Numerical Analysis
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Lecture No 41
Solution of Linear Systems of Equations -4 Gauss Elimination Method with Partial Pivoting

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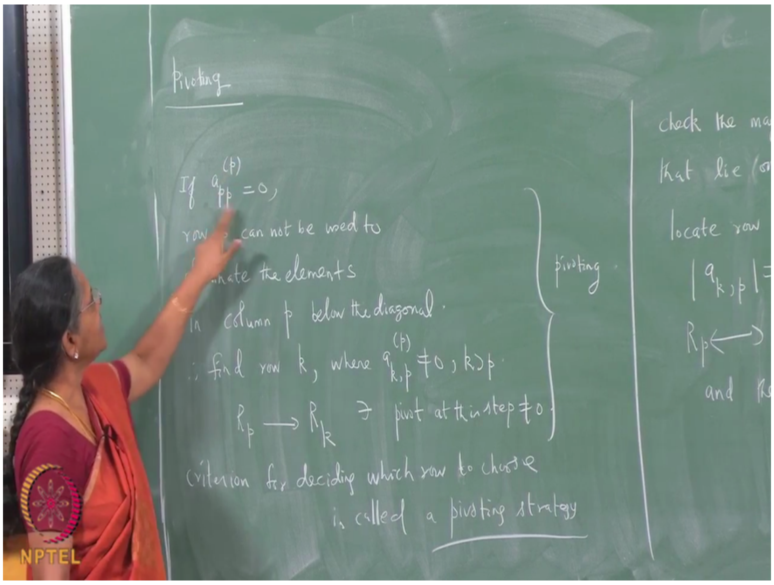

So we solve an example in which Gauss elimination method fails because the pivot at that stage which appears in the pivotal equation turns out to be 0. So while applying Gauss elimination method if such a thing happen namely say the pivot in the pth row at step p turns out to be 0 then row p cannot be used to eliminate the elements in column p below the diagonal that is what happened in the previous example. Coefficient of x 2 turned out to be 0 and therefore we cannot use row 2 as the pivotal row and eliminate the elements in all the other rows which lie below that row and make the coefficient of x 2 in the other rows to be 0 and therefore what do I do, what did we do there?

We observed all the equation which lie below that equation which was the pivotal equation at that step in which the pivot turned out to be 0 and check the weather the efficient of x 2 into the equations are non-zero. In our example there was only one equation so we observe that in that equation the coefficient of x 2 was nonzero and so we interchanged that row with the second row namely the third row with the second row. But if we have say N equations in N unknowns and say the pivot in the second row which is the pivotal equation at that step turns out to be 0 then we have to look at all the equations which lie the second row namely the second pivotal equation and the coefficient of x 2 in all these equations have to be looked at and get the magnitude of these coefficients of x 2.

From among these coefficients chose that which has the largest magnitude and select that particular row say if it appears in the kth row, choose the kth row and interchanged the kth row with the pth row where the pivot turned out to be 0 at the pth stage when it has to act as a pivotal equation. If you do that then you are essentially using a partial pivoting strategy and the criterion for desiring which row to choose is called a pivoting strategy, so let us just go through this that will explain what I had indicated above.

So if a pp turns out to be 0 at the pth step, row p cannot be used as the pivotal row so that you can eliminate the elements in column p below the diagonal and therefore you have to find row k in which a k p is different from 0 for k greater than p so you have to look at only rows which lie below that row which is the pivotal row for that step in which the pivot turned out to be 0 so you look for rows k which are bigger than p. And what do you have to do, you have to interchange row p with row k in such a way that pivot at this step is different from 0 after the exchange is done and now we have to see which of the rows can be called as the kth row, there must be some condition of criterion that must be used to select the row so that I can interchange that row with the pth row.

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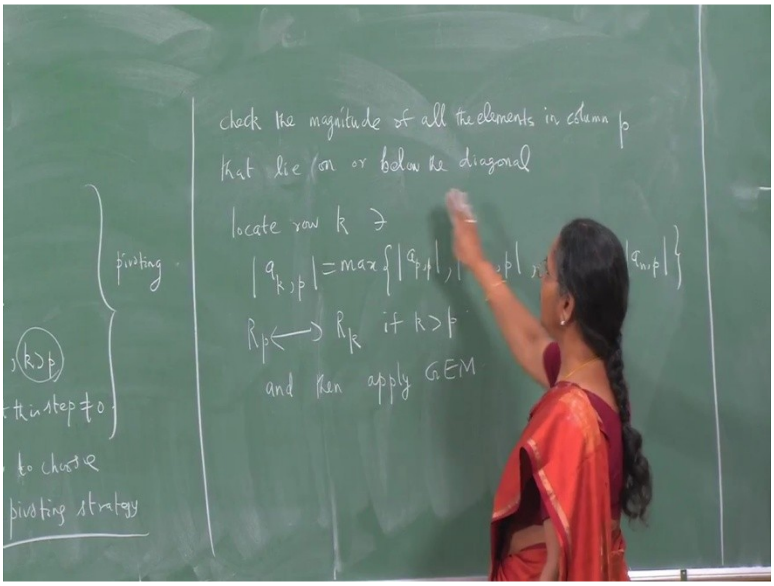

So you what you do is you check the magnitude of all the elements in column p that lie on the diagonal or below the diagonal and locate that row as the kth row such that its magnitude a k p is maximum of the magnitude of a pp a p + 1 p , etc a n p. Having selected a k p which satisfies this condition you interchange the pth row and the kth row and bring the kth row to the pth row with k greater than p.

And now you have a new row at this stage whose pivot is different from 0 so you can take this as the pivotal equation and then apply Gauss elimination procedure and whenever Gauss elimination procedure fails because pivot at any stage turns out to be 0 then employ pivoting strategy as described here and then continue to apply Gauss elimination method and complete the process and reduce the given coefficient matrix A in the system to an upper triangular matrix and follow it by back substitution procedure and solve the system of equations.

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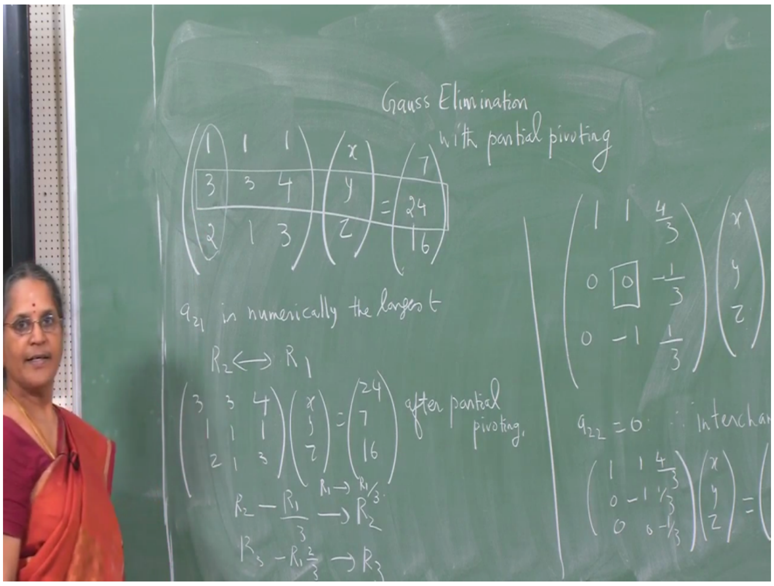

Let us consider this example and apply Gauss elimination with partial pivoting. So I am given a system A x = b, I would like to apply Gauss elimination with partial pivoting so I would like to ensure that the pivotal equation that I choose at any step is such that the pivot in that equation is numerically the largest. So I look at selecting my first equation in such a way that the coefficient of x 1 in that equation should be numerically the largest so I scan all the elements which appear in the first column namely, the elements 1, 3 and 2, I observe from among these coefficients which are coefficients of x 1 this is numerically the largest and therefore I would like to take this equation as my pivot equation at step 1 and therefore I perform the interchange of this equation with the first equation which appears in the system.

So what do I do? I interchange R 2 with R 1 so my equation is such that the second row 3 3 4 appear because of this interchange now 3 3 4 appear in the first row and therefore 24 is interchanged with 7 so 24 comes here and 7 is brought to the second row. Now I retain the third row as it is so I take this to be my system that is to be solved after partial pivoting. What did I do? I just wanted to ensure that at any stage of applying those elimination procedure I take care see that the pivot at that stage is numerically the largest and therefore, I interchange the second equation with the first equation to bring numerically largest coefficient for x 1 right in the first equation which is the pivotal equation.

Now what do I do? My procedure is to reduce this coefficient matrix to an upper triangular system, so I have to make this entry 0 and this entry 0 that is by step 1 in Gauss elimination procedure. So what do I do, I have to multiply this row by 1 by 3 and subtract it from this row so that is what I have written R 2 – R 1 by 3 is the new second row. I also have to make this coefficient 0 and therefore I subtract 2 by 3 of R 1 and subtract it from the third equation so R 3 – R 1 into 2 by 3 gives me the third equation, so my system becomes one with the first row as 3 3 4 and the second row and third row are such that the coefficients of x 1 in both of them turned out to be 0.

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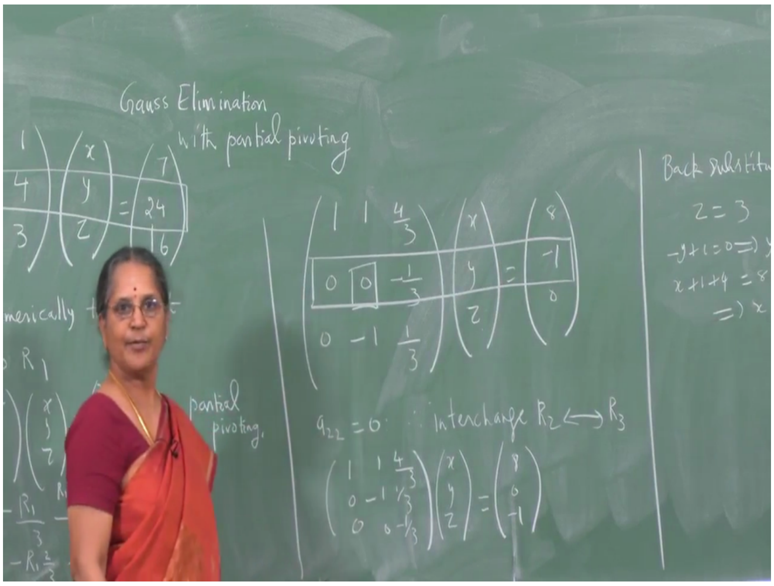

Now at this stage I also make the first row in such a way that my new first row is the old first row by 3 so that is my new first row so I have 1 1 4 by 3 and the right-hand side vector is right-hand side element is 24 by 3 which is 8. And having done this I observe what my second and third rows are, the second row is turns out to be this and the third row turns out to be this and I observe that the coefficient of x 1 in these 2 rows are 0. So my step 1 is complete in Gauss elimination procedure so I have to apply the second step. So in the second step I have to take the second equation as the pivotal equation and the pivot must be the coefficient of x 2 namely this entry, but I observe that this is 0 and therefore I cannot apply Gauss elimination procedure at this step because my pivot is 0.

So I have to use some pivoting strategy, what is it that I should do? I have to look at all the entries in that column which lie below this diagonal element in that column. I observe that I have only one equation and the entry that appears here is – 1 and it is a nonzero entry and therefore I would like to interchange this row with the second row and bring this here and that is what we have done here. So interchange row 2 with row 3 so I get the first equation as it is, the third equation has come as the second equation and the second equation now becomes the third equation so my R 1 remains as it is at this stage, my R 2 is exchanged with R 3.

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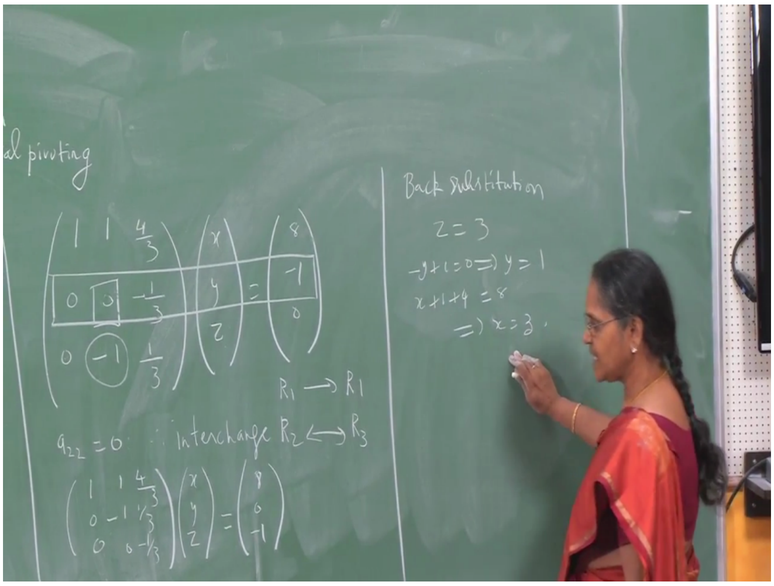

So now I look at this system, I observe that I have an upper triangular matrix as the coefficient matrix and therefore, step one in Gauss elimination procedure is completed namely, reducing the coefficient matrix A to an upper triangle matrix so we now use back substitution and obtain the solution of system of equations. So the third equation gives me z as 3, use the second equation which gives you – y + 1 = 0 so y is 1 and now use the first equation that give you x = 3 so the solution of the given system of equation is obtained where you have used partial pivoting strategy so that you have overcome the difficulty while applying Gauss elimination procedure. Let us take some more examples and see what happens if we apply partial pivoting and if we do not apply partial pivoting.

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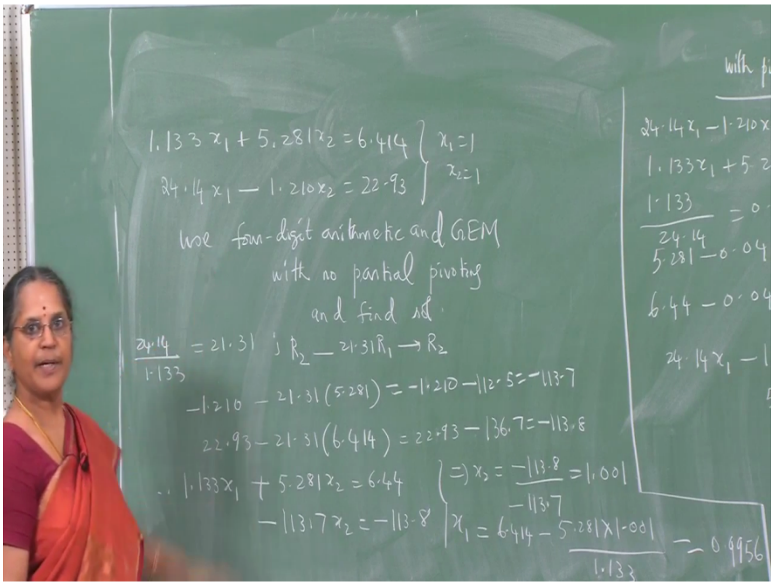

Let us consider this example and see the advantage of using Gauss elimination procedure with pivoting so let us first workout the solution for this system without pivoting. So we are given the system of equations for the 2 unknowns x 1 and x 2. It is immediately clear that x 1 is equal to 1 and x 2 equal to 1, solve this system. So substitute x 1 = 1 x 2 = 1 then 1.133 + 5.281 = 6.414 first equation is identically satisfied then 24.14 into 1 – 1.210 into 1 is again 22.93 the second equation is also identically satisfied. So it is clear that x 1 = 1, x 2= 1 is the exact solution of this system of equations, let us now work out this problem using Gauss elimination without partial pivoting.

So what do I do? I have to take the first equation as the pivotal equation and then make the coefficient of x 1 in the second equation to be 0 as. What do I do, I have to multiply the first equation by 24.14 by 1.133 and subtract it from the second equation, when you do that the coefficient of x 1 will be 0. And what is it, it is 21.31 so what is the elementary row operation that you perform, the new R 2 is obtained by subtracting some old R 2 21.31 times the row R 1 so the coefficient of x 1 will be 0. Now let us see what will be the coefficient of x 2. So it will be – 1.210 – 21.31 times 5.281 and that simplifies to – 113.7. Let us see what happens to the right-hand side, so that is 22.93 – 21.31 times 6.414 and when you simplify that turns out to be – 113.8 so let us now right down the system of equations.

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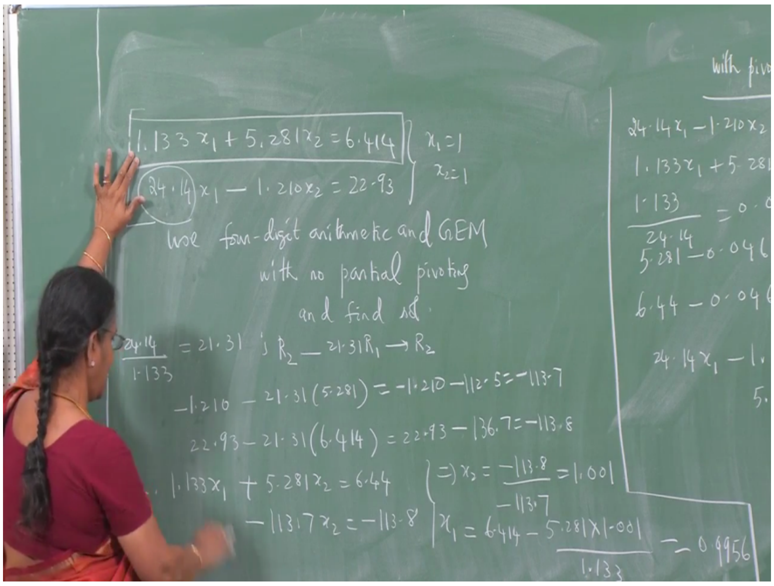

The first equation being the pivotal equation remains as it is and the second equation has now been obtained by making the coefficients of x 1 to be 0 and the resulting equation is – 113.7 x 2 = – 113.8 and therefore, I see that I have an upper triangular system and hence I can solve by back substitution and get what x 2 is from the last equation. So it is – 113.8 by – 113.7 and therefore it is 1.001 and I use the first equation and determine what x 1 is so that gives me 6.414, I see that x 1 is 6.414 – 5.281 into x 2 which we have computed as 1.001 by 1.133 and that turns out to be 0.9956 and we know that the exact solution is x 1 = 1 and x 2 = 1.

We have used Gauss elimination procedure without partial pivoting strategy and we have obtained the solution and we observe that there are some rounds of errors which are incorporated in during the computations. And we observe that we have for x 2 1.001 which is more than what is the exact solution for x 2 and for x 1 it is 0.9956 which is less than the exact solution which is x 1 = 1. So we see that we are unable to get the exact solution, some round of errors have been incorporated into our computations because we have not taken into consideration Gauss elimination method with partial pivoting. So let us see what happens if we use Gauss elimination method with partial pivoting right.

What should we do? Given a system of equations I have to decide at the first step which equation should I use as the pivotal equation. I must choose that equation as the pivotal equation for which the pivot is the numerically the largest at that step. Now I am starting the computations so my first equation should be the pivotal equation and for the position of x 1 is the pivot at that step and therefore I must have that equation as my pivotal equation having coefficient of x 1 to be numerically the largest, so I look at the coefficients of x 1 in the given system. I observe that the positions are 1.133 and 24.14, so this is numerically the largest and therefore I must take this as my pivotal equation at step 1 and take 24.14 as the pivot at that step.

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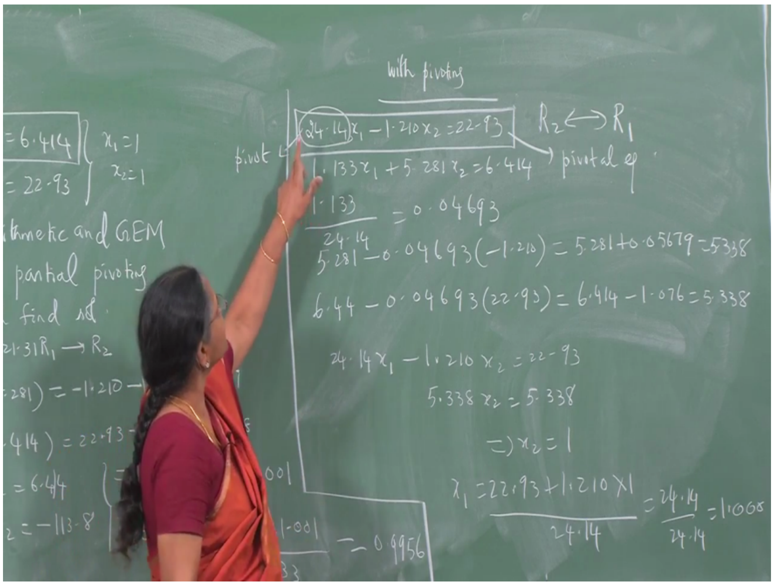

So what should I do? I should interchange row 2 with row 1 and rewrite the system properly after applying partial pivoting that is what I have done here so my system now becomes 24.14 x 1 – 1.210 x 2 = 22.93 so I have exchanged the second equation with the first equation, this is the elementary row operation that we have performed here and the first equation appears here, what should I do now? I must take this as my pivotal equation at this step and this coefficient as my pivot and therefore I must make the coefficient of x 1 in all the equations which lie below the pivotal equation to be 0. So there is only one equation so I should make this entry 0, how do I do it? I must multiply the first equation which is the pivotal equation by 1.133 by 24.14 and subtract it from the second equation.

So let us compute what is 1.133 by 24.14 that turns out to be this, so I would like to see what is the coefficient of x 2 now, that will be 5.281 – 0.04693 multiplied by – 1.210 and that turns out to be 5.338. Let us see what is the right-hand side, that is 6.414 – 0.04693 into 22.93 and that simplifies to 5.338 so we write down the system at this stage after performing step one. So the equations are 24.14 x 1 – 1.210 x 2 = 22.93, the second equation becomes 5.338 x 2 = 5.338 and therefore I see that I have an upper triangular system so I apply back point substitution that gives me x 2 to be this by this and so x 2 is 1.

I use x 2 as 1 and determine what x 1 is, so that will be 22.93 + 1.210 into 1 by 24.14 and that simplifies to 24.14 by 24.14 which is 1, so I get x 1 and x 2 to be equal to 1 and therefore the exact solution of the equation has been obtained and this is obtained by Gauss elimination method with partial pivoting. So whenever a pivot at any particular stage that appears in the pivotal equation turns out to be 0 or the pivot turns out to be a small quantity as compared to the coefficients of that unknown in the other equations which lie below it to be numerically mauler then take the pivoting strategy and then work out the details then you will avoid incorporating a round of errors and solution will turn out to be more accurate.

Now the question comes, are there systems where you can apply Gauss elimination method without partial pivoting confidently so that your solution will be correct to the desired degree of accuracy, the answer is yes. The results says that if you have the system of equation A x = b where the coefficient matrix A is a strictly diagonally dominant matrix then Gauss elimination method preserves the diagonal dominance throughout the computations and therefore in that case you need not have to use partial pivoting strategy. So we write down the result and we will also explain what we mean by a diagonally dominant matrix is.

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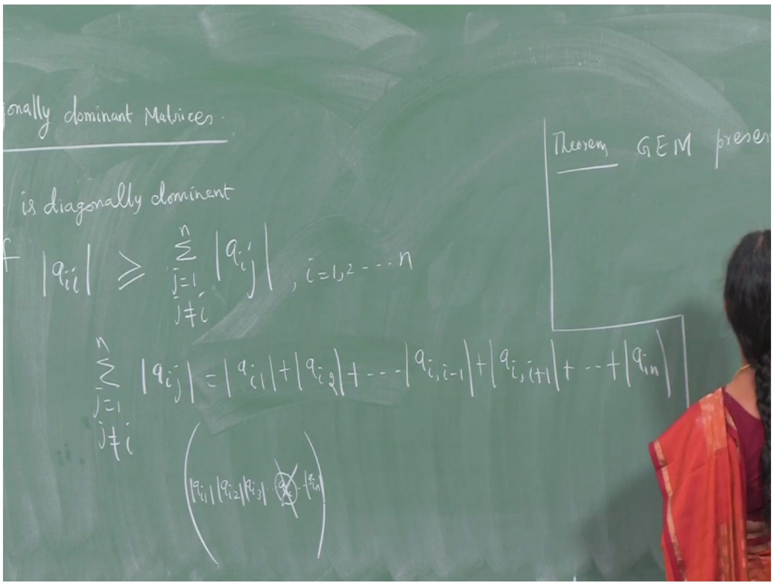

So there are certain classes of matrices for which Gauss elimination method can be safely used without partial pivoting, one such class is the class of Diagonally dominant Matrices. So a matrix A is said to be diagonally dominant if the following condition is satisfied, namely if mod a ii is greater than Sigma j = 1 to n j not equal to i modulus of a ij. What does the right-hand side tell you? The right-hand side is modulus of a ij for j not equal to y and j = 1 to n this must be summed up. So the right-hand side is going to be modulus of a i1 + a i2 + etc +

modulus of a ii – 1 + a ii + 1 + etc + a i n. And you observe that these entries are the entries which appear in the ith row namely the entries are a i1, a i2, a i3, etc, there will be an entry a ii then there will be an entry a i n in the nth column.

The right-hand side tell you that take the absolute value of all these entries if the absolute value of a ii which appears on the diagonal is greater than the sum of the absolute values of the other entries in that row. If this happens for $i = 1, 2, 3$ up to n so for all rows this should happen then you say that matrix A is a diagonally dominant matrix. If the strict inequality is satisfied then you say that it is strictly diagonally dominant and there may be some rows for which equality can happen, in that case you say that A is a diagonally dominant matrix. So we have the result which tells us that Gauss elimination method preserves diagonal dominance without partial pivoting.

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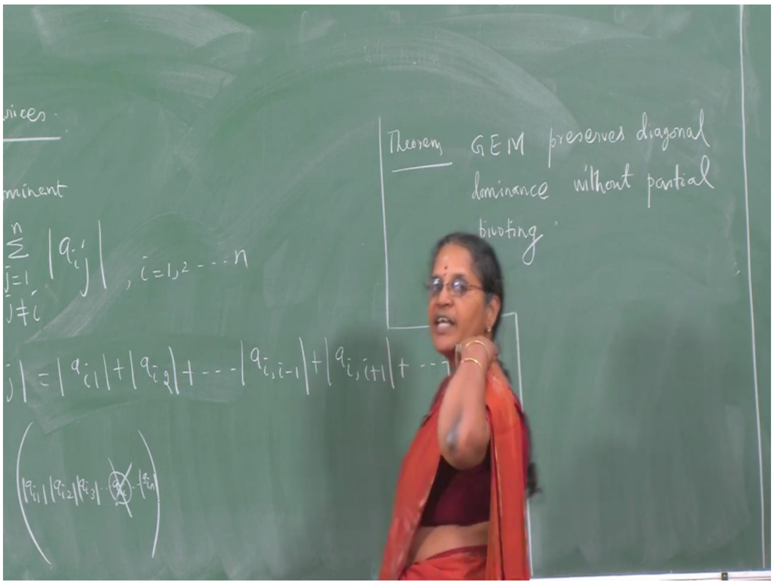

So if the given system $A x = b$ is such that A is a diagonally dominant matrix then application of Gauss elimination method preserves this property of the system throughout and you can safely use Gauss elimination method without partial pivoting in such cases that is what the results says so I will give you an example of a matrix which is diagonally dominant. Matrix A given by 5, – 1, 1, 2 , 4, 0, 1, 1, 5, we observe that in the first row 5 is greater than modulus of – 1 + 1 which is 2, in the second row a 22 is greater than 2 + 0 and in the third row 5 is greater than 2 and so the matrix A is strictly diagonally dominant.

And you can solve the system of equation of the form $A x = b$ where the coefficient matrix is A which is strictly diagonally dominant without partial pivoting and you can safely use Gauss elimination method and you will not come across any difficulty where the pivot element at any step will become 0 because the strictly diagonally dominant property of this matrix is preserved by Gauss elimination method. So I want you try to solve system A x = b where A is given by this and vector B is given by 10, 12 say – 1, you do not have to use partial pivoting strategy at any step here because the matrix A is a diagonally dominant matrix.

If you are asked to solve a system of equations of the form A x = b by Gauss elimination method, please look into the system and find out whether the coefficient matrix A is a diagonally dominant matrix, in that case you need not have to apply partial pivoting strategy and directly apply Gauss elimination method and obtain the solution. And in case you see that the matrix is such that you will have to apply Gauss elimination method with partial pivoting, take care to interchange the rows properly so that at any step your pivotal equation is such that the pivot has numerically the largest value in that column and that will ensure that the method is going to give you solutions which are very accurate.

So we will stop our discussions here and continue with another direct method namely Gauss Jordan method in which system of equations A x = b with coefficient matrix A is reduced to a diagonal matrix so that system turns out to be system of the form D into x = B cap where D is the diagonal matrix so the solutions can be immediately obtained, so we shall continue our discussion in the next class.