Numerical Analysis

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Lecture No 38

Solution of Linear Systems of Equations -1

Decomposition Methods-1

Good morning everyone, in the next few classes we shall focus our attention on the numerical development of solution of system of linear equations.

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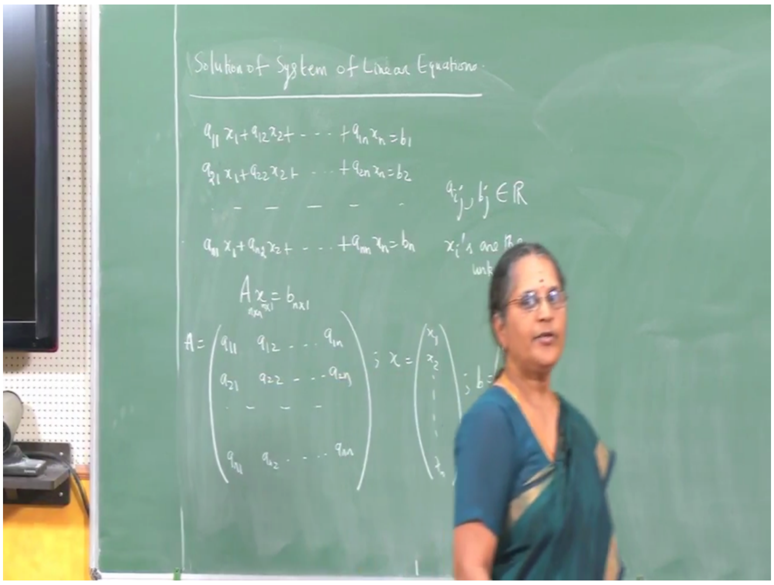

The system of linear equations in n variables can be written in the form a 11 x 1 + a 12 x 2 + etc + a 1n x n = b n, a 21 x 1 + a 22 x 2 + etc + a 2 n x n is b 2 and so on, the nth equation is a n1 x n + a n2 x 2 + etc + a nn x n = b n. Here the a ij and the b j they are all real numbers and the x i are the unknowns to be determined and we see that the system of equations can be represented in matrix form as a x = b, where a is a N Cross N matrix and x is an N cross 1 vector and b is an N cross 1 vector where a can be given by a 11, a 12, etc a 1n, a 21, a 22, et cetera a 2n and so on a n 1, a n2 etc a nn. x factor is given by x 1, x 2, et cetera, x n and the right hand side vector b has n components b 1, b 2, et cetera, b n.

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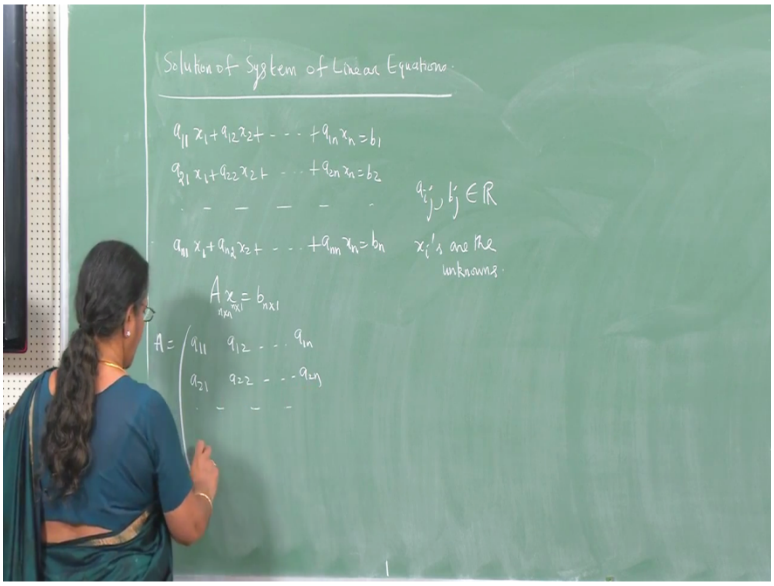

So our objective is now to discuss Numerical aspects of solving the system of the equations given in matrix form say A x = b. So let us see some simple cases where we can immediately present the solution of such a system, the system of equation A x = b is such that A is a diagonal matrix namely A has elements d 1, d 2, etc, d n along a null and the rest of the entries in the matrix are all 0s, so A has entries only along the diagonal, then in that case the system A x = b will be such that A multiplied by we call the vector x 1, x 2, et cetera, x n will be b 1, b 2, etc b n and we observe that we have d 1 into x 1 is b 1, so x 1 is b 1 by d 1. The next equation gives you d 2 into x 2 is b 2 and so x 2 is b 2 by d 2 and so on. The last equation gives you d n into x n is b n and therefore x n is b n by d n, so in general x i is b i divided by d i and this is for i = 1, 2, 3, et cetera up to n.

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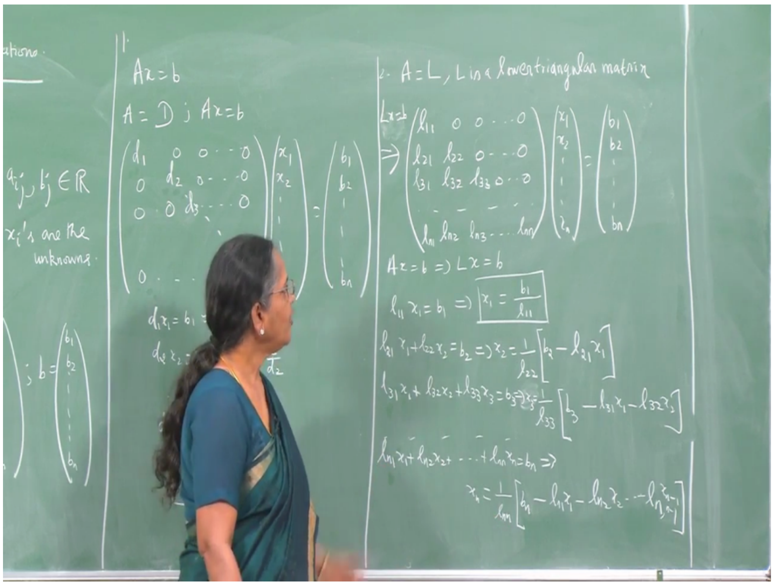

So if A is a diagonal Matrix and you are asked to solve the system A x = b then the solution is immediate, that is a very special case. Let us consider another case where say A is a lower triangular matrix. So L is of the form l 11, 0 etc 0, l 21, l 22 and then 0s, l 31, l 32, l 33, 00, et cetera and the last equation will be l n1, l n2, l n3, et cetera l nn, so this is your matrix a, it is a lower triangular matrix. So I have to solve the system A x = b or in other words I have to solve the system L x = b, so that will give me this L multiplied by the column vector x 1, x 2, et cetera, x n to be equated to vector b 1, b 2, et cetera b n, so my system is L x = b and it is written in this form.

So we observe immediately that the 1^{st} equation is l 11 into x 1 and it is b1 that gives you what x 1 is namely, b1 by l 11. Then the 2^{nd} equation gives you l 21 into x 1 + l 22 into x 2 is b 2, you already have computed x 1 here so this gives you x 2 to be = 1 by l 22 multiplied by b 2 – l 21 into x 1, let us solve from the 3^{rd} equation what is x 3, so l 31 x 1 + l 32 x 2 + l 33 x 3 is b 3. So that immediately gives you x 3 because you already have computed x 1 and x 2 from the 1^{st} 2 equations, so you have b 3 to be given by 1 by l 33 multiplied by b 3 – so this gives you x 3 as 1 by l 33 into b 3 – l 31 x 1 – l 32 into x 2, so it is clear how we can get the successive unknowns and so finally we have, from the last equation which is l n1 x n + l n2 x 2 + etc + l nn x n = b n and this will give you x n to be = 1 by l nn into b n – l n1 x 1 – l n2 x 2, etc – l n – 1 in to x n - 1.

So you have computed all the components of the unknown vector x namely x 1, x 2, et cetera, x n. How would you do it? You obtained x 1, you substituted in the next equation so you moved forward to the next equation got your x 2 with the knowledge of x 1 then you further moved forward to the 3rd equation with the knowledge of x 1 and x 2 you compute x 3 and so on, you moved forward to the last equation and with the knowledge of x 1, x 2, et cetera, x n - 1 you computed the value of x n and all the unknowns are determined. This method is called the forward substitution method and therefore if you have system of linear equations in which the coefficient matrix a is a lower triangle matrix then its solution can immediately be obtained using forward substitution method as has been explained here, so let us now see what happens if A is an upper triangular matrix.

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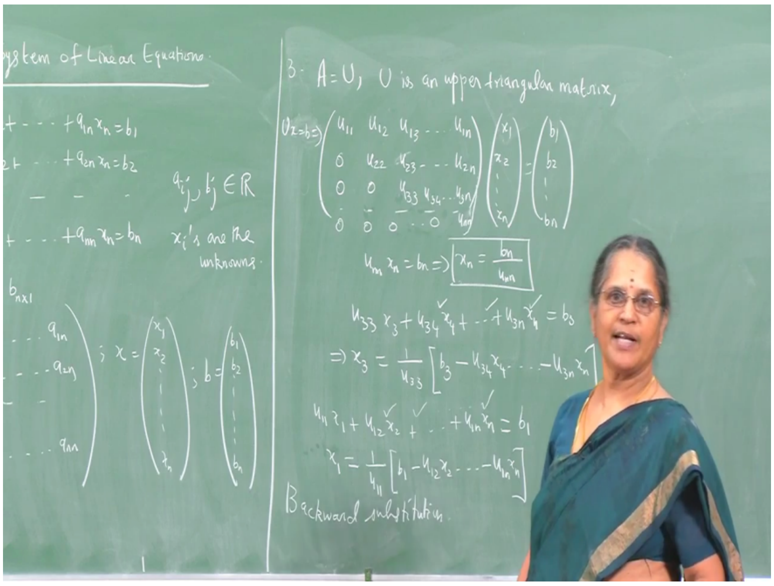

Let us consider A to be an upper time matrix U given by, U = u 11, u 12, u 13 and so on u 1n, 0 u 22, u 23 and so on u 2n, 0, 0, u 33, u 34, et cetera, u 3n and so on, and the last equation we have 0 everywhere and the diagonal entry will be u nn, so u is an upper triangular matrix and you now have to solve the system U x = b. So this multiplied by x 1, x 2, x 3, etc, x n should be = b1, b2 and so on b n. So now you observe that you should go backward because you have an equation of the form u nn x n is b n, which will immediately give you what x n is, namely it is b n by u nn, and then you go backward and look at the equation which appears in the last one row. So that gives you u where does it appear? It will appear in the n – 1th row and the entry here and the entry in the diagonal will be nonzero and other entries will all be 0.

So let us see what happens in the case when we go to the 3rd equation, so typically the 3rd equation will give you u 33 multiplied by x 3 + u 34 x 4 + et cetera + u 3n x n and that will be b 3, but you have already obtained x n, x n - 1, etc up to x 4 because you have moved backward and so this will immediately tells you what x 3 is, namely x 3 will be 1 by u 33 multiplied by b 3 – u 34 x 4 and so on – u 3m x n. And so the 1st equation if i move backward and reach the 1st equation that is u 11 x 1 + u 21 x 2 + et cetera + u 1n x n and that is b1, you have computed all these coefficients x 2, x 3, et cetera, x n so you require x 1 and the 1st equation will give you x 1 to be 1 by u 11 into b 1 – u 12 x 2, et cetera – u 1n x n.

So you are being able to obtain the unknowns by moving backward from the last row of the system of equations to the 1st row and so this is referred to as backward substitution and therefore, if you are given a system of equations in the form A x = b, where A is an upper triangular matrix then immediately its solution can be obtained by backward substitution method. So we may not come across such simple equations all the time namely the coefficient matrix is a diagonal matrix or an upper triangular matrix or a lower triangular matrix, we will have to solve a system of equations of the form A x = b, where A is neither a diagonal matrix nor an upper triangular matrix nor a lower triangular matrix.

But this suggests that if we can reduce the coefficient matrix A of the given system to a diagonal matrix or either a lower triangular or an upper triangular matrix then the solution can immediately be obtained. So we now look for from some general-purpose algorithms that will help us to solve a system of equations A x = b, so our objective is going to be discuss numerical algorithms that will help us to solve a system of equations and then discuss the errors associated with the computed solutions and study methods for controlling and reducing these errors and finally see if we can understand some iterative methods for solving some of equations, so our system is something like this.

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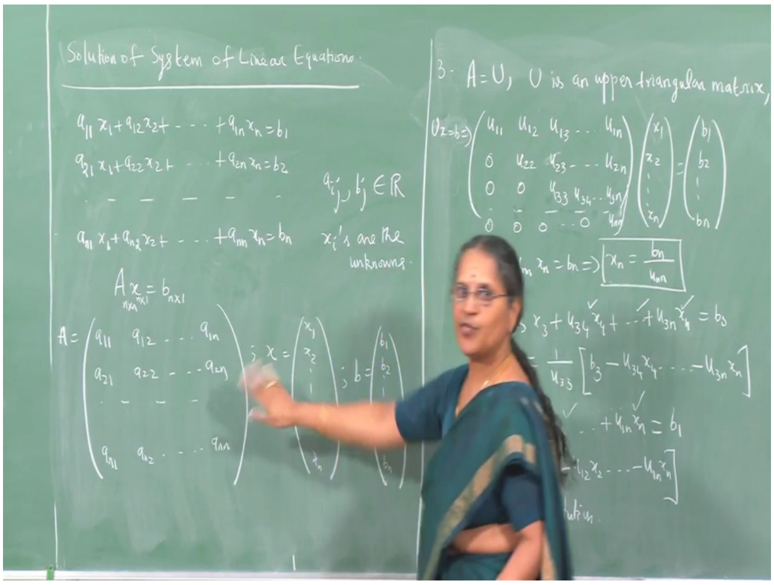

We have n equations in n unknowns which can be put in the matrix from namely A x = b. So while deriving such numerical algorithms, the elementary operations are very useful to us so we look at what are these elementary operations and using elementary operations we try to develop some numerical algorithms for solution of a system of equations. So let us write down what are the elementary operations that we will use in our computations.

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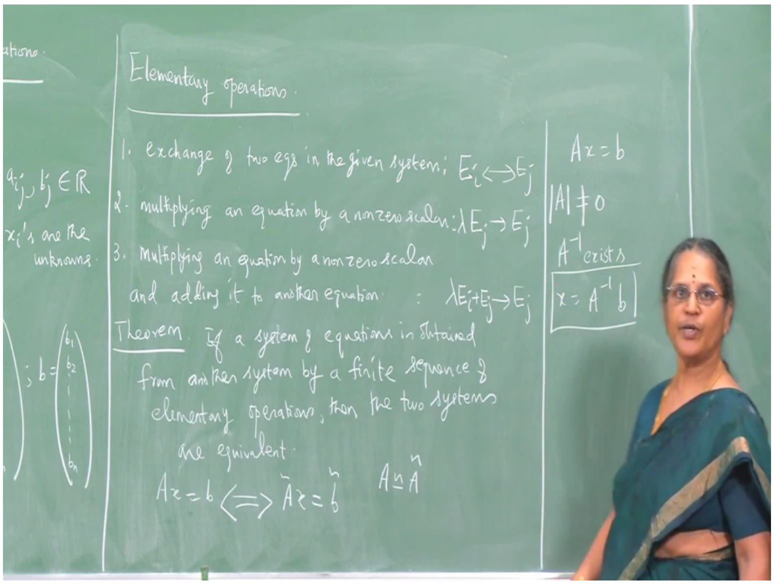

So it is exchange of 2 equations in the given system namely I take the ith equation of the system and exchange it with the jth equation for example, I can take the 1st equation, exchange it with the nth equation namely, put the 1st equation here and nth equation as the 1st equation. Secondly, multiplying an equation by a nonzero scalar namely I take the equation say E j multiply it by Lambda and replace E j by Lambda E j, where Lambda is a scalar. Multiplying an equation by a scalar which is nonzero and adding it to another equation namely, I consider the equation E j and multiply an equation E i by Lambda and adding it to E j and replacing this E j by Lambda E i + E j, so anyone of these operations is called elementary operations.

Now the question is, when you perform such elementary operations on a system of equations like this then if we determine the solution of the new system, will the solution be different from the solution of the original system? The answer is no and it is given by the following result which says, “If a system of equations is obtained from another system by a finite sequence of elementary operations then the 2 systems are equivalent, what does that mean? I have a system A x = b that is given to me and I make use of a finite sequence of these elementary operations and I obtain a new system namely, say A is changed to some A tilde, the unknown vector is x and b is changed to b tilde. So this new system and the old system are equivalent to each other that is what the theorem says.

And we write A is equivalent to A tilde when elementary operations are applied to A and we have a finite sequence of elementary operations that has converted A to A tilde. So the system obtained after finite sequence of elementary operations is equivalent to the original system and therefore, if we solve the new system and obtained its solution then the solution of this system is the same as the solution of the original system that is what this result says.

So given system A x b, what is it that we have done all along? We will see if A is a non-singular matrix namely determinant of A is different from 0 then A inverse exists and therefore we will write down the solution as A inverse into b, this is what we have been doing all along. So given a system if A inverse is available then use this where the solution x is given by A inverse b and obtain the solution. If suppose A inverse is not available to you, then do not try to obtain A inverse because there are many efficient procedures which are available for solving a system of equations of the form A x = b and that we will derive or we will develop these efficient algorithms now so that you can make use of them in solving the system of equations A x = b.

And these methods belong to 2 categories of methods namely, they belong to either direct methods or iterative methods. And indirect method we will discuss methods such as decomposition methods and Gauss elimination method and Gauss Jordan method. Whereas, in the case of iterative method we shall consider methods such as Gauss Jacoby and Gauss Seidel methods, so these are different methods that we are going to develop in the next few classes and try to see how these methods can be used in solving a system of equations A x = b. Along with that we are also going to discuss the errors associated with the computed solution and see ways by means of which we can control the error and reduce this error. So let us now look into the decomposition method 1st.

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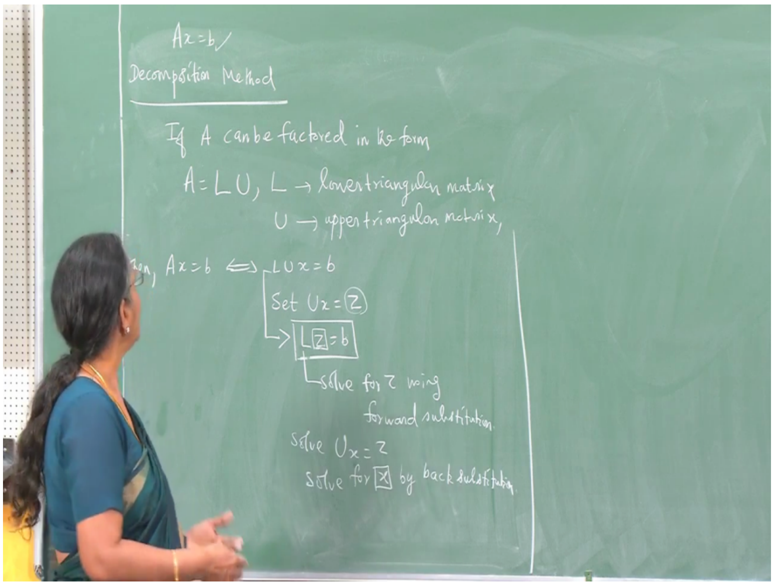

We now consider decomposition method which is a direct method for solving a system of equation A x = b. So given a system A x = b, we would like to develop a method called decomposition method by means of which the system can be solved, so if a matrix A can be factored or decomposed in the form A = L into U, where L is lower triangular matrix and U is an upper triangular matrix then in that case solving the system A x = b is equivalent to solving L U x = b because A can be factored in the form L into U, where L is lower triangular and U is upper triangular matrix. This suggests that if I set U x = z, then this L U x = b becomes L z = b, so let us see how we can obtain our solution x. In this L z = b we know what b is because we are given the system A x = b so the right-hand side is a known vector.

And z is an unknown I do not know what z is, but L is a lower triangular matrix because we have been able to factor A in the form L into U where L is lower triangular and so we have now a system L z = b with coefficient matrix has a lower triangular matrix, we have already discussed. In that case the system solution can be immediately written down by what method? By forward substitution method because it is a lower triangular coefficient matrix, so solve sub z using forward substitution, so what will you get? You will get what z is. Once you know z you substitute here, so you now can solve U x = z, the only unknown is this vector x. You know what the matrix U is, it is an upper triangular matrix and z has just now been computed by forward substitution method so the unknown vector is x.

So at this stage you solve for x by what method? The system is U x = Z, where the coefficient matrix is an upper triangular matrix and therefore, you can solve for x by that substitution. So what have you got? You now have the solution namely the unknown vector x has been obtained because you are asked to solve the system A x = b, so you should be in a position to factor the matrix A in the form L into U or decompose the matrix as a product of 2 matrices L and U, where L is lower triangular and U is upper triangular. If that is done then the solution of the system can immediately be obtained by 1st doing forward substitution and obtaining z from the system L z = b and then using backward substitution method and solving the system U x = z.

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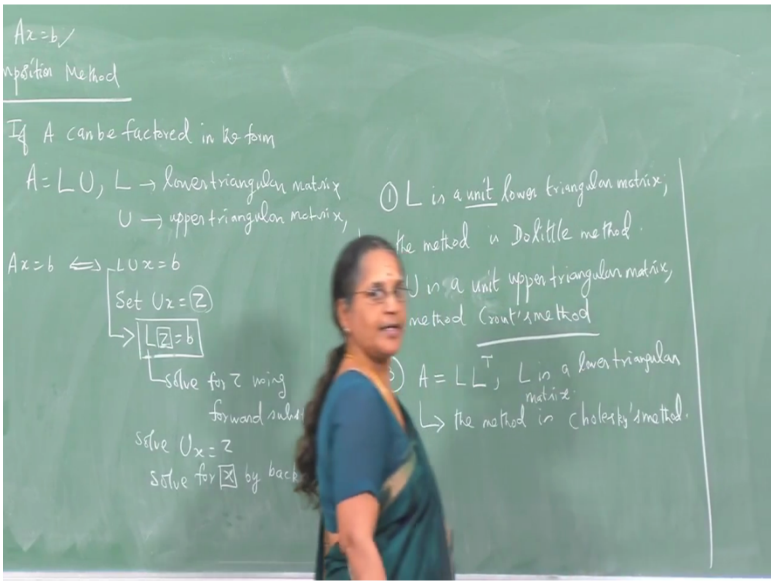

Suppose say we are able to factor A in the form L into U and in case that matrix L is a unit lower triangular matrix then you say that the method is Dolittle method, namely factorise A in the form L into U with L as a unit lower triangular matrix and then solve the system by forward substitution followed by backward substitution then you have used a Dolittle method for solving the system of equations. Secondly, if U is an upper triangular or U is a unit upper triangular matrix then the method is called Crout's method namely, you factorise A in the form L into U with U as a unit upper triangular matrix. And thirdly, if you can write down A in the form L into L transpose, where L is a lower triangular matrix then solve the system of equations, the resulting method is called Cholesky's method

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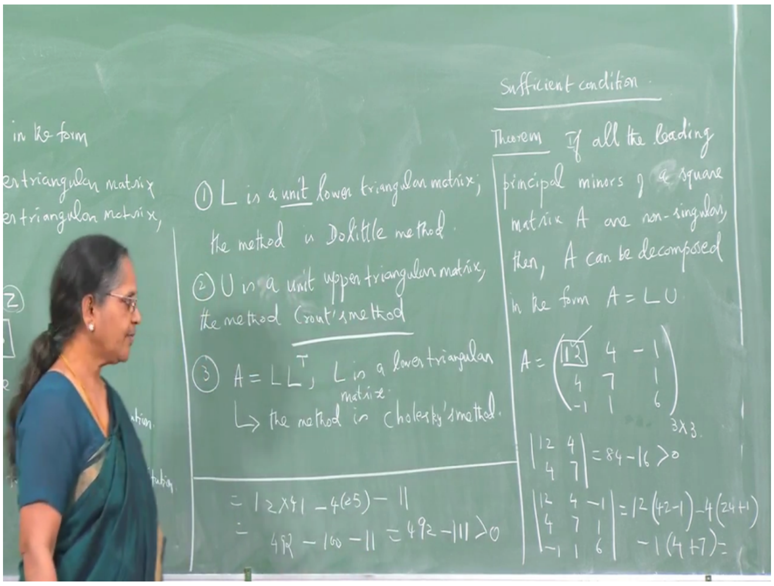

The question now is, if we can decompose the matrix in the form L into U then we can apply forward substitution followed by back substitution to obtain the solution of the system of equations. Is it always possible to factorise A in the form L into U? The following result gives us a sufficient condition under which a matrix A can be decomposed in the form L into U, so let us write down the sufficient condition. It is given by the result that if all the leading principle miners of a square matrix A are non singular then A can be decomposed in the form A = L into U. So what do you mean by leading principal miners of a square matrix A are non-singular? Let us take an example and understand this.

Let us consider the matrix A as this; it is a 3 by 3 matrix. The principle leading miner of order 1 is this entry so it is positive then the principle leading miner of order 2 is determinant of 12, 4, 4, 7 and that is 84 - 16 which is again positive. And the principle leading miner of order 3 is determinant of the matrix A because A is a 3 cross 3 matrix, so determinant of 12, 4, – 1, 4, 7, 1, - 1, 1, 6, so let us evaluate the determinant that gives you 12 into 42 - 1 – 4 into 24 + 1 and - 1 into 4 + 7, so that turns out to be… so I shall write down here 12 into 41 – 4 into 25 – 1 into 11, so it is 492 – 111 which is again greater than 0.

And we observe that all the leading principle miners namely principle miners of order 1 order 2 and order 3 are all positive and therefore, all the leading principal miners are non singular and therefore the given matrix A is such that it can be decomposed in the form L into U or A has a decomposition in the form L U. So we given a system A x = b and we will check whether the coefficient matrix satisfies this sufficient condition and then we decompose the matrix A into form L U and either apply a Dolittle decomposition for a Crout's decomposition and then solve the system by forward substitution followed by backward substitution. And then we will focus on the other case namely when A can be decomposed in the form L into L transpose, is it always possible?

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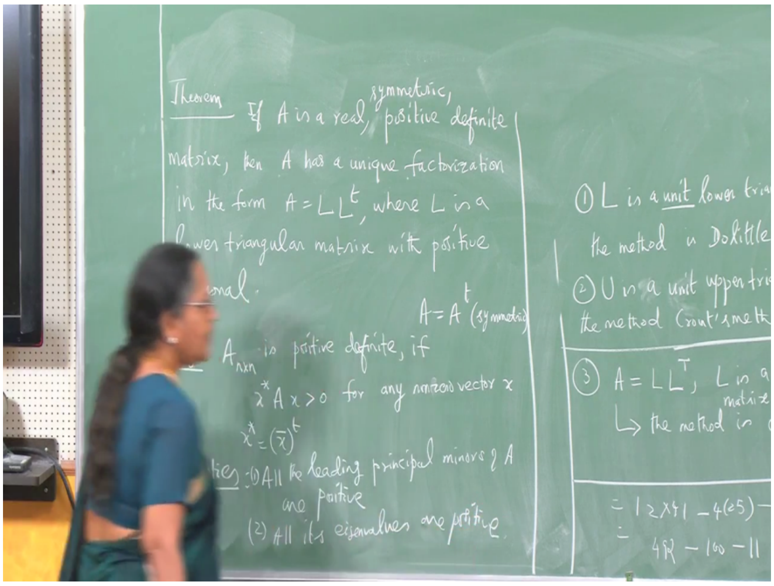

The results says, if A is a real positive definite matrix then A has a unique factorisation in the form A = L into L transpose where L is a lower triangular matrix with positive diagonal, so what are the conditions which are given? A is a real so it is also a symmetric matrix, A is a real symmetric positive definite matrix then A has a unique factorisation of the form L into L transpose. What is L? L is a lower triangular matrix and it has the property that the diagonal elements in L are positive, so we understand when we are given that A is a real matrix so all its entries are real, and what do you mean by A is a symmetric matrix? A must be = a transpose then it is a symmetric matrix. And then what do we mean by a positive definite matrix, so let us give the definition of the positive definite matrix.

So a square matrix A is said to be positive definite so an N cross N matrix a is positive definite if x star A x is greater than 0 for any nonzero vector x. What is x star? x star is transpose of the conjugate, so if this condition is satisfied namely take a nonzero vector x and if x star A into x is greater than 0 for any nonzero vector x then you say that A is a positive definite matrix, and we list down the properties of these positive definite matrices namely, all the leading principle miners of the matrix A are positive that is one of the properties of that a positive definite matrix possesses and secondly, all its Eigen values are positive.

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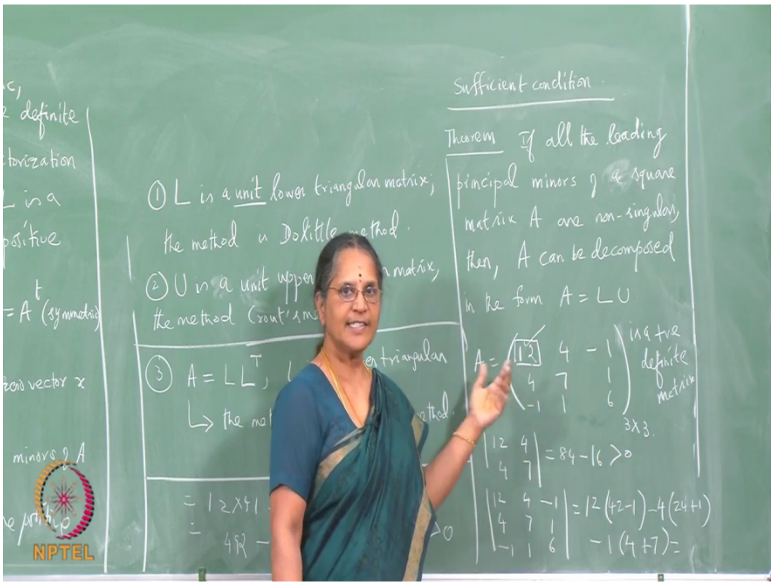

So we observe that the given matrix A that we have taken here has all its leading principle miners to be positive and so this matrix A is a positive definite matrix, and let us check whether it is a real matrix? Yes. Is it a symmetric matrix? Yes what do we see, this is a 12 and that is a 21, this is a 13 and this is a 31 it is same as that, and a 23 is the same as a 23 so this matrix is also a symmetric matrix, so it is a real symmetric matrix in addition all its leading principle miners are positive and therefore, the matrix A is a positive definite matrix and therefore, by this theorem this matrix A possesses a unique decomposition of the form A = L into L transpose, where L will be all over triangular matrix and the diagonal entries on this lower triangle matrix will all be positive.

So now that we know the conditions under which A can be decomposed in the form L into U, let us work out some examples and see how the solution of a system A x = b can be obtained if A can be factored in the form L into U.