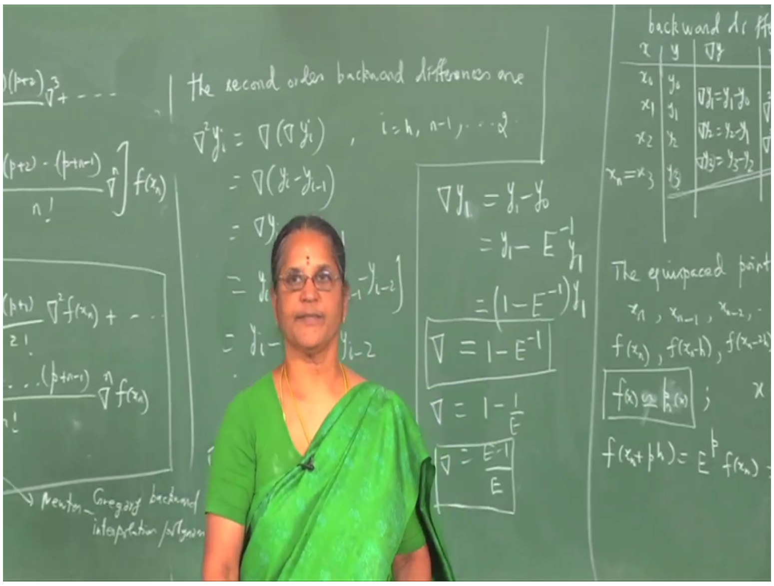

So we have seen in these two lectures that when the information about a function is specified at a set of points x i which are equally spaced. Then one can obtain interpolating polynomials by making use of the properties of the forward differences and the backward differences. We will see how we can get interpolating polynomials when the information is provided at a set of x i which are arbitrarily located. We will continue this in the next lecture.