Numerical Analysis
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Lecture No 36
Root finding Methods 8
Fixed Point Iteration Methods 2

Good morning, in the last class we considered fixed point iteration method for solving equations of the form f of x = 0. Before connecting the fixed point iteration method with the root finding problem we define what we mean by fixed point for a given function in an interval. We said that p is a fixed point for a given function g if g of p = p where g is defined in the interval a b.

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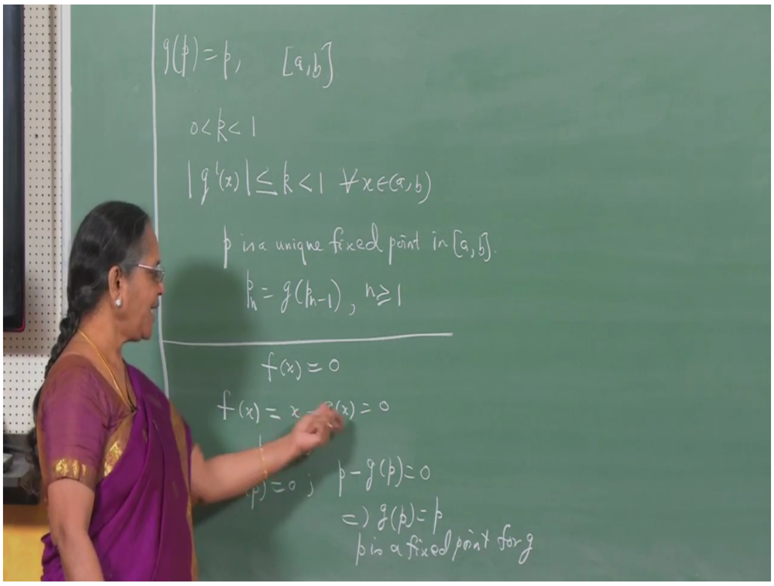

And we represented the sufficient conditions for existence and uniqueness of a fixed point in an interval a b for this function g and said that g must be a continuous function in the interval a b and g takes the values in that interval a b then it has a fixed point, and in addition if g dash of x exists in the open interval a b and there is a constant K with 0 less than k less than 1 such that modulus of g dash of x is less than or = K less than 1 for every x in the total interval a b then these fixed point p is a unique fixed point for the function in that interval. Then we presented the error bounds on the approximation to this fixed point which we obtained by generating a sequence of successive iterates using p n = g of p n – 1 for n greater than or = 1 where g satisfies all the conditions that we have listed here namely g satisfies conditions of the fixed point theorem.

Then we showed that the sequence of iterates generated by this converge to a unique fixed point p for the function G, so having understood the fixed point problem we try to connect this problem with that of root finding problem, what is a root finding problem? In that we will be given an equation of the form f of x = 0, where we are required to find a root of this equation or a 0 for the function f. So we said that we will regret this equation f of x = 0 in the form f of x = x – g x = 0, if p such that f of p = 0 and p is the root of the equation then we observe that this p – g of p = 0 implying that g of p = p so that p is a fixed point for this function g.

So if p belongs to an interval a b then we have a fixed point for the function g in that interval and it exists and is unique provided g satisfies certain conditions; namely if g satisfies the sufficient conditions specified then the sequence of iterates generated by p n = g of p n – 1 will converge to this fixed point p for the function g and that unique fixed point is going to be a 0 for the function f root of this equation f of x = 0. So we try to see how can we write f of x in the form x – g of x, can it be written in a unique way? We just considered an example and said that there are several ways in which f of x = 0 can be written in the form x = g of x.

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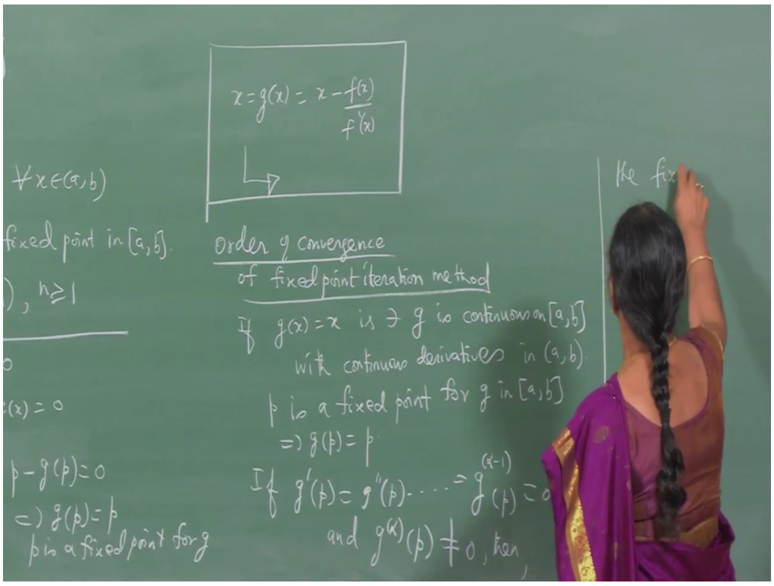

Then we considered each of these forms and said that in some there is no reason why we can believe that the fixed point will give a sequence of iterates which will converge because g did not satisfy the conditions of the theorem. And in some other forms we were able to show that g satisfies the conditions and we also made a remark that if we take our g of x to be in the form x – f of x divided by f dash of x then this form will converge rapidly to a unique fixed point for the function g and therefore rapidly to a 0 of the equation f of x = 0. So we would like to look at this form in this class and see whether this method converges rapidly and what is the order of convergence of this method when we write the given equation f of x = 0 in the form x = g of x where g of x is given by x – f of x by f dash of x.

So before doing all these lets define the order of convergence of a fixed point method. So if g is such that g of x = x and that g is continuous on the interval a b with continuous derivatives in the interval a b, which is an open interval. Further (())(6:58) point for function g in the interval a b, what does that mean? (())(7:13) = p. And if this g is such that g dash of p = g double dash of p etc up to g to the Alpha – 1 at p they are all 0 and the Alpha derivative of g at p is different from 0.

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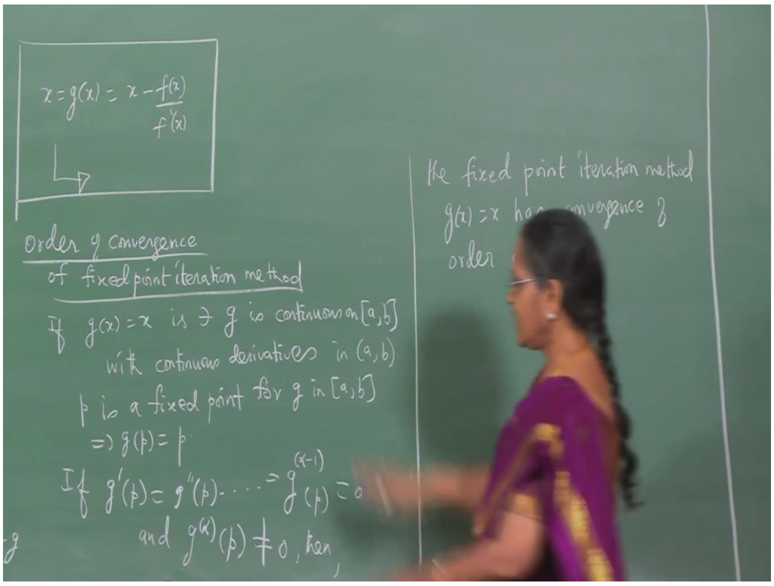

Then we say that the fixed point method given by g of x = x has convergence of order Alpha. So let us just see the definition again, so you are given a fixed point method g of x = x and you observe that g is continuous on the interval a b and it has continuous derivatives so we can also say that it has continuous derivatives g dash, g double dash, etc say up to order Alpha in the open interval a b and p is the fixed point for g in the interval so that g of p is p. If g is such that its derivative of 2 Alpha – – 1th derivative evaluated at p is 0 and the peak order derivative is different from 0 then we say that this fixed point method g of x = x has convergence of order Alpha or the order of convergence of the fixed point iteration method is Alpha, let us consider the following result.

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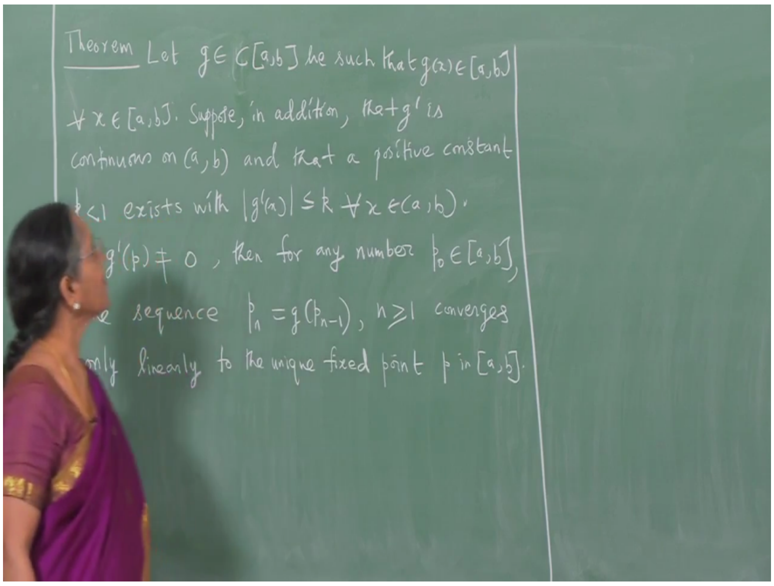

Let g be a continuous function in the closed interval a b and it is such that g of x takes values in the closed interval for every x in a b. In addition, g is such that g dash is continuous on the open interval and there exists a positive constant k less than 1 such that mod g dash of x is less than or = K for every x in the open interval a b. We now show that if g dash of p is different from 0 then for any number p 0 belonging to the closed interval a b, the sequence p n given by g of p n – 1 for n greater than or = 1 converges only linearly to the unique fixed point, this is what we have to show, so let us work out the details.

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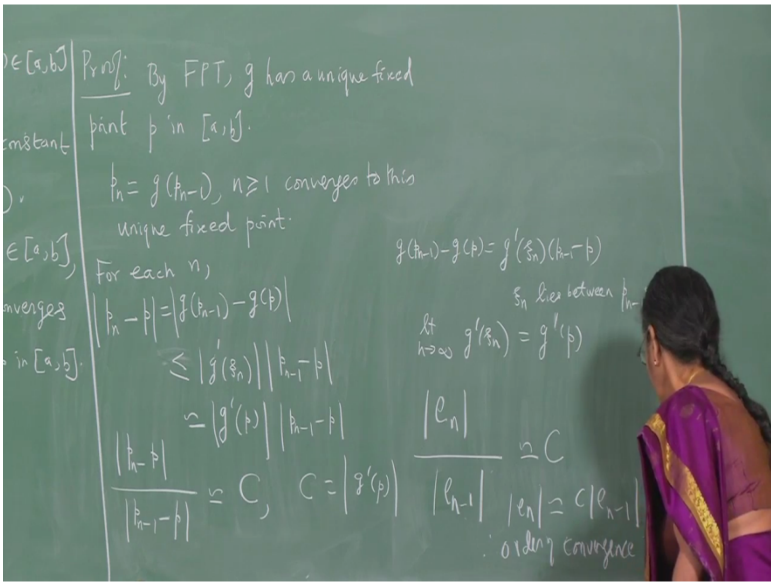

So what is it that is given, the first 2 conditions show that they are the conditions listed out in the fixed point theorem so we immediately conclude that by fixed point theorem g has a unique fixed point p in the interval a b. And the sequence p n = g of p n – 1 for n greater than or = 1 converges to this unique fixed point. We now have to show that the rate, the order of convergence of this method is 1 namely the method has linear convergence; this is what we have to show. What is it that is given to us? It is given that g dash of p is different from 0, so let us make use of this and then show that the method has order of convergence to be 1. So now for each n, we know that p n – p will be such that it is g of p n – 1 – g of p, g is the continuous function on the closed interval.

G dash exists in the open interval and therefore I can use mean value theorem and write g of p n – 1 – g of p is some g dash at Si n into p n – 1 – p where Si n lies between p n – 1 and p. And what do we know about the sequence p n – 1, it converges to p so limit as n tending to infinity of p n –1 is p and therefore, limit as n tending to infinity of g dash of Si n is g dash at p. So we shall make use of this and take the absolute value of mod p n – p absolute value of g of p n – 1 – g of p, so that will be less than or = modulus of g dash of Si n into modulus of p n – 1 – p, so in the limit we know that g dash of Si n converges to g dash of P. So this is approximately g dash of p in absolute value multiplied by p n – 1 – 1 p, so what is it that we have shown, we have shown that p n – p in absolute value by mod p n – 1 – 1 is approximately a constant C, where C is mod g dash of p.

So we recall the definition of order of convergence of a method. What is it that we have shown? We have shown p n – p is the error at the nth step, so that by the error at the n – 1 step is a constant C, so by definition the power to which this is raised to gives you the order of convergence of the method. So what we have shown is mod e n is C times mod e n – 1 and therefore, order of convergence of this method is 1 so the method which generates a sequence of successive iterates and which converges to a unique fixed point for the function g converges linearly to the unique fixed point, what is the condition that we have used, is g dash of p is different from 0. So the result immediately implies that if we want higher order convergence for fixed point iteration method then this can happen only when g dash of p = 0.

The question now arises, is this condition g dash of p = 0 alone is enough to ensure that we have higher order convergence for fixed point iteration methods, let us see the following result and understands the conditions under which the fixed point iteration methods will have higher order convergence when the conditions that has to be satisfied by g include the condition that g dash of p = 0, so let us write down the result in the form of the following theorem.

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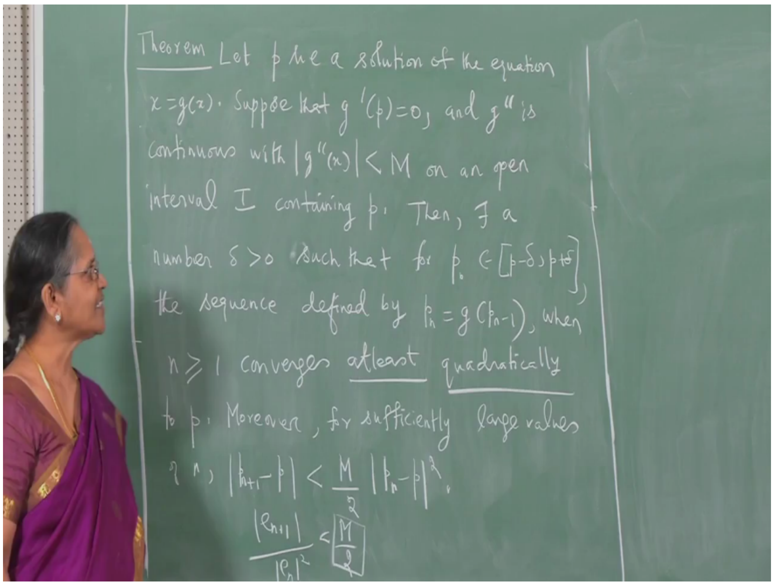

The result is given by this theorem, which says that let p be a solution of the equation x = g of x, what does that mean, g of p is p so p is a fixed point for the function g. In addition, g dash of p is 0 so g is such that its first derivative vanishes at p which is the solution of the equation and x = g of x. And its 2nd derivative is continuous with modulus of g double dash of x to be strictly less than m on an open interval i, which contains this p. Then there is a number delta positive such that you start from any p 0 which lies in the interval p – Delta to p + delta. Namely you start with a p 0 in a neighbourhood of this p which is the solution of x = g of x and then consider the sequence of iterates generated by p n = g of p n – 1 then the results says that this method converges at least quadratically.

And where does it converge to? Converges to p and what is p, it is a solution of the equation x = g of x or it is the fixed point for the functiong. In addition it tells you that the error at the n + 1 step e n + 1 is by modulus of e n square, is less than capital M by 2. So that also gives you the asymptotic error constant C in terms of capital M and what is this M, M is such that modulus of g double dash of x is strictly less than M in an open interval that contains this p which is the solution of the equation x = g of x. So you also have some information about what the asymptotic error constant is. So the theorem tells us that the best way to construct a fixed point iteration problem associated with the root finding problem namely f of x = 0 is to consider the following.

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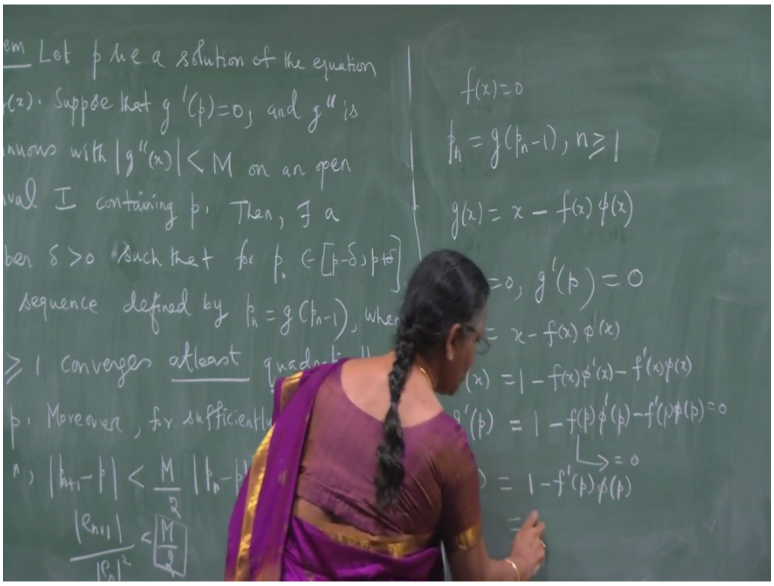

Namely, subtract multiple of f of x from x so that if you consider the sequence of iterates generated by p n = g of p n – 1 for n greater than or = 1 with g of x given by x – f of x into a function Phi of x, where Phi of x is an unknown function, it is a differentiable function that has to be determined such that the method g of x = x has a fixed point p to which this sequence will converge to will have higher order convergence or will have quadratic convergence, so let us try to find out this function Phi of x which is a differentiable function such that this method has quadratic convergence, what are the conditions that it should satisfy, when f of p = 0, g dash of p must be = 0.

You want the function g to be such that g must be of the form x – f of x into Phi of x where Phi of x is a differentiable function, it is unknown to be determined in such a way that g dash of p = 0 and p is a fixed point for function g, so let us write down what g is. So g of x is x – f of x into Phi of x, so g dash of x is going to be 1 – f of x into Phi dash of x (())(21:32) so g dash of p is 1 – f of p into Phi dash of p – dash of p into Phi of p and this must be 0. So we know that f of p is 0 because p is the root of the equation f of x = 0, so this tells us that g dash of p is 1 – f dash of p into Phi of p and we want this to be = 0.

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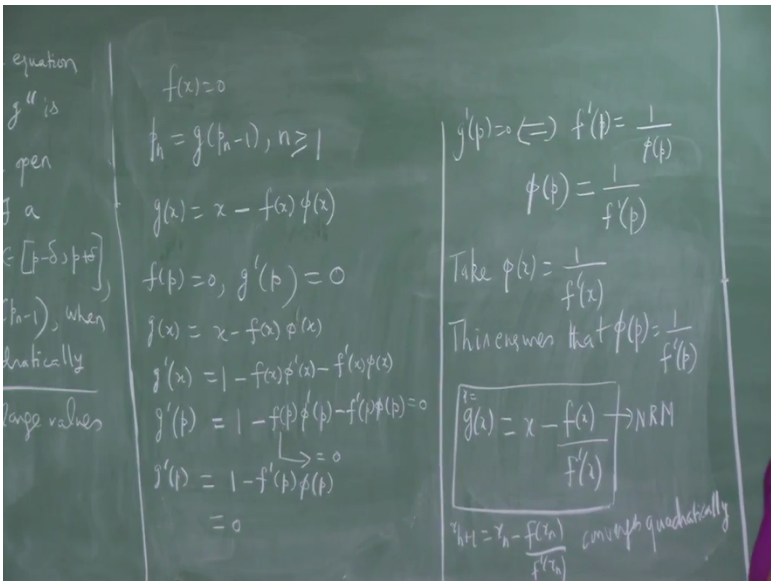

So g dash of p is 0 if and only if your f dash of p is 1 by Phi of p, but what do we want? We want to determine this function Phi, so Phi of p is 1 by f dash of P. So we take the function Phi of x to be 1 by f dash of x and this ensures that Phi at p is 1 by f dash at p, so we have determined this differentiable function Phi of x so we can take g of x to be in the form x – f of x and what is Phi of X, it is 1 by f dash of x and what do you see, you observe that this is nothing but your Newton Raphson Method why, x = g of x and g of x is x – f of x by f dash of x so this is essentially Newton Raphson Method. And we have already established the order of convergence of Newton Raphson Method namely the method (())(23:33) x n – f of x n divided by f dash of x n converges quadratically, namely it has order of convergence to be = 2 and that is why we made a remark earlier that…

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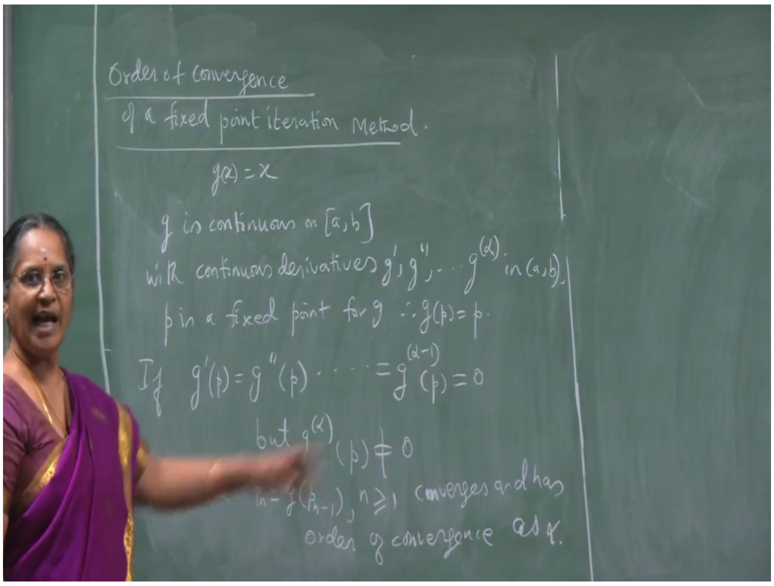

So let us now define order of convergence of a fixed point iteration method. So fixed point iteration method is of the form g of x = x, if g is continuous on the open interval a b with continuous derivatives g dash, g double dash and so on and Alpha derivative of g in the open interval a b, p is the fixed point for g so therefore g of p is p. And if the derivatives are such that g dash at p, g double dash at p and so on, the Alpha – 1th derivative at p, they are all 0 but the Alpha derivative of g at p is different from 0, then the sequence given by p n = g of p n – 1 for n greater than or = 1, this is a fixed point iteration this converges and has order of convergence as Alpha, so the conditions are that the 1st Alpha – 1 derivative of g at p must be 0, the Alpha derivative of g at p should be different from 0 then this method has convergence of order Alpha.

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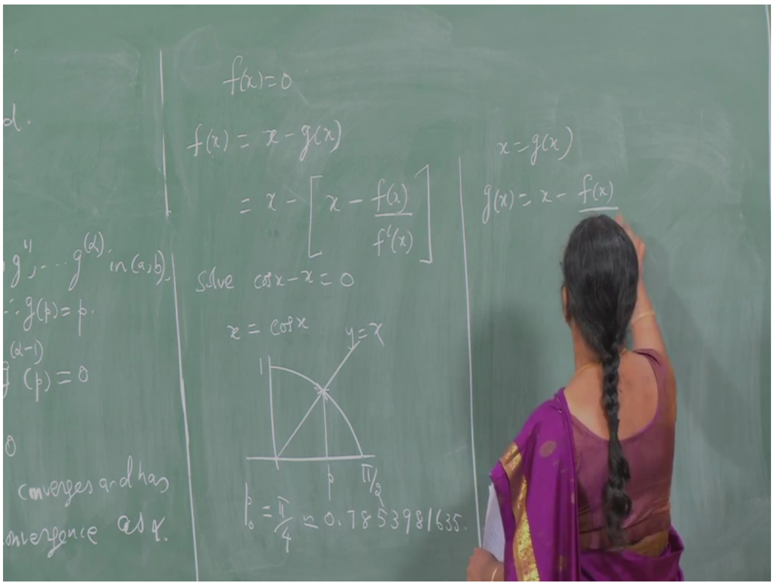

So this is how we find the order of convergence of a given fixed point iteration method. And our discussion tells us that the Newton Raphson Method belongs to the class of x point iteration method, so let us solve the equation cos x – x = 0. So solving this problem is equivalent to solving the corresponding fixed point problem namely find that x for which x = cos x or find that p for which p = cos p. What is that point P? If I draw the graph of y = cos x and the graph of y = x, the point where the line Y = x crosses the graph of Y = cos x is the required point at which p = g of p. So if I draw the graph of Y = cos x so at x = 0 it is 1 and at x = pie by 2 it is 0, so this is the straight line y = x and this is the point at which they cross each other.

This point p is a root of the equation cos x – x = 0, so let us take an initial approximation p 0 to be pie by 4 bitches 0.7853981635 start with this initial approximation. So I would like to solve it by fixed point iteration method namely, the method is going to be x = g of x, where g of x is x – f of x by f dash of x.

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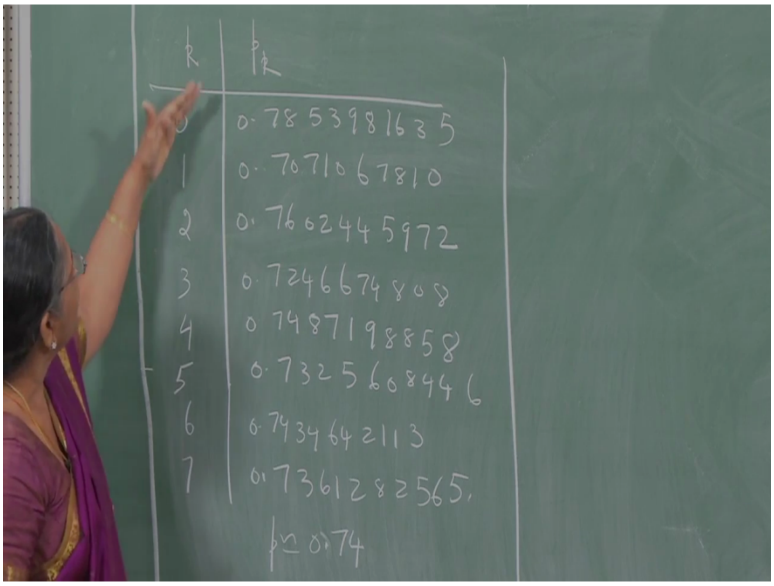

We write down the successive iterates generated using p n = g of p n – 1 and we observe that if we require the result correct to 2 decimal accuracy then we observe that we are able to achieve the desired accuracy at this step so that at the 6^{th} step we have correct to 2 decimal places, the answer is 0.74. In the 7 step correct to 2 decimal accuracy the answer is again 0.74, so the fixed point to which the sequence of iterates converge to is given by p = 0.74 correct to 2 decimal places and it is an approximation to the solution of the equation f of x = 0 namely cos x – x = 0 in the interval 0 to Pie by 2. So we observe that Newton Raphson Method belongs to the class of fixed point iteration methods.

So in the beginning we said that we are going to develop numerical methods which belongs to the following 2 categories namely, Enclosure methods and fixed point iteration methods. And we have developed some methods which belong to enclosure methods namely the bisection method and the secant method and we have now a method which is Newton Raphson Method which belongs to the class of fixed point iteration method. So given an equation of the form f of x = 0 and we are interested in solving this equation numerically, then we can make use of any one of these methods which belong to either enclosure category or fixed point iteration category and solve this problem correct to the desired degree of accuracy.

And we have discussed what is going to the order of convergence of each of these methods, we have discussed that analysis and so the complete details about the error that occurs at each step of computation is known to us by the analysis that we have performed for each of the methods that we have discussed, so in the next class we shall consider some problems based on these methods and see how we can solve these problems, we will continue in the next class.