Numerical Analysis
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Lecture No 43
Solution of Linear Systems Of Equations -6 Error Analysis 1

So we consider the error analysis for direct method. So let us first give some examples to show that small errors can produce large deviations in the solution, so let us consider the following example.

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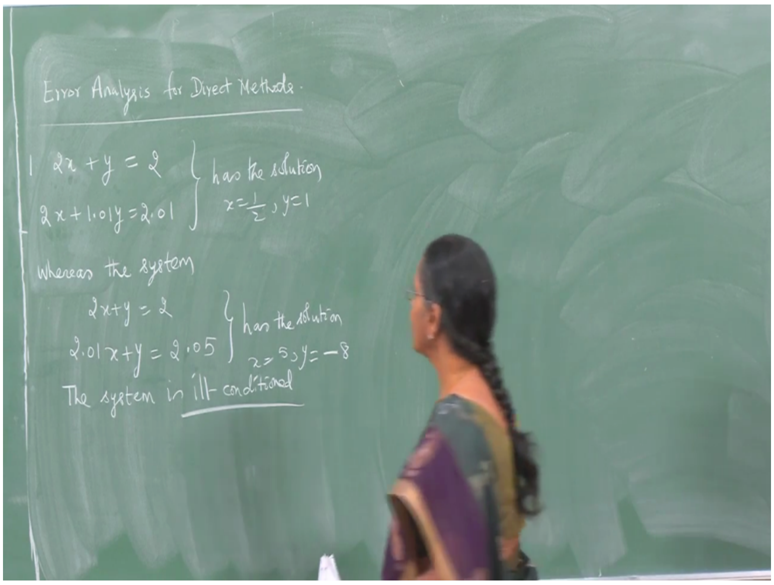

Suppose say I am given a system of equations of the form 2 x + y = 2, 2 x + 1.01 y = 2.01, it is immediate that this has the solution x = half and y = 1. So you can substitute x as half and it will give you 1, y is one so this equation satisfies. x is half so it is 1 + y is 1 so 1.01 that give you 2.01, so half comma 1 is the exact solution of the system of 2 equations that we have written down. If suppose I consider the following system namely, the first equation is as it is, I observe that there is a small change in the coefficient of x in the second equation, there is a small change in the second component of the right-hand side vector. I would like to see what is the solution of this system, and I observe that the system has the solution given by x = 5 and y = – 8.

So small changes in either the coefficient matrix or on the right-hand side vector yield very large deviations in the solution of the system. In that case we say that the system is ill conditioned, namely when there are small deviations in the input resulting in large deviations in the final solution, we call the system to be an ill conditioned system, let us consider another example.

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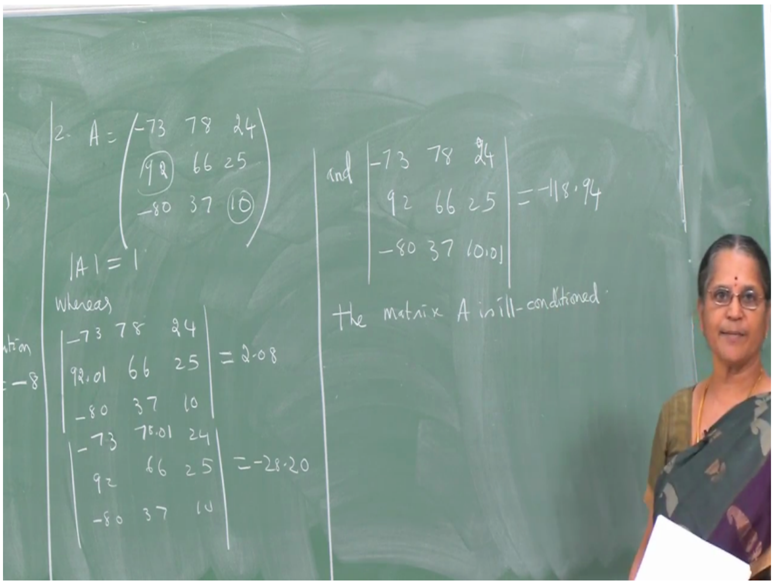

Suppose say the matrix A is – 73, 78, 24, 92, 66, 25, – 80, 37, 10. And if I compute the determinant of A, it turns out to be 1. There is a small change in this entry say it is 92.01, the rest of the entries are as there, there is no change. And if I compute the determinant of this, it turns out to be 2.08 this change is very very small, 92 has been changed to 92.01, but the determinant value has changed from 1 to 2.08. Let us take the case when this entry 78 is changed to 78.01, the other entries remain as there and if I compute the determinant then it turns out to be – 28.20.

And if I consider say the determinant with entries as – 73, 78 and 24, 92, 66, 25, – 80, 37 say I make a small change in this entry then you observe that the determinant turns out to be – 118.94. We have made only small changes in one of the entries in any one of the 3 rows in each of these cases that we have considered. But we observe that the value of the determinant A is such that there are large deviations from the value namely determinant A = 1 for the given matrix A. So small changes in any one of the entries in matrix A which is the coefficient matrix for a system has resulted in large deviations in the determinant value, and therefore we say that this matrix is ill conditioned.

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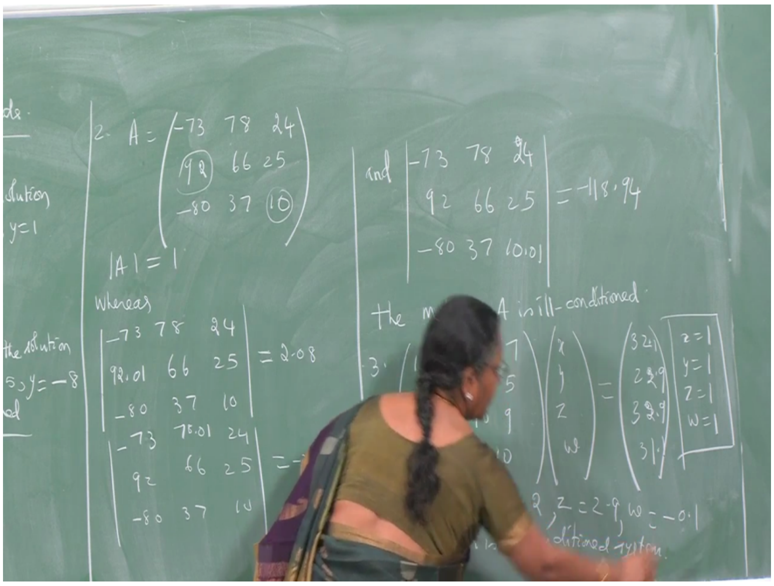

And therefore if this matrix has been the coefficient matrix of a system of equation A x = b if small changes occur due to round of errors in anyone of its entries then it is going to give large deviations in the solution of that system and therefore the system is going to be an ill conditioned system. Let us consider another example where I solve the system whose coefficient matrix has entries given by these and I have to solve for the unknowns x, y, z, W so that the right-hand side vector is 32, 23, 33, 31 and I immediately observe that my solution is x = 1, y = 1, z = 1 and w = 1. you can simply verify, 17, 25, 32, 12, 18, 23, 14, 23, 33, 12, 21 + 10 = 31 so these are the solutions of this system of equations.

Now I would like to make a small change on the right-hand side vector. Say due to some round of errors, the right-hand side vector is say changed so I shall introduce those changes here. Suppose the right-hand side vector changes to 32.1 then 23.9 or 22.9, 32.9 and 31.1 right. So the first and the right entries, some increase is incorporated and in the second and third entries are small decrease is introduced so that all the entries in the right-hand side vector are changed.

And if you solve the resulting system you will see that x turns out to be 6, y is – 7.2 and z is 2.9 and w turns out to be – 0.1 and you observe such large deviation in the solution of the system for very small changes on the right-hand side vector and therefore the system is again an ill conditioned system because small changes result in a large deviation in the output. And therefore your final solution cannot be a reliable solution, why does it happen? There are round of errors which are incorporated while carrying out the steps which are involved in the methods that we have discussed namely, in the direct methods that we have discussed. And due to these errors, the final solution turns out to be 1 which is such that there are large deviations in the solution from the actual solution and therefore, the system is an ill conditioned system.

So we must know a priory whether the system is an ill conditioned system alright, so we must have a knowledge of the magnitude of the error that is being incorporated at each step of our computation and that is why we need to perform the error analysis for direct methods and we shall see how we can obtain the magnitude of the error that is incorporated at each step from our discussions which follows. So to discuss the errors in the numerical methods involving vectors, it is useful to introduce the concept of or ideas of norm of a vector, so let us define what we mean by a norm of vector.

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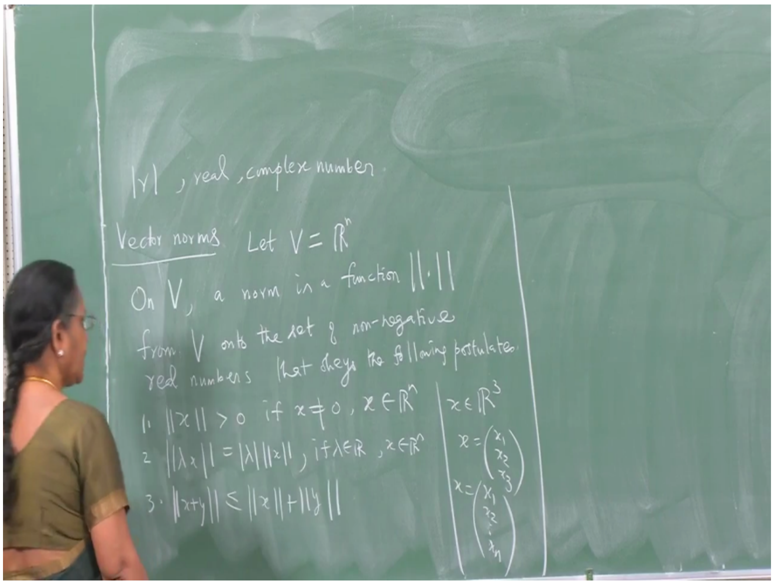

The norm that we are going to introduce generalizes the notion of magnitude say of r of a real number or a complex number is, so the norm is a notion which realises the magnitude of a real number for a complex number. When we consider norm, we consider norm of a vector so it gives you the length or the magnitude of that vector so let us define what we mean by a norm of a vector. So we introduce what are called vector norms, so let us call the n dimensional Euclidean space as V, then on this set V we introduce a norm, so a norm is a function from V onto the set of nonnegative reals that obeys the following postulates. So it satisfies the postulate that norm x is greater than 0 if x is different from 0, where does x belongs to?? x belongs to R n.

So is n is 3 then when we say x belongs to are 3 then x has components x 1, x 2, x 3, so x is a vector in the 3-dimensional Euclidean space that is what we mean. So here x belongs to R n so x will have components x 1, x 2, etc, x n, so it has n components or it is an n topple. So it is a point in the n dimensional Euclidean space, what does it satisfy? It satisfies the conditions that norm of x is positive if x is different from 0. Secondly, norm lambda x is going to be mod lambda into mod x if lambda is a real number and x is R n and thirdly, norm x + y is less than or equal to norm x + norm y.

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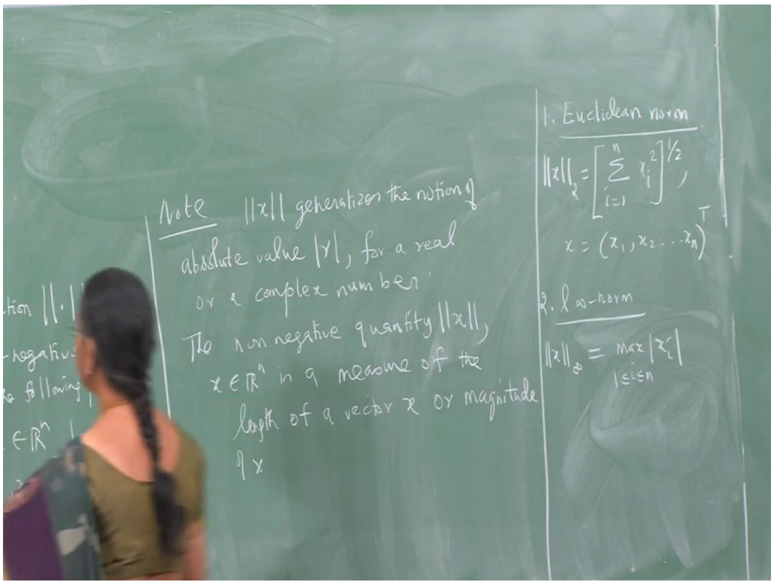

So as we remarked, norm x generalises the notion of absolute value for a real or a complex number. So this nonnegative quantity norm x is positive, the nonnegative quantity norm x where x belongs to R n is is a measure of the length of this vector x or it is the magnitude of this vector x. The most commonly used norms are as follows, so one can introduce the Euclidean norm which we denote by norm x suffix 2, it is just a notation. What does that mean? It is Sigma i = 1 to n x i square whole to the power of half, and what is x? x has components x 1, x 2, et cetera, x n.

I take a point having components x 1, x 2, x 3 and measure this length which I call as R then it is nothing but square root of x 1 – 0 the whole square + x 2 – 0 the whole square + x 3 – 0 the whole square namely, root of x 1 square + x 2 square + x 3 square that represents the length of this vector or the distance of the point p from the origin. This is extended, you consider all the components of the vector, square them, add them, take the square root and that is denoted by norm x 2. So this represents the length of the vector x in R n. Another commonly used norm is the L infinity norm, so you denote that by norm x infinity what is it?

It is maximum of modulus of x i for i lying between 1 and n. So what is norm? Norm is a function from V that is R n on the set of nonnegative real numbers, so you will end up with a real number that is the norm of x.

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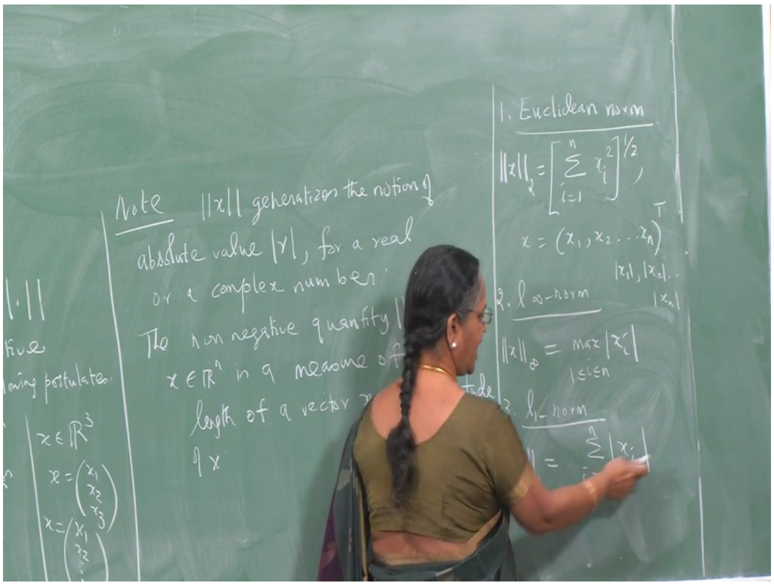

Take an x in R n then norm x is a real number which is a nonnegative real number. And therefore, I must have a number here when I write down norm x infinity what is it, earlier I took it to be the distance of the point p from the origin where the point p is in R n. Now I take the norm as follows, what is it, it is maximum of the absolute value of the x i so I look at the vector x which is given to me and I look at all the components of vector x, take the absolute value of each of these components, from among them I see which is the maximum, I take that and I call that as norm x infinity. If I do that then I have defined L infinity norm on V. Thirdly I have what is called L 1 norm, I denote it by norm x 1 and what is it? It is Sigma i = 1 to n modulus of x i.

So in this case what do I do? Given a vector x in R n, I look at the components of that vector and consider the absolute value of each of these components, I add them all that is what I have done Sigma i = 1 to n mod x i, so I have a real number and this is what I call has norm x 1 so given R n and vectors x belonging to R n, I can use any one of these definitions of norm on R n and find out the magnitude of that vector x. Is it clear now? So let us try to understand the notion of norm by taking the following set and trying to sketch that set in R 2.

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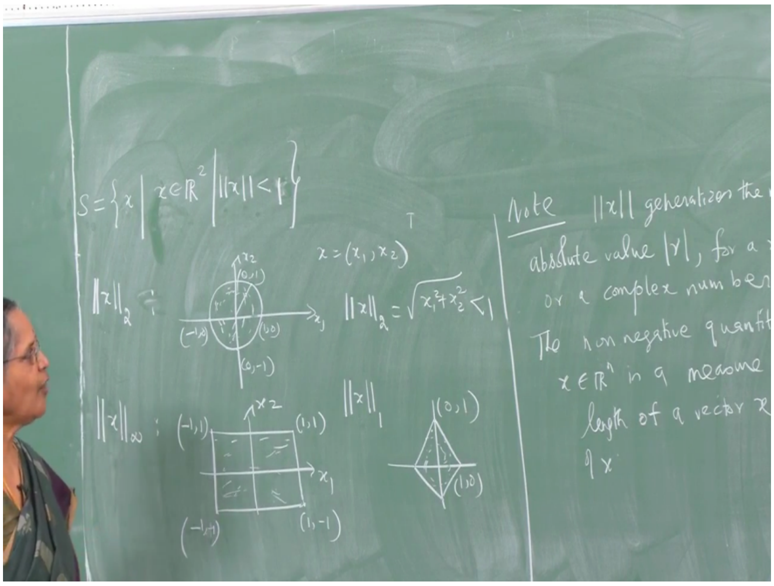

Set of all x such that x belongs to R 2 and I want the set to be such that it should consist of all those points x in R 2, R 2 is a plane so in R 2 such that norm of x should be less than 1. Suppose I want to sketch this set, I call this S and I want to use the norm as the Euclidean norm, what is the set? So if I use the Euclidean norm then what do I want? I want all those members x in the two-dimensional plane such that x is now having components x 1, x 2, so I want to use norm x 2, by definition it is root of x 1 square + x 2 square. So any typical x which is given to me having components x 1, x 2 then it will belong to this set if its distance from the origin is less than 1. So what all those points which will satisfy this criterion will all lie within a unit disc in the 2-dimensional plane.

So all those points within this circle where suppose this is x 1 axis, this is x 2 axis, this point will be 1 comma 0 and this is – 1, 0, this will be 0, 1 and this will be 0 comma – 1. So this is what the set describes in R 2 if I make use of the Euclidean norm. Let us use the L infinity norm for the same set, so if I use L infinity norm then in the Euclidean plane which is a two-dimensional plane, I would like to mark all those points x such that norm x infinity must be less than 1. What is norm x infinity? It is the maximum of modulus of x i and that must be less than 1, so what are those points, they are going to be points which lie within this rectangle so x 1, x 2, so this point is – 1 comma – 1 and this point is going to be 1 comma 1 so this will be 1 comma, – 1 and this will be – 1 comma 1.

So all those points within this rectangle will belong to this set S in which we use infinity norm. What about the third case? What if you use the L 1 norm? So you must have all those points in the two-dimensional plane such that the sum of the absolute value of the components of each of these vectors that you take must be less than 1 and therefore, in the two-dimensional plane you will have a set which consists of all those points which satisfies the conditions, so this will be 1, 0 and this will be 0, 1. So if you use this norm and mark all the points which lie in the set S, so these are the points which lie within this rhombus. I hope how you compute norm 2 or norm infinity or norm 1 maybe we can take some examples so that we have a clear understanding of the notion of these norms.

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Suppose I give you the following vector namely, x has components 4, 4, – 4, 4 and then y is another vector having components 0, 5, 5, 5 and say V is a vector or W is a vector (())(24:20) 0, 0, 0. So I would like to find the norm of x if I use each of these definitions, so I want to use norm 1, norm 2 and finally norm infinity for the vector x which is given to me. What is norm 1? If I use norm 1 to compute the L 1 norm for the vector x, it is take the absolute value of each of the components and submit up, so it is 4 + 4 + modulus of – 4 + 4 that is going to be 16. What about norm 2 for this vector, it is going to be square root of 4 square + 4 square + 4 square + 4 square and therefore that is going to give me the value as 8. What about norm infinity, norm x infinity? It is the maximum of modulus of x i, so it is the maximum of mod 4, mod 4, mod – 4, mod 4 so that is going to be 4.

So let us do the same thing with the vector y, so norm 1 is going to be the sum of the absolute values of its components, so it is going to be 5 + 5 + 5 so 15, norm 2 root of 5 square + square + 5 square so root of 75, it is 25 + 25 + 25 and that is going to be 8.66. What about norm infinity for y, it is going to be maximum of modulus of 0, mod 5, mod 5, mod 5, so that is going to be 5. Let us now take the same definitions of the norms for the vector W and compute each of these norms, so for W it is going to be norm W 1 so it is the sum of the absolute values of the entries so it is 6. Norm 2 that is square root of 6 square + 0 square, etc so 6, and then norm infinity maximum of the absolute values of the entries which is again 6.

So now we know given a vector x in R n for any value of n, we will we know how to compute the norm of that vector with respect to either the Euclidean norm or L 1 norm or the L infinity norm. So we will solve the system of equations with direct methods, we come across a matrix also namely, the system is given by A x = b, where x and B are vectors and A is a square matrix. So we should also have some knowledge about the matrix norm, so let us introduce the notion of matrix norm that is associated with the vector norm.