Numerical Analysis
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Lecture No 35
Root finding Method 5
Fixed Point Methods 1

In this class we shall see how we can relate the solution of fixed point problem to that of solution of an equation of the form f of x = 0, so let us consider the following equation is given to us namely f of x = x cube + 4 x square – 10 = 0 in the interval say 1 to 2.

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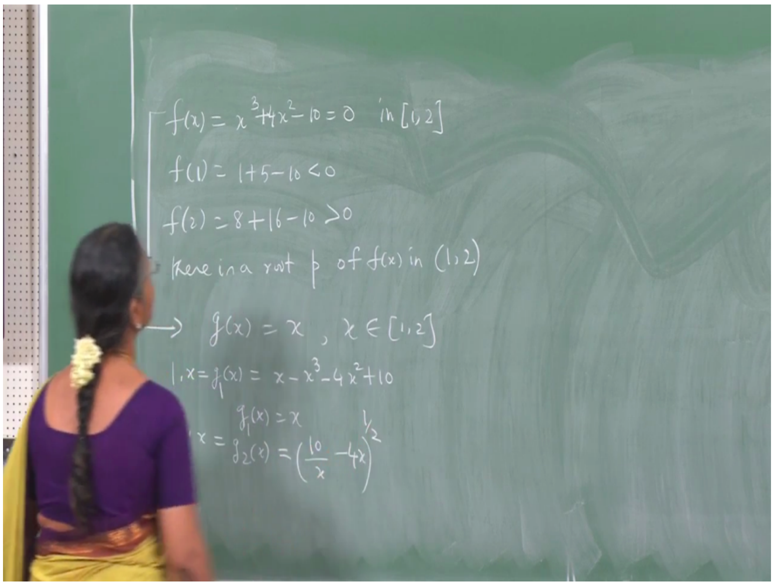

I observe that f at 1 is negative and f at 2 positive, so there is a root p of f of x in this interval 1 to 2. Of course we know some methods by means of which we can obtain this root numerically namely Bisection method, Newton Raphson Method, Secant method and Regula Falsi method are the methods that the already have learned of which the Bisection method and the Secant method belong to the class of enclosure method. Let us now relate the root finding problem to that of fixed point iteration problem, so if I have to connect this root finding problem to that of fixed point iteration problem then I must write this f of x in the form g of x = x for x in the interval 1 to 2, if I am able to do that then I have a fixed point problem.

And I should be able to determine a fixed point p for this function and if that is p then I should be able to relate that we beat a root of the equation f of x = 0 can this be done is the question and we try to answer this question. So let us see in what way we can express f of x given by this as g of x = x. Say I can do it as follows, namely I can take g of x to be = x – x cube – 4 x square + 10 alright, g of x = x while the rest of the terms is f of x and that is 0, so this is one way in which I can write down g of x = x so this gives me g of x as x because f of x is 0. Then I can also write it in the form maybe I shall call this as g 1 of x, the 1st form in which I can write f of x = 0 as g 1 of x = x. Then secondly I consider the form x = say g 2 of x and that is 10 by x – 4x raise to the power of half.

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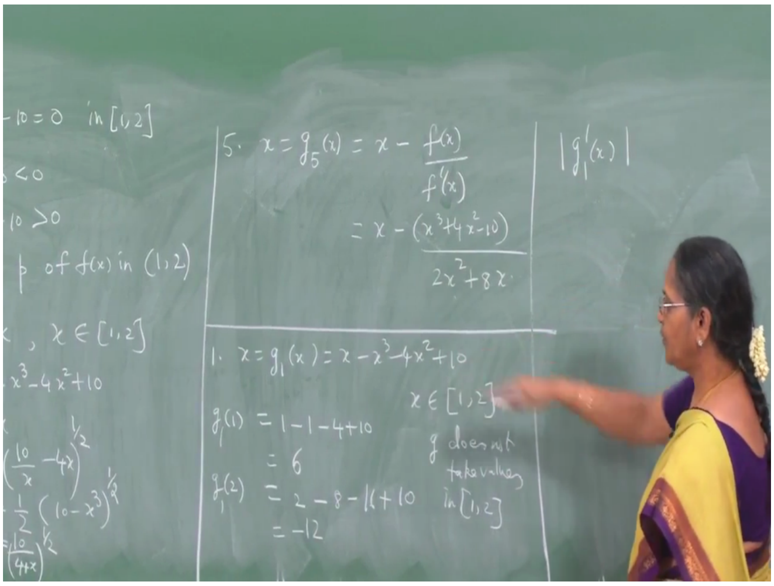

I can also write this equation in another form namely, x = g 5 of x which is x – f of x by f dash of x, why? f of x is 0 so this term will be 0 so x = x, so I take g5 of x to be x – f of x by f dash of x so that is another form so of course one can write f of x = 0 in other different forms also. We have just presented 5 such forms when we look at the equation f of x = 0, each of these forms are written as g of x = x for x in the interval 1 to 2. The question now is, some among these forms which one should be choose in such a way that we generate a sequence of iterates to that vertical form and this sequence converges reliably and rapidly to a fixed point of the equation g of x = x and that this gives us the root of the equation f of x = 0, so we try to answer this question.

So let us now consider the 1st form namely x = g 2 of x given by x – x cube – 4 x square + 10 for x in the interval 1 to 2. So we now check whether this g satisfies the sufficient conditions for existence and uniqueness of a fixed point for g what is it, g must be a continuous function, yes it is a continuous function being a polynomial. Second condition says that g takes values in the interval 1 to 2 so let us just find out what is g of 1, so it is 1 – 1 – 4 + 10 that is 6, g at 2 is going to be 2 – 8 – 16 + 10, it is going to be – 12 and therefore we see that g of 1 is 6 and g of 2 is – 12 and therefore g does not take values in this interval 1 to 2 in the interval 1 to 2. And in addition if you see what is g dash of x I have called this as g 1 so I shall continue to denote it by g 1.

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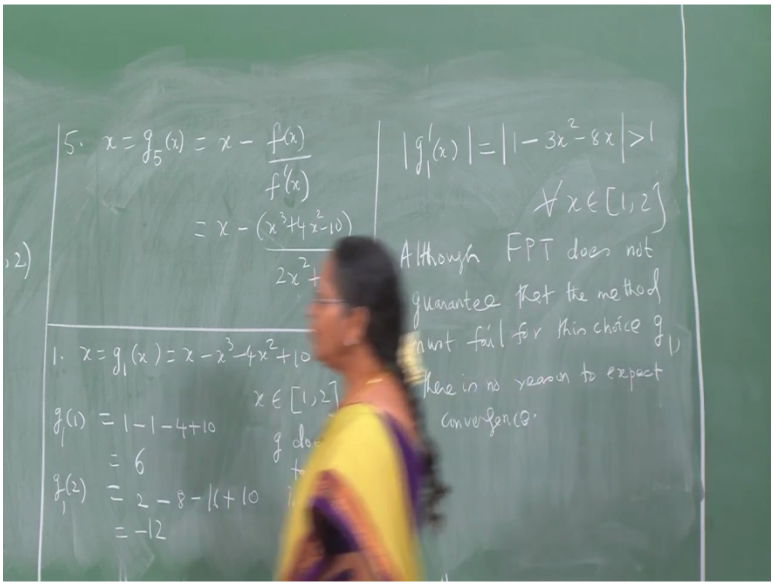

What is modulus of g 2 dash of x? It is modulus of 1 – 3 x square – 8 x and for x in this interval 1 to 2 this is going to be greater than 1 and so we observe that modulus of g dash of x is greater than 1 where again the condition of the theorem is not satisfied. But we know that those conditions are sufficient conditions therefore, there is no reason to expect convergence although the fixed point theorem does not guarantee that the method must fail for this choice of g 1, so we rule out this possibility and take up the 2nd form and discuss, so let us now consider the 2nd form that we had taken.

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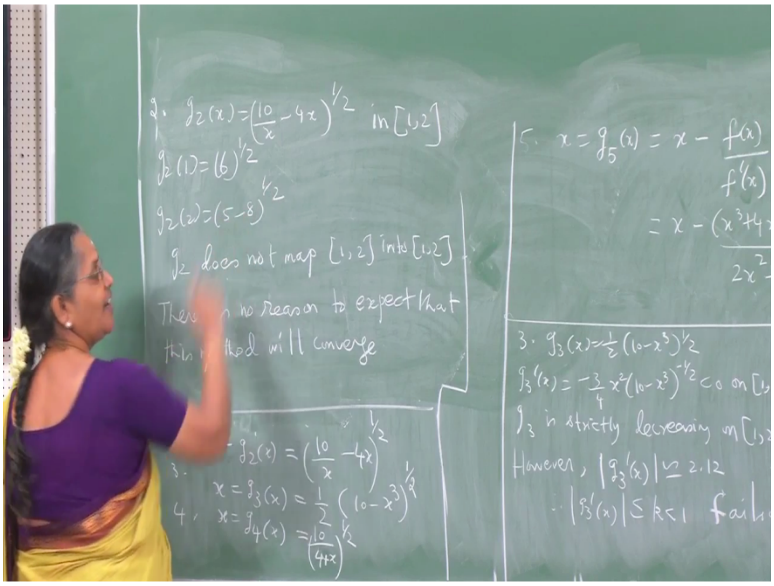

So when x = g 2 of x given by this in the interval 1 to 2, g 2 of 1 is root 6 and g 2 of 2 is 5 – 8 to the power of half, so not even a real number so g 2 does not map 1 2 into itself, so there again there is no reason to expect that this method will converge, let us move onto the next form namely, x = g 3 of x which is half of 10 – x cube to the power of half.

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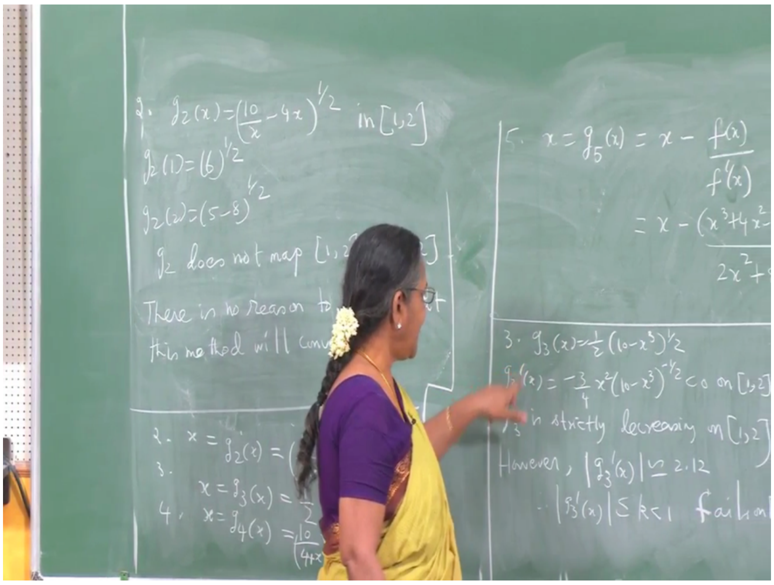

So let us find out g 3 dash of x, it is – 3 by 4 x square into 10 – x cube power – half, so g 3 dash of x is negative for all x in the interval 1 to 2, so g 3 is strictly decreasing in that interval 1 to 2. But modulus of g 3 dash of x is approximately 2.12 and that is less than or = K less than 1 therefore fails on the interval 1.2, so the condition that modulus of g dash of x is less than or = K strictly less than 1 is not satisfied by this g 3 where, in the interval 1 to 2. But let us now see if I start a p 0 in the interval 1 to 2 what happens to the successive iterates generated using p n = g of p n – 1, I am just trying.

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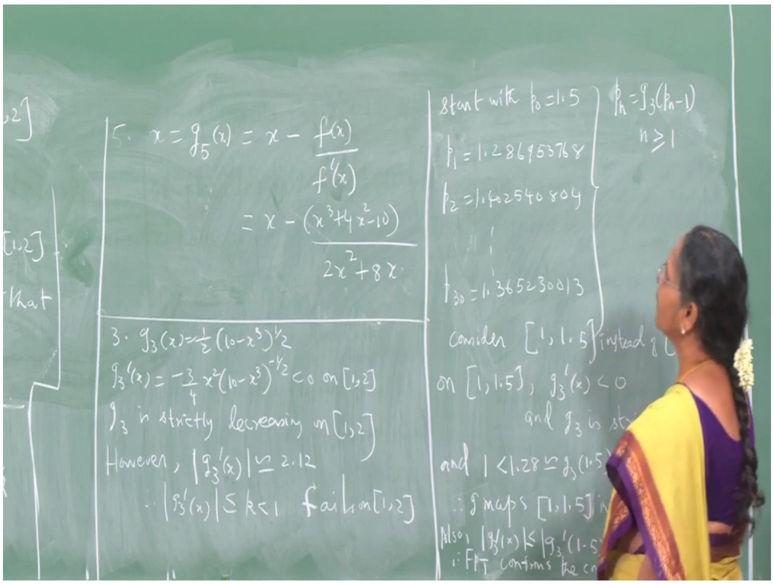

I start with an initial point p 0 which is the midpoint of the interval say 1.5 and use the method which is given by p n = g 3 of 3 n – 1 for n greater than or = 1. So with p 0 = 1.5 I find what p 1 which is g 3 of p 0, I get p 1 then with p 1 of this p 2 will be g 3 of p 1 so I find p 2 and generate the successive iterates say I go onto up to 30th iteration and I observe that correct to the desired degree of accuracy I am able to get the value of p at 30th iteration. So I had started with 1.5 and I get a sequence of iterates and at the 30th iteration I have the value to be 1.365230013, so this is this that instead of considering the interval 1 to 2 with this form I shall consider the interval 1 to 1.5 because p 30 is less than 1.5. So if I consider the interval 1 to 1.5 instead of the interval 1 to 2, then I see some properties of the function g 3.

On this interval 1 to 1.5, g 3 dash of x is negative that is clear from here, so g 3 is strictly decreasing so I have some information about g 3 that it is strictly decreasing, where on the interval 1 to 1.5. So if I consider g 3 at 1.5 than that is approximately 1.28 and that is greater than 1 and g 3 at 1.5 is less than or = g 3 of x that is less than or = g 3 at 1, why? Because g 3 strictly decreasing.

And what is g 3 at 1, it is 1.5 so I observe that g 3 takes values in the interval 1 to 1.5 and therefore, g maps the interval 1 to 1.5 into itself. In addition, modulus of g 3 dash of x that is less than or = modulus of g 3 dash at 1.5 and that is approximately 0.66 which is a than 1 in the interval 1 to 1.5 so the conditions of fixed point theorem are satisfied by g 3 in the interval 1 to 1.5 and therefore, fixed point theorem confirms the convergence of the method p n = g 3 of p n – 1 for n greater than or = 1 in the interval 1 to 1.5 whereas g 3 is given by half of them – x cube to the power of half.

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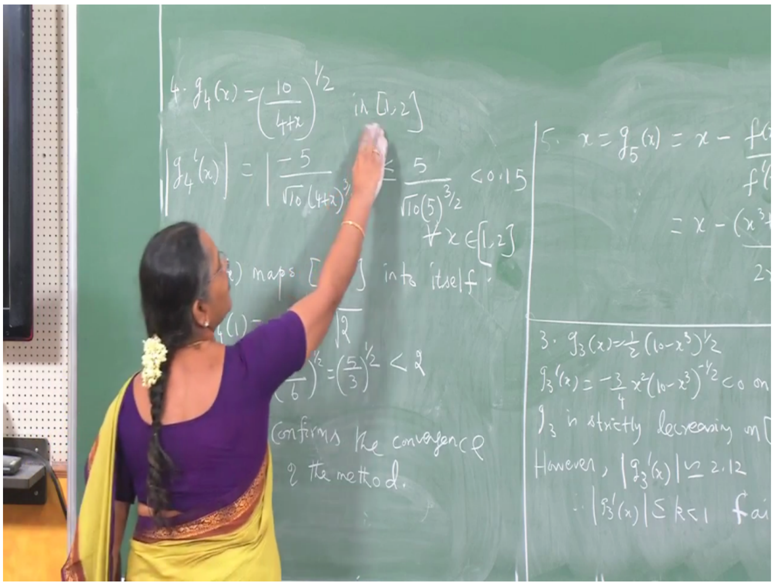

So now let us consider the 4th form which is given by x = give 4 of x which is 10 by 4 + x to the power of half and see what happens in that case, so let us consider the 4th form that we have taken namely x = g 4 of x given by this in the interval 1 to 2. We observe that g 4 of 1 is square root of 2, which is greater than 1 and g 4 at 2 is less than 2, so g takes values in the interval 1 to 2. In addition we observe that modulus of g 4 dash of x is absolute value of – 5 by root 10 into 4 + x to the power of 3 by 2 and that is less than or = 5 by root 10 into 5 power 3 by 2 for x in the interval 1 to 2 and this is less than 0.15 so there exists a k which is strictly less than 1 such that modulus of g 4 dash of x is less than 1, so the conditions of fixed point theorem are satisfied by g 4 and therefore the fixed point theorem confirms the convergence of the sequence of iterates if you take this form x = g 4 of x.

We also observe that modulus of g 4 dash of x is less than 0.15 for this form and if you take x = g 3 of x then modulus of g 3 dash of x is 0.66 that is also less than 1 but here this is much smaller than what appears here and therefore if you take x = g 4 of x and then use the method p n = g of p n – 1 then convergence will be much faster with x = g 4 of x as compared to convergence with x = g 3 of x. And finally we will show that when we take this first form given by x = g 5 of x then the convergence is very rapid and we will show why this happens later on. So we shall consider an example and apply this fixed point iteration method and understand how the fixed point iteration method can be related to the root finding problem in the next class