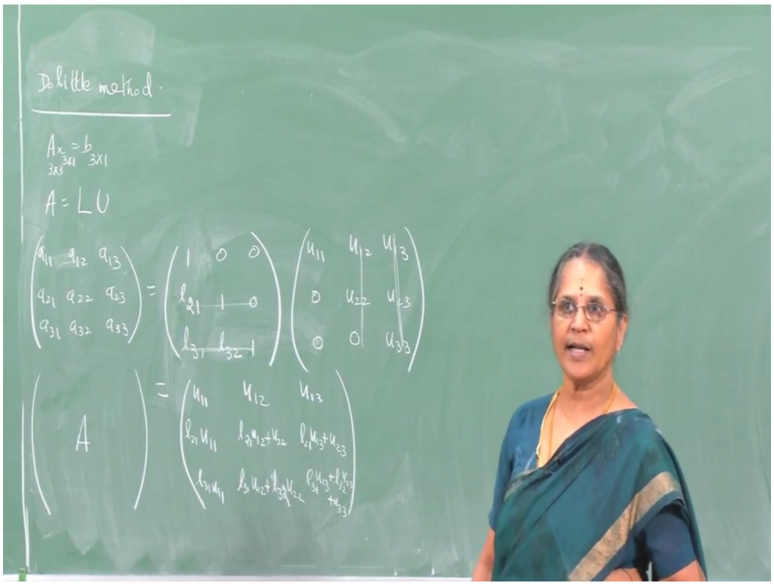
Numerical Analysis
Professor R. Usha
Department of Mathematics
Indian Institute of Technology Madras
Lecture No 39
Solution of Linear Systems Of Equations –1
Decomposition Methods–2

So let us first discuss Dolittle method, what is Dolittle method? If A can be decomposed in the form L into U with L as a unit lower triangular matrix then the system A x = b can be solved by using forward substitution method followed by backward substitution.

(Refer Slide Time: 1:03)



So let us illustrate this Dolittle method for a simple system A x = b where A is a 3 by 3 matrix. So x is 3 by 1 vector and b is a 3 by 1 vector, the procedure is the same whatever be the value of n is when we are given an N Cross N square matrix A. So if A is a 3 cross 3 matrix and A has a decomposition of the form L into U, where L is a lower triangular matrix but we would like to use Dolittle method, so L is a unique lower triangular matrix. So A has entries a 11, a 12, a 13, a 21, a 22, a 23, a 31, a 32, a 33, this = L into U, we would like to apply Dolittle method in which L is a unit lower triangulat matrix. So it has entries 1 on the diagonal it is a lower triangular matrix so the other entries will be l 11 in the 2nd row, and l 31, l 32 below the diagonal on the 3rd row and the other entries will be 0.

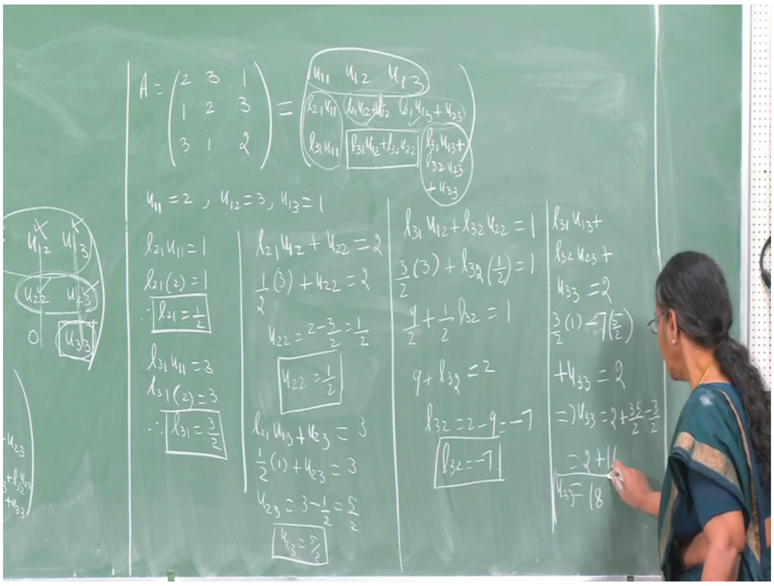
So you observe that this L is a unit lower triangular matrix with entries only lying below the diagonal and U is an upper triangular matrix, so u 11, u 12, u 13, 0, u 22, u 23, 0, 0, u 33. So let us multiply this and right down, so the 1st row into the 1st column will give you u 11, 1st row 2nd column u 12, 1st row 3rd column will give you u 13, then the 2nd row into 3rd column l 21 u 11 + 1 into 0 + 0 into 0, then 2nd row into the 2nd column will give you l 21 into u 12 + 1 into u 22, 2nd row into the 3rd column will give you l 21 into u 13 + 1 into u 23. Now 3rd row into the 1st column will give you l 31 into u 11 + l 32 into 0 + 1 into 0, then 3rd row into the 2nd column will give you l 31 into u 12 + l 32 into u 22, and finally the 3rd row into the 3rd column will give you l 31 into u 13 + l 32 into u 23 + 1 into u 33.

So we have obtained the product of L into U and that must be = the entries here in the matrix A, when do we say that the 2 matrices are equal? When the i jth entry in this matrix is same as the i jth entry in the matrix A, so we equate the i jth matrix for i = 1, 2, 3, j = 1, 2, 3 and obtain the unknowns which appear here. What are the unknowns? There are 1, 2, 3 unknowns appearing in the lower triangular matrix and there are 3 + 2 + 1 so 6 unknowns appearing in the upper triangular matrix, totally there are 9 unknowns and we have 9 entries here which can be equated to the corresponding entries in the A matrix which are known to us, so we will be in a position to obtain these unknowns which appear on this matrix L into U.

And you observe that you 1st equate u 11 to a 11, so u 11 is determined, then you use the next equation as letter l 21 into u 11 is a 21 because u 11 is determined, l 21 is determined so we 1st determine this then we determine this. Now equate l 31 u 11 to a 31, u 11 is determined so this will immediately give you l 31, so what did you do? You obtained the 1st entry in the upper triangular matrix and you completed the 1st column entries in the lower triangle matrix by making use of these. Now you move over to u 12 that is a 12 that is immediate so you get u 12 and then u 13 is a 13 so determine that also. Now you move over to this, right|? You know u 12 just now you computed, l 21 is already obtained and therefore, you can compute what u 22 is by equating this entry to a 22 and continue this way namely determine the entries in the 1st row of the upper triangular matrix in the 1st column of the lower triangular matrix.

Then go to determining the entries in the 2nd row of upper triangular matrix and go to the entries in the 2nd column of the lower triangle matrix and finally the entry in the 3rd row here and that completes determining all the unknowns which appear on the right–hand side and so we have with us the matrix L and the matrix U. So let us take an example and then illustrate this procedure. So let us consider the matrix A given by this and let us try to factorise A in the form L into U, where L is a unit lower triangular matrix and U is an upper triangular matrix.

(Refer Slide Time: 8:10)



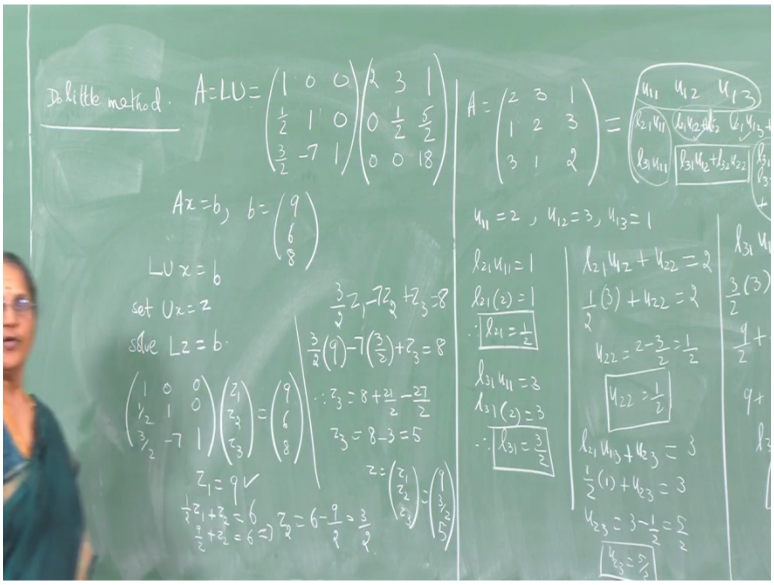
So now that all that is needed has been done, I directly equate this to u 11, u 12, u 13, l 21 u 11, l 21 u 12 + l 22, l 21 u 13 + u 23 then l 31 u 11, l 31 u 12 + l 32 u 22 and finally the entry here is l 31 u 13 + l 32 u 23 + u 33. So what should we do? We first determine the 1st row here by equating it to this, so that immediately tells us u 11 is 2, u 12 is 3, u 13 is 1, so these are determined. Now I move over to the column here, so l 20 u 11 is 1, therefore l 21 into 2 is 1 so l 21 is half, then I have l 31 into u 11 is 3 so l 31 into u11 is 2 and that is 3 and therefore, l 31 is 3 by 2, so we have determined these entries in the 1st column of the L matrix. We now move on to determining the entries on the 2nd row here, namely determining the entries on the 2nd row of the upper triangular matrix.

So we have l 21 u 12 + u 22 = 2, l 21 is known, u 12 is known, u 22 is unknown, so this gives you u 22 as 2 – 3 by 2 and so u 22 is half, so you have determined this entry in the 2nd row. Now we equate this, l 21 into u 13 + u 23 = 3, in this l 21 is half, u 13 is 1 and u 23 is unknown so this gives you u 23 to be 3 – half and so it is 5 by 2 and therefore this entry is determined, so the unknowns in the 2nd row of upper triangle matrix have been computed. What should we do now? We should try to determine now the unknown in the 2nd column of the lower triangular matrix namely, compute what is l 32 so we take this entry and equate that to the value 1 here, l 31 is 3 by 2, u 12 is 3, l 32 is what we want, u 22 has been computed and that must be = 1, so 9 by 2+ half of l 32 is 1 so 9 + l 32 will be 2 and so l 32 is – 7.

So we have computed the entry in the 2nd column of the lower triangular matrix, only unknown that remains is the entry in the 3rd row of the upper triangular matrix. So we equate this entry to the corresponding value in the A matrix, so l 31 u 13 + l 32 u 23 + u 33 must be = 2, l 31 is 3 by 2, u 13 is 1, l 32 is computed by us just now and it is – 7 into u 23 that is computed as 5 by 2 and u 33 is not known and that must be = 2, this gives you therefore u 33 as 2 + 35 by 2 – 3 by 2 so 2 by 35 – 3, 32 by 2 so 16, so U 33 is 18, so the unknown in the last row of the upper triangle matrix has also been computed.

I hope the procedure is clear, when you apply Dolittle method and decompose A in the form L into U with L as a unit lower triangular matrix then follow these steps is systematically to compute the entries in L and U. Start with computing entries on the first row of the upper triangular matrix, step 1. Step 2, compute the unknowns in the 1st column of the lower triangle matrix then the entries which are unknown in the 2nd row of the upper triangular matrix, at the 4th step compute the entries in the 2nd column of the lower triangular matrix, then finally move over to computing the entry in the 3rd row of the upper triangular matrix. So we now have all these values so we write down what A is and how A can be factored in the form L into U.

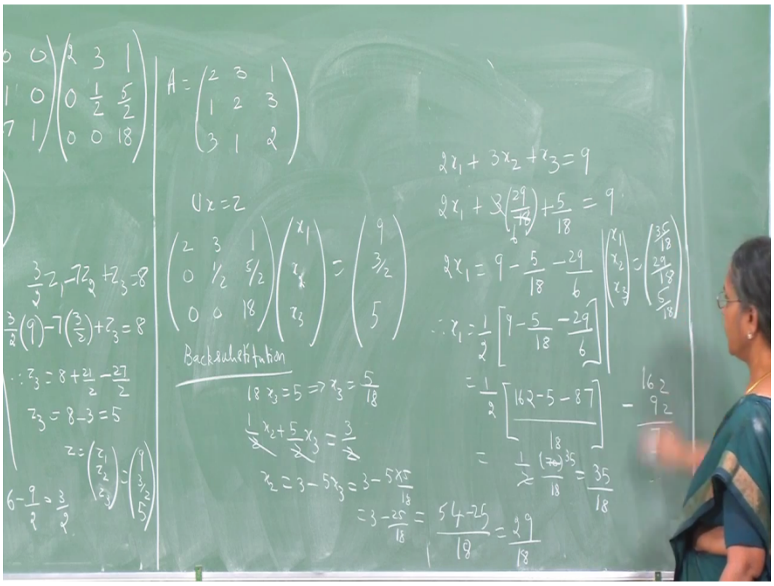
(Refer Slide Time: 15:49)



We will write down the matrix in the form A = L into U; 1, l 21 is half, l 31 is 3 by 2, then 0, 0, 1, 0, l 32 is –7, 1, so that is our l. We write down u, u 11 is 2, u 12 3, u 13 is 1, it is an upper triangular matrix so 0, u 22 that is half and then u 23 5 by 2, then we have 0, 0, u 33 and that is 18. So A has a value decomposition of the form with L as a unit lower triangular matrix and U given by these entries. Now that we have decomposed A in the form A = L U, let us solve the system of equations in which A is a coefficient matrix.

Suppose say we are asked to solve the system A x = b with b as 9, 6, 8 when A is L U into x = b. I said U x to be = z and therefore, I have to solve the system L z = b, so I write down the matrix L, which has entries 1, 0, 0, half, 1, 0, 3 by 2, – 7, 1. Let us take z to be a vector with components z 1, z 2, z 3 in the right–hand side is b and that is 9, 6, 8, so we have a system in which the coefficient matrix is a lower triangular matrix so be we can solve this system by forward substitution. What is my 1^{st} equation? 1 into z 1 = 9 so immediately z 1 is obtained, what is the 2^{nd} equation give you? Half into z 1 + 1 into z 2 = 6 so z 1 is 9 so 9 by 2 + z 2 is 6 and therefore, this gives you z 2 as 6 – 9 by 2 so it is going to be 3 by 2. Now I move to the last equation, what does it give? It gives you 3 by 2 into z 1 – 7 z 2 + z 3 = 8.

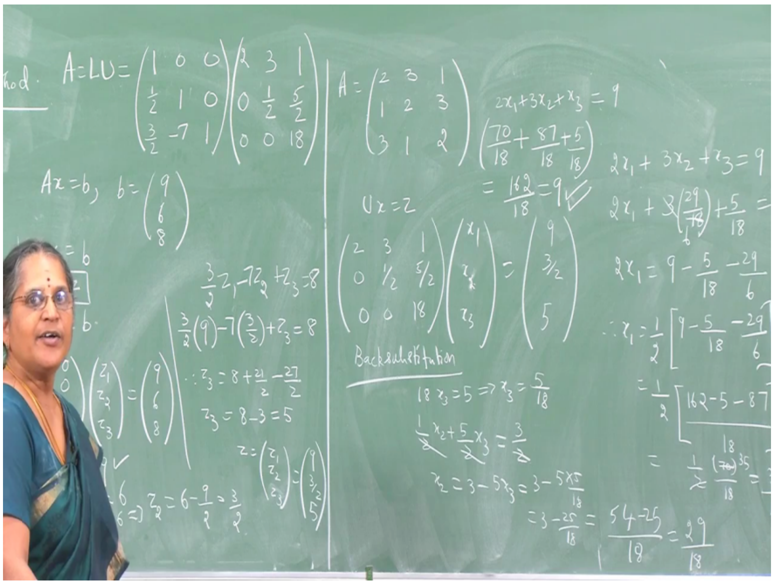
(Refer Slide Time: 20:05)



We already have z 1 and z 2 with us so 3 by 2 into z 1 is 9 – 7 into z 2 is 3 by 2 + z 3 is 8, so therefore z 3 will be 8 + 21 by 2 – 27 by 2, so 8 – 6 by 2 so it is going to be 5, so z 3 is 5. So the vector z 1 having components of z 1, z 2, z 3 is given by 9, 3 by 2 and then 5. So vector z is now known and therefore we can use the system U x = z and solve for what x is. Write down the system u x = z, what is u? 2, 3, 1, 0, half, 5 by 2, 0, 0, 18. X has components x , x 2, x 3 and the right–hand side vector z has components 9, 3 by 2, 5. What do you do? We see that we have a system whose coefficient matrix is an upper triangle matrix, so we employ back substitution and solve this system so we start from the last equation and we observe that we have 18 x 3 as 5 so x 3 is 5 by 18.

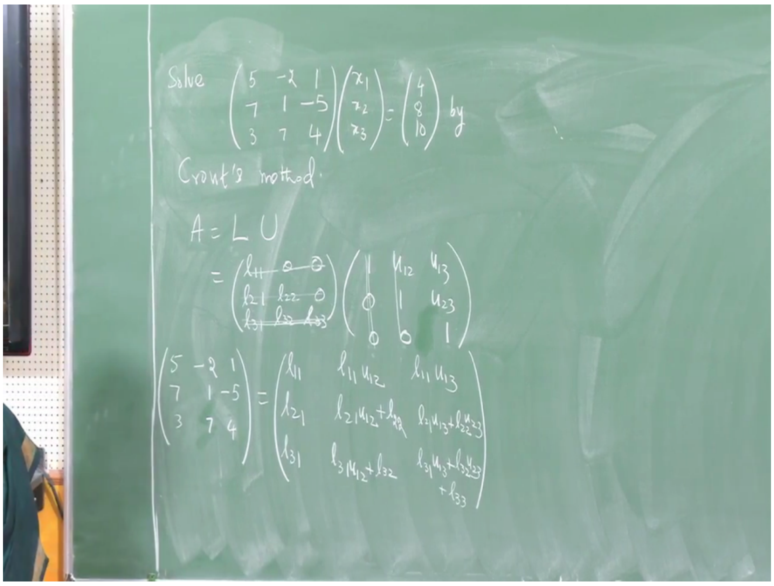
Then half of x 2 + 5 by 2 x 3 is 3 by 2, so cancelling 2 throughout x 2 is 3 – 5 x 3 so it is 3 – 5 into 5 by 18, so 3 – 25 by 18 so 54 – 25 by 18 so 29 by 18 that is x 2. Now that we have x 2 we move to the 1^{st} equation which gives us 2 x + 3 x 2 + 3 x 3 is 9, x is an unknown to be determined so 3 times x 2 is 29 by 18 + x 3 which is 5 by 18 and this must be = 9. This gives you 2 x 1 as 9 – 5 by 18 – 29 by 6 and therefore, x 1 will be = half of 9 – 5 by 18 – 29 by 6, so let us simplify this and obtain what x is, and we see that x = 35 by 18. So the solution is given by vector x =, so the solution is given by vector x having components x 1, x 2, x 3 and the values are given here. So we have been able to solve the system of equations A x = b using Dolittle factorisation method so you try to see whether your computed solution is correct by substituting the solution in the given system.

(Refer Slide Time: 24:27)



So you must have 2 times x + 3 times x 2 + + x 3 should be = 9 and you obtained x 1 as 35 by 18, so 70 by 18 + 3 x 2 so 87 by 18 + x 3 so 5 by 18 and that is your left–hand side, check whether this gives you 9, this is 9 so that = the right–hand side. So when you verify that your solution is correct then you can be confident that your calculations are correct, so it is always better that you ensure that at each step of your computation your computed result is a correct result, so let us now move over to the next method namely Crout's method. In Crout's method, A has to be factored in the form L into U with U as a unit upper triangular matrix, the procedure is the same so we will quickly demonstrate how Crout's method can be used to solve a system of equations.

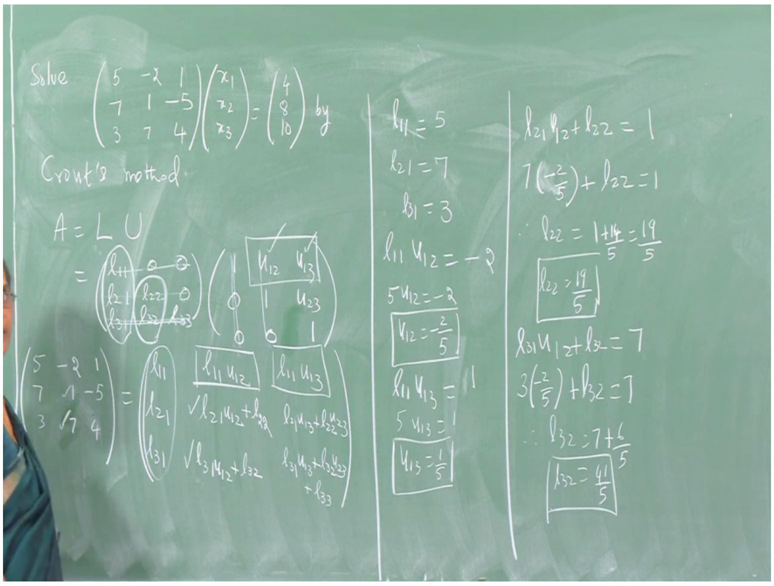
(Refer Slide Time: 26:06)



So we shall consider an example and then work out the details and that will explain to us how one can solve the system of equations using Crout's method. So let us solve the system 5, – 2, 1, 7, 1, – 5, 3, 7, 4 multiplied by x 1, x 2, x 3 = 4, 8, 10 by Crout's method, so in this we have to decompose A in the form L into U. L is a lower triangle matrix and U is a unit upper triangular matrix so on the diagonal we have entries as 1, so this will be u 12, u 13, and this is 0, u 23, 0, 0, 1 that is the upper triangular matrix. And L is a lower triangular matrix so l 11, l 21, l 22, 0, l 31, l 32, l 33, so L is a lower triangular matrix. And what is A? A is given to be the matrix with entries 5, – 2, 1, 7, 1, – 4, 3, 7, 4 and this must be = the product of these 2 matrices, so let us multiply these 2 matrices and right down the entries.

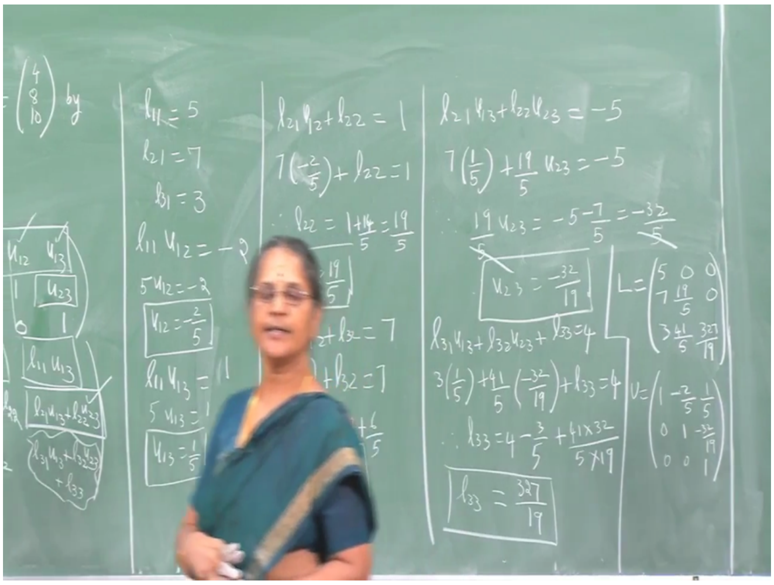
So first row into 1^{st} column so l 11, then l 11 u 12, 0 into 1 + 0 into 1, l 11 u 13 + 0 into this + 0 into 1, then 2^{nd} row into 1^{st} column l 21 into 1 + 0 + 0, l 21 into u 12 + l 22 + 0 into 0, then l 21 into u 13 + l 22 into u 23 + 0 into 1, then the 3^{rd} row into the 1^{st} column so l 31 into 1 + 0 into l 32 + 0 in 1 33 then 3^{rd} row into the 2^{nd} column so l 31 u 12 + l 32 into 1 + 0, and finally l 31 into u 13 + l 32 into u 23 + l 33.

(Refer Slide Time: 29:31)



So now we equate the corresponding entries in both the matrices, so that will give you l 11 to be 5, l 21 to be 7 and l 31 to be 3, so we have determined the 1st column elements in the lower triangular matrix so we move over to determining the 1st row elements which are unknowns in the upper triangular matrix, so what should we do? We should equate l 11 into u 12 to be – 2, just now we computed l 11 so 5 u 12 is – 2 so u 12 is – 2 by 5. So I move over to the next entry because that will determine u 13 which is in the 1st row of the upper triangular matrix so that gives you l 11, u 13 to be 1, l 11 is 5 and so we have u 13 to be 1 by 5. Having determined the unknowns in the 1st row of upper triangular matrix we should move to the unknowns in the 2nd column of the lower triangular matrix.

(Refer Slide Time: 32:50)

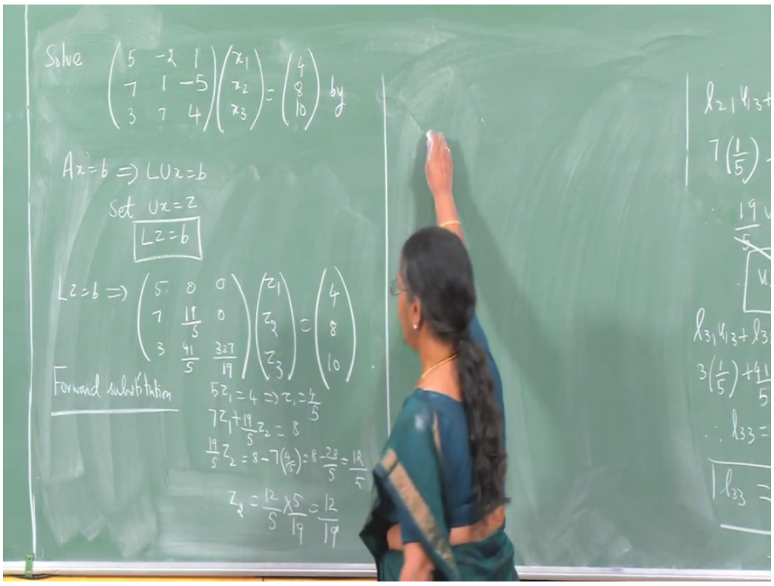


And therefore, I make use of these 2 and equate the corresponding values here so that will give me l 21 into u 12 + l 22 to be = 1. Do I know l 21? Yes 7, what about u 12? It is computed as – 2 by 5, what about l 22 that is what I am trying to determine and that must be = 1 so l 22 is 1 + 14 by 5 and so it is 19 by 5. Then we move over to this, l 31 into u 12 + l 32 = the corresponding entry which is 7, I know what is l 31 and I also know u 12 which is – 2 by 5 but I do not know what is l 32 so that gives you l 32 to be 7 + 6 by 5, so 41 by 5. So we have determined the unknowns in the 2nd column of the lower triangular matrix so we moved to the unknown in the 2nd row of the upper triangular matrix so we have to equate this entry to the corresponding entry in the A matrix.

That gives you l 21 into u 13 + l 22 into u 23 and so equate this to – 5, so in this l 21 is known as 7, u 13 is 1 by 5, l 22 in 19 by 5, the unknown is u 23. So 19 by 5 into u 23 is – 5 – 7 by 5 so – 32 by 5 and therefore, u 23 is – 32 by 19 so we have determined the unknown in the 2nd row of the upper triangular matrix so I have to move to the 3rd column of the lower triangular matrix and determine the unknown which appears there namely l 33 and so I make use of this entry in the product and equate it to the corresponding entry here. This will give me l 31 into u 13 + l 32 u 23 + l 33 is 4, l 31 is 3, u 13 is 1 by 5, l 32 is 41 by 5, u 23 is – 32 by 19, l 33 is an unknown to be determined so therefore, l 33 is 4, – 3 by 5 + 41 into 32 divided by 5 into 19, we simplify this and write down the value of l 33, it turns out to be 327 by 19.

So all the unknowns have been computed and we write down the matrix L and matrix U. L has entries l 11 5, 0, 0, l 21 is 7, l 22 is 19 by 5 and then 0, l 31 is 3, l 32 turns out to be 41 by 5 and l 33 is 327 by 19. We also write down U, U has entries 1, u 12 which is – 2 by 5, u 13 is 1 by 5 and then 0, 1, u 23 is – 32 by 19, the last row in u has entries as 0, 0, 1, so we now know the matrix L and the matrix U. So at this stage I would suggest that you multiply the 2 matrices L and U and check that you get the elements as the given matrix A so that computations up to this stage can be checked and verified. Now that we have the 2 matrices L and U and we are asked to solve the equation system A x = b let us work out the details by 1st using forward substitution followed by backward substitution.

(Refer Slide Time: 37:33)



So the system A x = b will now become L U x = b and we said u x as z and that will give you L z = b. So I 1st have to solve for the unknown vector z and then substitute here to get the unknown vector x, so what is my system L z = b? L has components 5, 0, 0, 7, 19 by 5, 0, 3, 41 by 5 and 327 by 19 that has to be multiplied by the unknown vector having components z 1, z 2, z 3 and that = b, b has components 4, 8 and 10 so use forward substitution and right down the solution. The 1st equation is 5 z 1 = 4 so z 1 is 4 by 5, the 2nd equation is 7 z 1 + 19 by 5 z 2 is 8 so 19 by 5 z 2 is 8 – 7 into z 1 that is 4 by 5 so 8 – 28 by 5 so 40 – 28 so 12 by 5 is z this gives z 2 to be 12 by 5 into 5 by 19 so it is 12 and 19.

(Refer Slide Time: 39:27)

And now we use the 3rd equation which gives 3 z 1 + 41 by 5 z 2 + 327 by 19 into z 3 = 10, so 3 into z 1 is 4 by 5 + 41 by 5 into z 2 is 12 by 19 + 327 by 19 into z 3 must be 10 so therefore you can compute z 3 from this equation which gives you 10 – 12 by 5 – 41 into 12 by 5 into 19 and so z 3 will be 19 by 327 times simplify this to obtain what z 3 is, 46 by 327 so now that you have the components of z vector, you can solve the system U x = z, the system is given by u x = z where U is 1, – 2 by 5, 1 by 5, 0, 1, – 42 by 19, 0, 0, 1 into unknown vector x 1, x 2, x 3 is z 1, z 2, z 3 and we have already computed z 1, z 2, z 3 as 4 by 5, 12 by 19 and 46 by 327.

So use backward substitution method and obtain the values of the unknowns x 1, x 2, x 3, 1st you will obtain x 3, 1 into x 3 is 46 by 327 so x 3 is 46 by 327 then next equation gives you 1 into x 2 – 32 by 19 into x 3 is 12 by 19 so x 2 is 12 by 19 + 32 by 19 into x 3 which gives 46 by 327 and the 1st equation gives you 1 into x 1 that is x 1 = 4 by 5 then + 2 by 5 into x 2 and x 2 is what you have computed here, then – 1 by 5 into x 3 that is 46 by 327, so simplify and you have the solution for the unknown vector x and you have used Crout's method to solve the system where you have decomposed A in the form L into U where U is a unit upper triangular matrix.

And once you have determined L and U, you have used the forward substitution method followed by backward substitution procedure to determine the unknowns which is the vector x. So we have demonstrated how we can obtain the solution of a system of equations by decomposition method namely Dolittle method and Crout's method. In the next class we shall discuss how we can solve this by Cholesky's method if we are given a real symmetric positive definite matrix and then move onto other direct methods such as Gauss elimination and Gauss Jordan techniques, we will continue with our discussions on direct methods for solving system of equations in the next class.