Numerical Analysis
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Lecture No 34
Root finding Method 5 Fixed Point Methods 1

Good Morning, in the previous classes we had discussed about methods such as Bisection method, Method of false position which belong to the class of enclosure method which helps us determine a solution of an equation of the form f of x = 0. Now we would like to develop numerical methods which belong to the second–class namely fixed point iteration method. We will define what is the fixed point for a function and relate the solution of this fixed point for a function to that of solution of an equation of the form F of x = 0 and develop numerical method which belong therefore to the class of fixed point iteration methods, so let us try to understand what we mean by fixed point for a given function.

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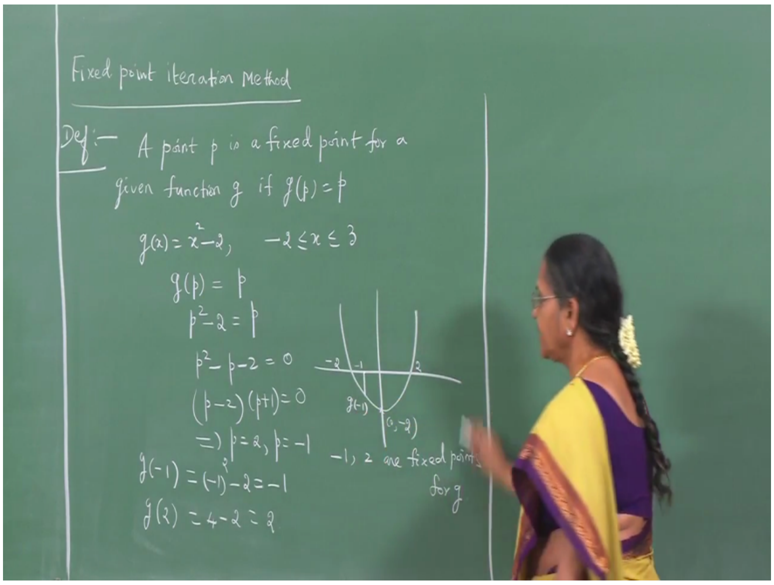

So a point p is a fixed point for a given function g if g of p = p, if this condition is satisfied by some p then we say that it is a fixed point or this given function G. Let us consider an example say g of x is x square – 2 in the interval – 2 less than or = x less than or = 3, so I am interested in determining the fixed point of this function. By definition I should be able to find out p such that g of p must be = p or in other words, this p must satisfy the equation p square – 2 = p, so p square – p – 2 must be 0 and so this is p – 2 into p + 1 = 0, which gives p = 2 and p = – 1 are points which have the property that g of p = P. Let us just check what happens at p = – 1, g at – 1 is – 1 the whole square – 2 so it is – 1 and what about g at 2? That is going to be 2 square – 2 which is again 2.

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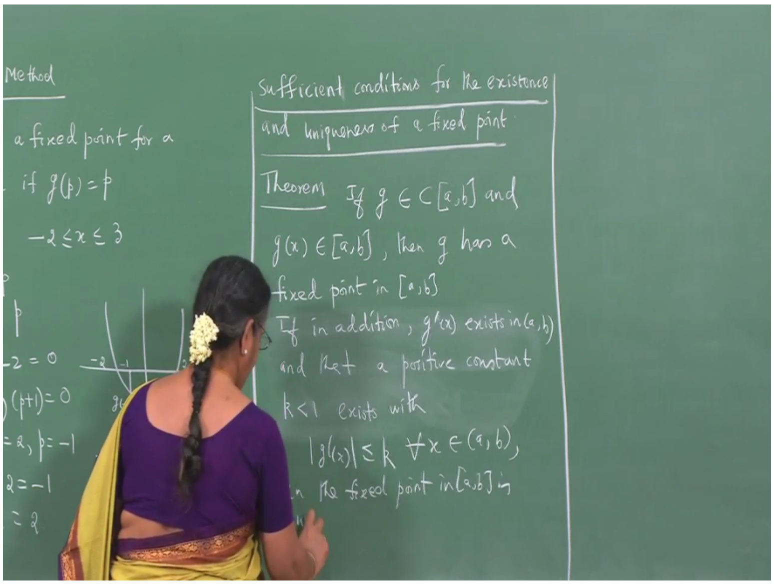

The question now arises, given a function g, does it always have a fixed point or are there conditions which should be satisfied by g so that the fixed point for that function in an interval are guaranteed. So let us now consider the following result which gives us the sufficient condition for so we now give the sufficient condition for the existence and uniqueness of a fixed point and it is given by the following result. It says if g is such that g belongs to C of a b, namely it is a continuous function on a closed interval a b and g takes values in that interval, so g of x belongs to a b. Then g has a fixed point in that interval a b so the conditions that g must satisfy are that g is a continuous function on the closed interval a b and g takes values on the interval a b.

And if these conditions are satisfied, then g has a fixed point in that interval a b, so if in addition g dash of x exists in the open interval a b and that a positive constant k less than 1 exists with modulus of g dash of x less than or = k for every x in this open interval a b, then the fixed point in that interval a b is unique. So consider the function g of x given by x square – 1 by 3 and the interval – 1 to 1, let us try to determine fixed point of this function which belongs to the interval – 1 to 1.

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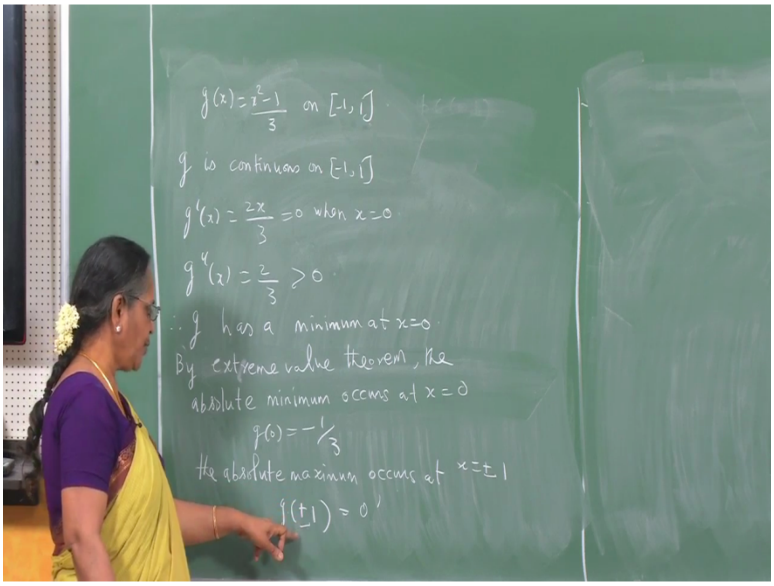

We observe that g is continuous on this closed interval – 1 to 1, let us find g dash of x and that is 2 x by 3 and it is 0 when x = 0 and what about g double dash of x, that is 2 by 3 and it is positive. Therefore, g has a minimum at x = 0 and extreme value theorem implies that the absolute minimum across at x = 0 for the function g and the value is g of 0 which is – 1 by 3. Also the absolute maximum occurs at x = + or – 1 and the value at those points + or – 1 are given by 0. So g has minimum value – 1 by 3 and its maximum is 0, so g takes values in the interval – 1 to 1, g is the continues function on the interval – 1 to 1 and therefore by the 1st part of the theorem that we have shown, g has a fixed point in the interval – 1 to 1.

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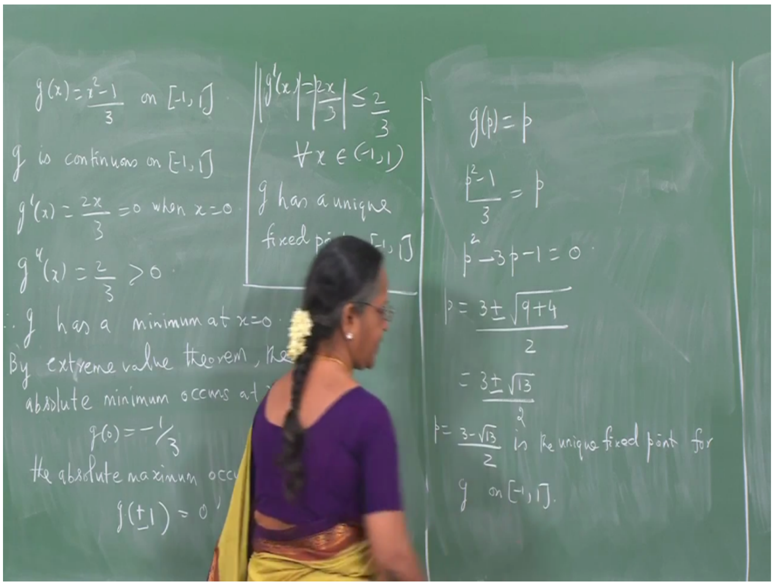

Now let us also see what happens to g dash of x, g dash of x is 2 x by 3 and the absolute value of g dash of x is less than or = 2 by 3 for every x in the interval – 1 to 1. So there exists a k such that modulus of g dash of x less than or = k, where k is 2 by 3 and this is strictly less than 1. So the 2nd part conditions are also satisfied by g and therefore, g has a unique fixed point in the interval – 1 to 1, so let us find out this unique fixed point so how do you compute that? The fixed point of this function is that p for which g of p is p, so we must have p square – 1 by 3 must be p so p where – 3 p – 1 is 0, so p is 3 + or – root of 9 + 4 by 2 so 3 + or – root 13 by 2. So p is 3 + root 13 by 2 so its value will be bigger than 3 and the other one is 3 – root 13 by 2 whose value will be less than 1. So p = 3 – root 13 by 2 is the unique fixed point for this function g in the interval – 1 to 1.

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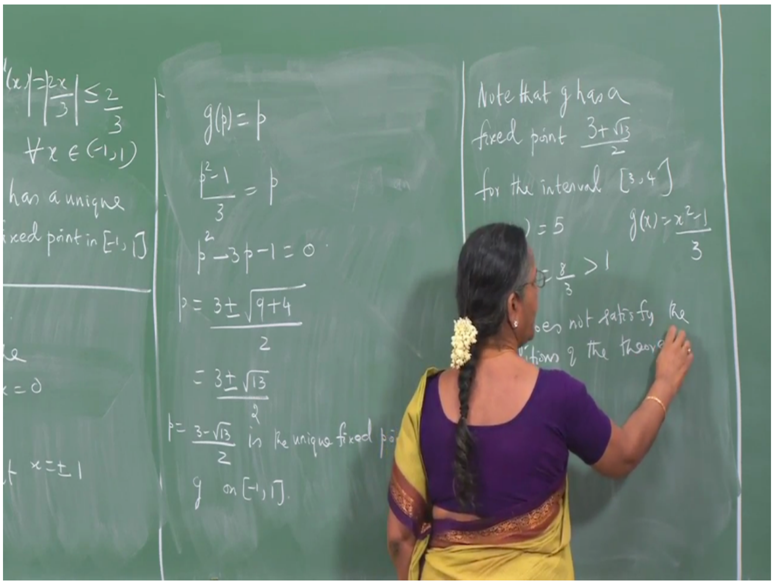

We note that g also has another fixed point 3 + root 13 by 2 and this fixed point belongs to the interval 3 to 4 and we observe that g of 4 is 5 because g of x is x square – 1 by 3 and g dash of 4 will be 8 by 3 and that is greater than 1. So we observe that g does not satisfy the condition of the theorem because g does not take values within this interval and in addition g dash of 4 is greater than 1 and therefore, g does not satisfy the conditions of the theorem that we have proved just now.

Now we know what is the fixed point for a function and what are the sufficient conditions for the existence and uniqueness of the fixed point for a function, so now we would like to know how to find this fixed point for a given function numerically, the following result gives us a way by means of which we can generate a sequence of iterates which converge to the fixed point for a function g provided g satisfies certain conditions, so let us write down the statement of the theorem and see what the result says.

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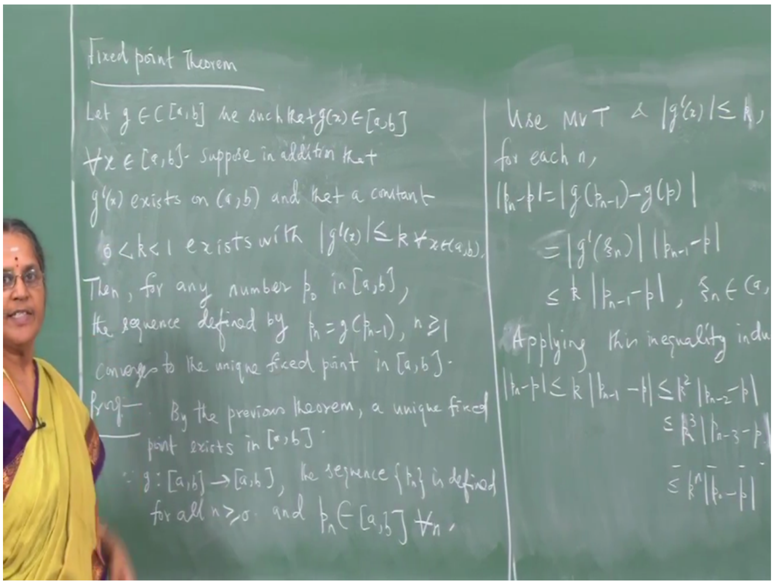

We look at the result which is given by fixed point theorem, this states that if g is a continuous function in the closed interval a b and g takes values in that interval for every x in that interval and in addition g dash of x exists in the open interval a b and a positive constant k which exists with k less than 1 such that modulus of g dash of x is less than or = k for every x in the open interval a b, these are the same as the conditions that we have given in the sufficient condition for existence and uniqueness of a fixed point for a function. So if those conditions are satisfied then the results says for any number p 0 in this interval a b, you consider the sequence of iterates generated by p n = g of p n – 1 for n greater than or = 1, what does that mean?

You start with any p 0 in that interval so that when n is 1 you know p 0, compute g of p 0 compute g of p 0 and called that as p 1, when you know p 1, find g of p 1 that will give you p 2 and so on you generate the sequence of iterates starting with p 0 in a b and using p n = g of p n – 1, this sequence of iterates p n converges to the unique fixed point in the interval a b. So if the conditions mentioned here are satisfied then the sequence of iterates generated by p n = g of p n – 1 starting from any p 0 which lies in the interval a b will converge to the unique fixed point in the interval for the function G, so this gives us a way to find a fixed point of the given function g numerically correct to a desired degree of accuracy, so let us see the proof of this result.

So by the conditions specified in the earlier theorem which are sufficient conditions for existence of uniqueness which are the same as given here, a unique fixed point for g exists in the interval a b. Now g takes values in a b and g is a function from a b to a b, so now I consider the sequence p n which is generated using p n = g of p n – 1, so g takes values in that interval a b and therefore, all the p n which are members of this sequence they are all defined for every n for n greater than or = 0. And p n belongs to a b because p n is g of p n – 1 and g takes values in the interval a b, so p n belongs to a b and this is so for each n.

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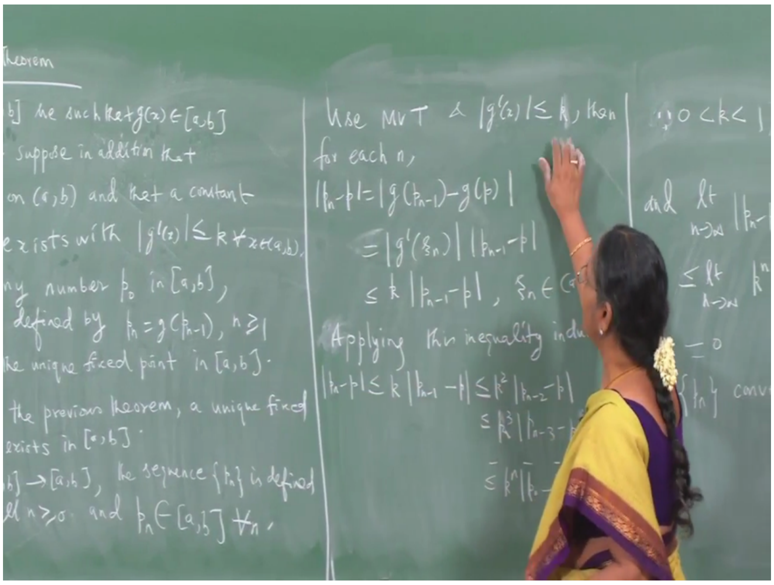

And now use the fact that modulus of g dash of x is less than or = k so that for each value of n let us find out the difference between p n and p in absolute value, so p n is g of p n – 1, p is g of p so mod p n – p is mod g of p n – 1 – g of p. So by mean value theorem there exist a Si n belonging to the open interval a b such that modulus of g of p n – 1 – g of p is modulus of g dash of Si n into mod p n – 1 – p. But this is less than = K, so k times mod p n – 1 – P, so I now have shown that mod p n – p is less than or = k into mod p n – 1 – p so I apply this result in that truly.

So what do I know about p n – 1 – p that will be less than or = k times p n – 2 – p in absolute value so this will be less than or = k square into mod p n – 2 – p so again that will be less than or = k cube into mod p n – 3 – p and continue this will be less than or = k power n into mod p 0 – p. And what is p 0, p 0 is any number in a b with which we have started our iterations, so what have we shown? We have shown modulus of p n – p is less than or = k power n into mod p 0 – p, but what we know about k, k is positive and it is strictly less than 1. So limit as n tending to infinity of k power n will be 0 and so I take limits n tends to infinity on both sides of the result that we have shown.

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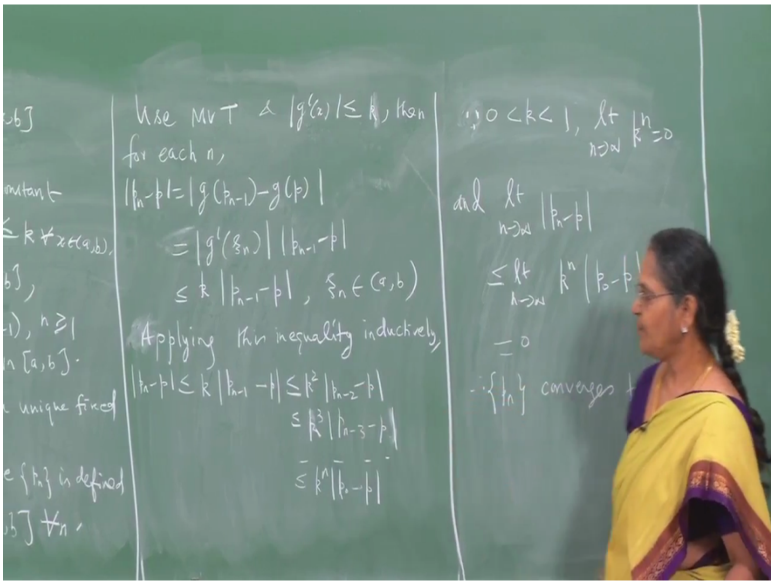

So limit as n tending to infinity of mode p n – p is less than or = limit as n tending to infinity of k power n into mod p 0 – p, but this goes to 0 so limit as n tending to infinity of mode p n – p is 0 what does that mean? It means that the sequence p n converges to p and what is this p; p is a unique fixed point for the function g which lies in the interval a b, how is it guaranteed? The conditions which are given here are nothing but the sufficient conditions for existence and uniqueness of a fixed point for the function g.

So p n converges to p and this p is a unique x point 4 the function g the interval a b so the sequence of iterates generated by p n = g of p n – 1 starting with p 0 in the interval a b converges to a unique fixed point for the function g in the interval a b and this theorem is called fixed point theorem and it gives us a way of finding a fixed point, for a given function which satisfies certain conditions by generating a sequence of iterates which converge to the unique fixed point for this function. This result also has the following corollary, so let us look into the result of the corollary.

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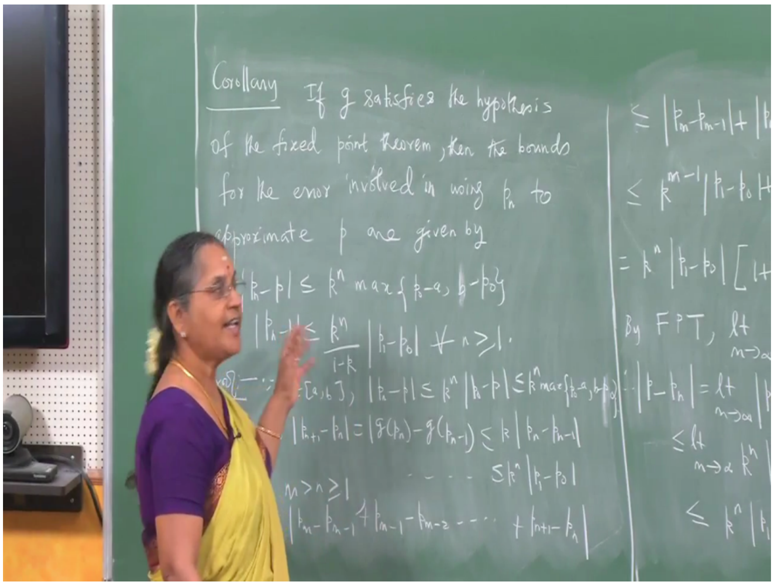

The corollary says if g satisfies the conditions given in the fixed point theorem which we have proved just now, then the bounce for the error involved in using p n to approximate the fixed point p are given as follows; namely the absolute error is less than or = k power n into maximum of p 0 – a, b – p 0 and the absolute error is less than or = k power n by 1 – k into modulus of p 1 – p 0 and this is so for every n greater than or = 1. So the corollary gives us abound on the absolute error when we approximate by p n the unique fixed point p for the function G, so let us try to understand the proof of this result. We know that the fixed point p belongs to the interval a b so p n – p in absolute value is less than or = k power n into mod p 0 – p, this we have shown in fixed point theorem so I am making use of that result.

So mod p n – p is less than or = k power n into mod p 0 – p so that will be less than or = k power n into this will be maximum of p 0 – a, b – p 0 so that gives us the 1st result, now let us consider the proof of the 2nd result. So let us start with mod p n + 1 – p n for n greater than or = 1 but p n plus 1 is g of p n and p n is g of p n – 1 so that will be less than or = be have already seen all this, I can use at this stage mean value theorem so the absolute value of this will be less than or = g dash of Si n into mod p n – p n – 1, modulus of g dash of Si n is less than or = k and therefore, this will be less than or = k into mod p n – p n – 1, I apply this result in that truly.

So that will be less than or = k square into p n – 1 – p n – 2 and so on so it will be less than or = k power n into mod p 1 – p 0. So what we have shown? We have shown for n greater than or = 1, mod p n + 1 – p n is less than or = k power n times mod p 1 – p 0. Now I consider m greater than n greater than or = 1 and consider that absolute value of the difference between p m and p n, so p m – p n is p m. Now I subtract p m – 1 add p m – 1, I subtract p m – 2 add p m – 2 and continue, I subtract p n + 1 and add p n + 1 and then write down which is – p n. So whatever I subtract, I add and therefore I do not any change in the left–hand side, but I only rewrite this in this form and now I apply triangle inequality.

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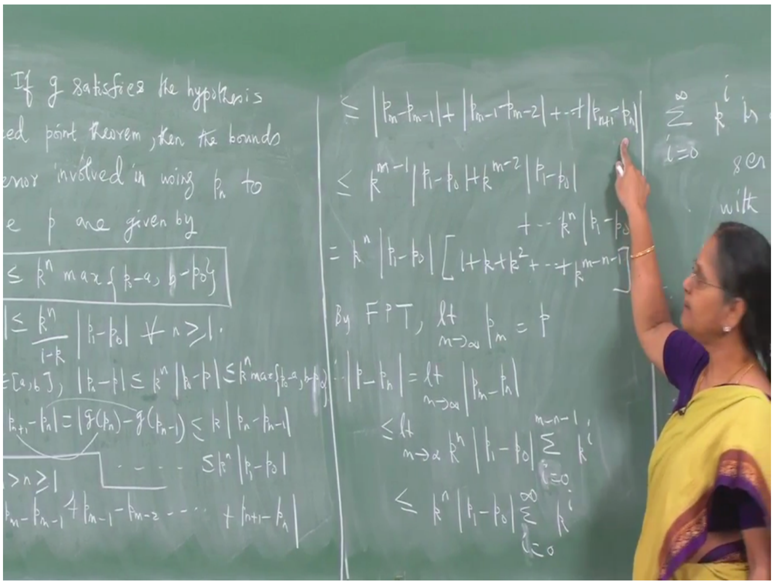

So modulus of a + b is less than or = mod a + mod b and extend this inequality and use that result here. So this will be less than or = mod p m – p m – 1 + mod p m – 1 – p m – 2 etc + mod p m – 1 – p m. But we have just now shown mod p m + 1 – p n namely the absolute value of the difference between successive iterates is less than or = k power n times mod p 1 – p 0. So we see that this is the absolute value of the difference between the mth and the m – 1 iterate, so it is less than or = k power m – 1 into mod p 1 – p 0. This is the absolute value of the difference between the successive iterates at m – 1 and m – 2 steps, so this is less than or = keep our m – 2 into mod p 1 – p 0. And continue in this argument we have mod p n + 1 – p n will be less than or = k power n into mod p 1 – p 0.

So we have k power n into mod p 1 – p 0, if I remove it as a common factor then I have one from this, the previous term will give me k, the last 2 terms will give me k square and so on and this term will give me k power m – n – 1. You just see k power n times k power m – n – 1 will be k power m – 1 times mod p 1 – p 0, which appears here. Now by fixed point theorem what do I know? The sequence of iterates p m generated using p m = g of p m – 1 converge to p, so limit m tending to infinity of p m is p therefore, what is mod p – p m? So that is limit as m tending to infinity of p m because the limit is p – p n. So this will be less than or = limit m tending to infinity of right, we require what mod p m – p n is and that is what we have computed.

What is it? Mod p m – p n has been shown to be less than or = k power n into mod p 1 – p 0 into this that is what I have written, k power n into mod p 1 – p 0 into this can be written as submission i = 0 to m – n – 1 into submission i = 0 m – n – 1 of k power i, when i is 0 you get 1, when i is 1 it is okay and so on. When i is m – n – 1 you get k power m – n – 1. But this series has finite number of terms so the sum of this series is less than sum to infinity of the series Sigma i = 0 to infinity k power i.

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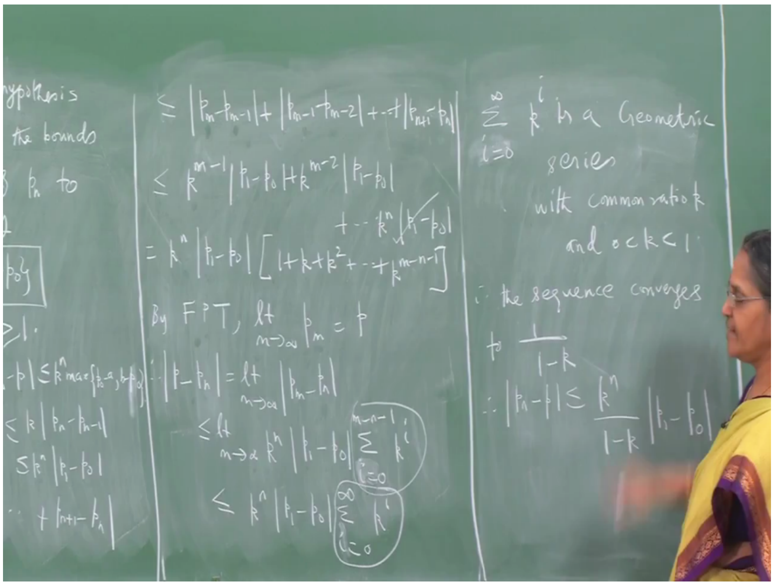

Now what do you know about the power i? k is such that k is less than 1 and k is positive, so Sigma k power i is a geometric series with common ratio k such that 0 less then k less than 1 and therefore I have sum to infinity of such a geometric series. So the geometric series converges to 1 by 1 – k so I substitute that value here and so I end up with mod p n – p less than or = k power n into mod p 1 – p 0 into sum to which this series converges to which is 1 by 1 – k. So I have the result that mod p 1 – p is less than or = k power and by 1 – k into mod p 1 – p 0 and that completes the proof of this part. So this gives you the error bound that is involved in taking p n as an approximation to the unique fixed point for the function g in the interval a b where g satisfied certain conditions namely the conditions listed in the fixed point theorem.

So what is it that we have done all along, we defined what is a fixed point for a given function g and how do we determine this fixed point numerically; namely if g satisfied certain condition in an interval a b, starting with any p 0 in that interval we can generate successive iterates using p n = g of p n – 1 and this sequence of iterates generated by p n = g of p n – 1 will converge to a unique fixed point for the function g provided g satisfies the condition in fixed point theorem. Then we wanted to determine the absolute error in taking p n an approximation to this unique fixed point and we have error bounds given in the results of this theorem.

So we now have completed our discussion about the fixed point problem, we know what is a fixed point, we know what is the sufficient conditions under which there is a unique fixed point for a given function and we know how to compute this numerically and we also have an error bound by taking p n as an approximation to p. So the problem that remains is to connect this fixed point problem to that of root finding problem and we shall do this in the next class.