Numerical Analysis
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Lecture -3 Part - 1
Polynomial Interpolation-2

In the last two lectures we saw some preliminary results some tools from calculus that will help us in understanding the topics in this course and that will also help us in understanding the topics in this course that will also help us in developing some numerical algorithm. We also saw an overview of the topics that we would be studying in this course. We said that we shall begin with the topic on polynomial interpolation.

So we shall see what we mean by polynomial interpolation and how we can reconstruct the function whose values are known at a set of discrete points with the help of polynomial interpolation. Or how we can obtain an approximation of a function whose values are all known in a close interval. Which again can be obtained using interpolation polynomials. So let us mathematically formulate this problem and see how we can develop the interpolation polynomial that approximates a given function.

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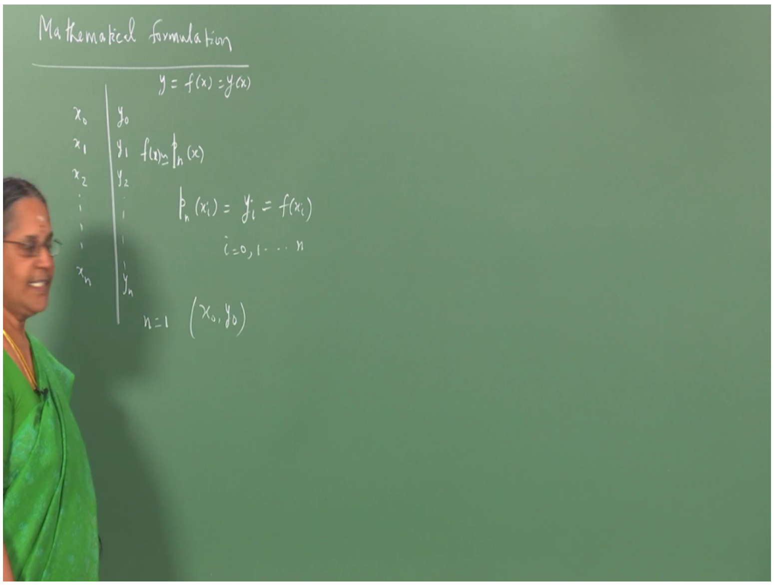

So let us formulate the problem first, so we are given a set of discrete values say at point x 0 x 1 etc x n and the corresponding values are say y 0 y 1 y 2 etc y n of some function y is equal to f(x) say which is y(x). Either the function values are given at these points and we do not know what the function is we want to reconstruct the function f(x) or we know this function f(x) in some closed interval and we would like to approximate this function say by a polynomial in which case we choose some n plus 1 points and list the set of values x i y i so we seek a polynomial say of degree at most n passing through these n plus 1 points such that p n (x) approximates the function f(x) such that p n takes values y i namely f(x i) at the points x i for i equal to 0,1,2,3etc n. So the problem is to seek a polynomial p n (x) of lowest degree possible that interpolates the function f(x) at a set of discrete points such that p n (x i) is y i for i equal to 0 to n. So this is the problem that we would like to address now.

So the question is the stair exists such a polynomial what happens if suppose I am given say information at 1 point x 0 y 0 namely n is equal to say 1. Then in that case the construction of the polynomial is obvious so I can take a polynomial say p 0 (x) to approximate the function such that it is the constant function.

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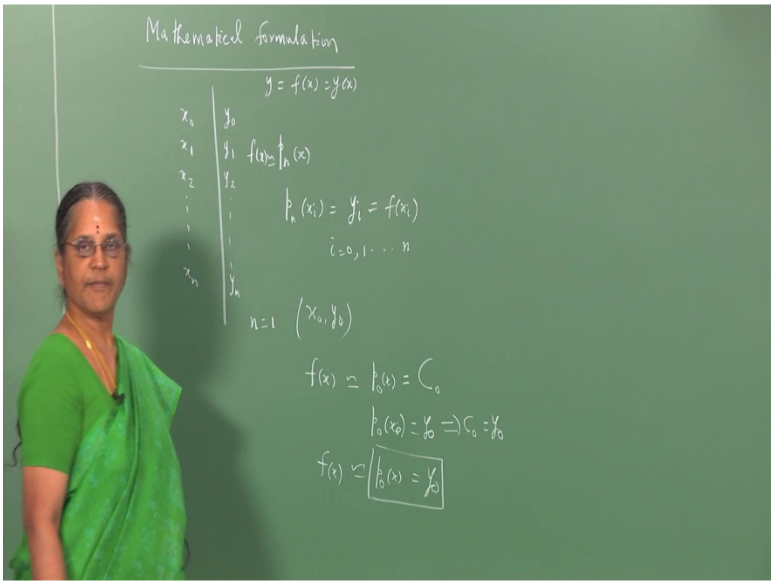

Say suppose I call that constant say c 0 what should be the value of c 0 so that the polynomials satisfies the given condition. So I want P 0 at x i or the x 0 to be equal to y 0. This immediately tells me that my C 0 is y 0 so my constant polynomial that approximates this function is P 0 (x) equal to y 0.

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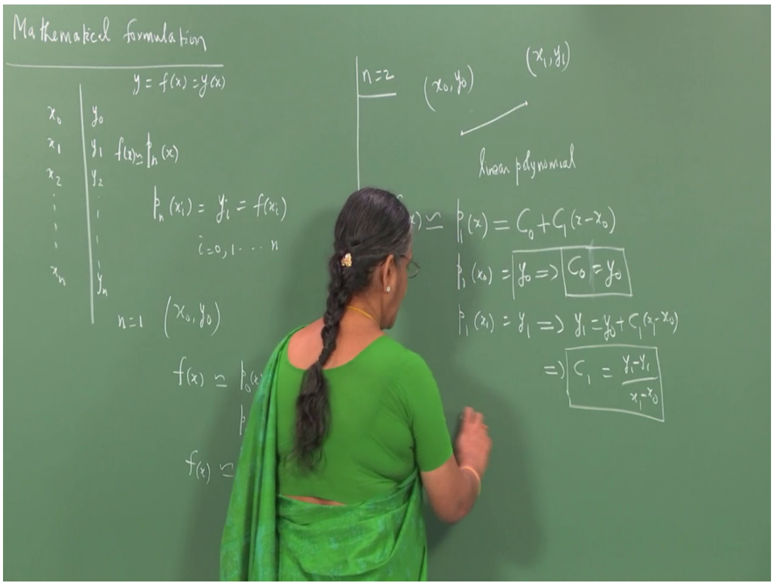

That is fine what happens say if n is equal to 2 let us understand first when n is equal to 2 we are given information at 2 points x 0 x 1 is corresponding values are y 0 and y 1. Say these are the two points then I want a polynomial so given two points I can fit a straight line passing through these two points so that I can look for a linear polynomial of degree 1. So in this case I can construct a function p 1 (x) of degree 1 that approximates the function f(x) with the property so let us write down this polynomial I shall take this polynomial as C 0 plus C 1 into say (x minus x 0) it is a linear polynomial.

What should be the property that should be satisfied by this polynomial. It is such that P 1 at x 0 must be y 0 and p 1 at (x 1) must be y 1. So this gives you see that at x 0 this factor will vanish so C 0 is y 0. And if the condition p 1 (x) is equal to y 1 should be satisfied then it means that y 1 is equal to C 0 that is y 0 plus c 1 into x minus x 0 so that immediately gives us the constant C 1 to be y 1 minus y 0 by x 1 minus x 0. So C 1 gets determined.

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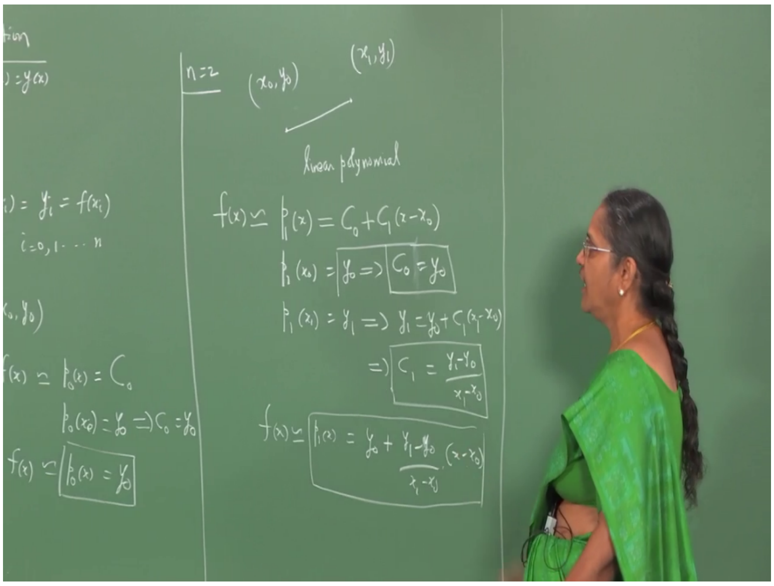

So what is the polynomial that approximates this function f(x) it is c 0 which is y 0 plus C 1 which is y 1 minus y 0 divided by x 1 minus x 0 that is C 1 into (x minus x 0) and that is a first degree polynomial that approximates this function f(x) and it interpolates the function at a set of discrete points namely (x 0 y 0) and x 1 y 1). So we see that once we know that the 0 degree polynomial exists the first degree polynomial can also be constructed and it exists. So one can continue now by taking a n is equal to 3.

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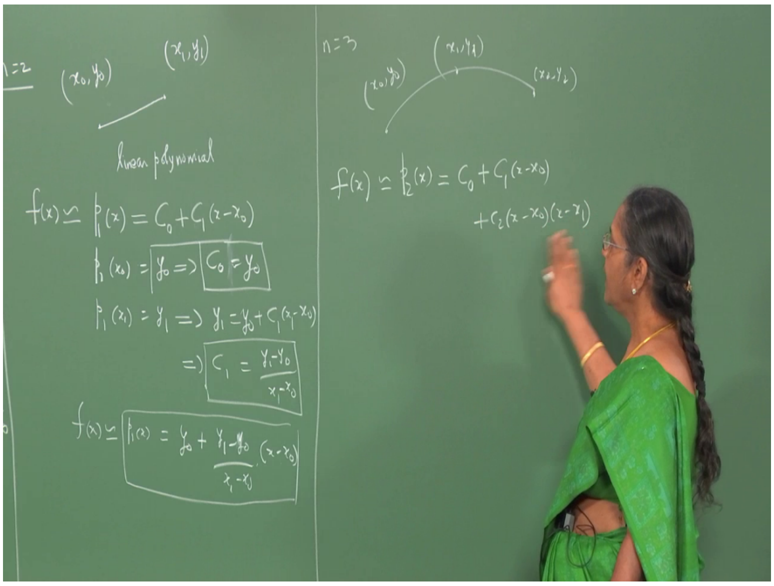

So there are 3points which are given to us. (x 0 y 0) say (x 1 y 1) and (x 2 y 2). So passing through these three points I must have a curve in the xy plane that is going to approximate the function f(x).So that the equation of this curve is going to be a second degree polynomial that is denoted by p 2 (x) so that f(x) is a approximated by p 2 (x) which is a second degree polynomial or a parabola passing through these three points. So what is its equation, I can take the equation of the parabola to be C 0 plus C 1 into (x minus x 0) plus C 2 into (x minus x 0) into (x minus x 1).

And you should see that the C 0 plus C 1into (x minus x 0) is essentially this P 1(x). So what have I done? I have added an extra term right to P 1(x) and construct a polynomial of degree 2 namely P 2 that is what I did here to get P 1 (x) to P 0 (x) which C 0 I added this term and I am able to show that this P 1(x) satisfies my requirement. Now to get P 2 (x) I add tp P 1(x) a term so that I am going to show that P 2 (x) is going to satisfy the properties that we want.

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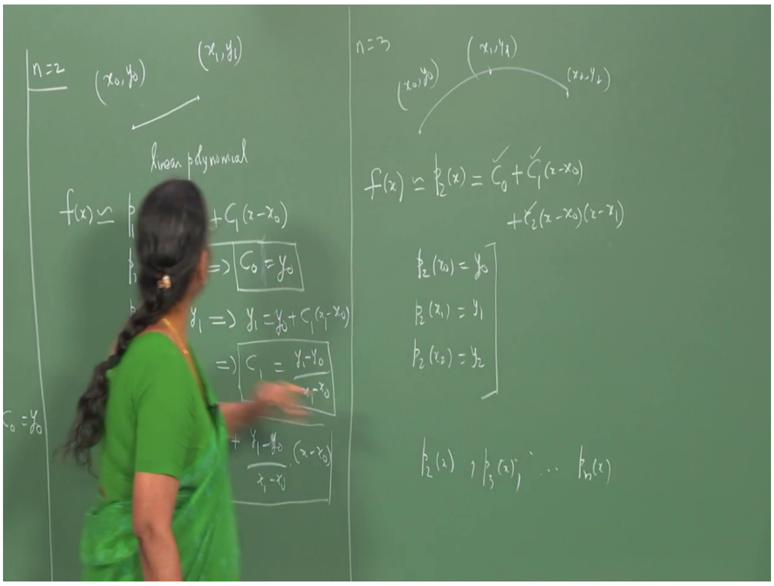

So what do we want we want P 2(x 0) to be equal to y 0, p 2 (x 1) should be y 1 and p 2 (x 2) should be equal to y 2.So there are three conditions to be satisfied and we have three constants C 0C 1 C 2 to be determined and these three conditions will give us a polynomial which is p 2(x) after determining C 0 C 1 C 2 subjects to these conditions which is a second degree polynomial.

So we can continue in this way and construct higher degree polynomials given n is equal to 4 p 3(x) etc so that finally given n plus 1 points namely x i , y i for i is equal to 0 to n we can get a polynomial p n(x) of degree n that interpolates the given set of points. We have shown that it is possible to get polynomials of appropriate degree depending upon the information that is available to us.

So the question that arises is yes by constructive proof we have shown that one can get a polynomial that interpolates a given data. The question now is, is this polynomial a unique polynomial that interpolates the given data. So we would like to now give a result that shows that a polynomial does exist having this property and that this polynomial so constructed is a unique polynomial.

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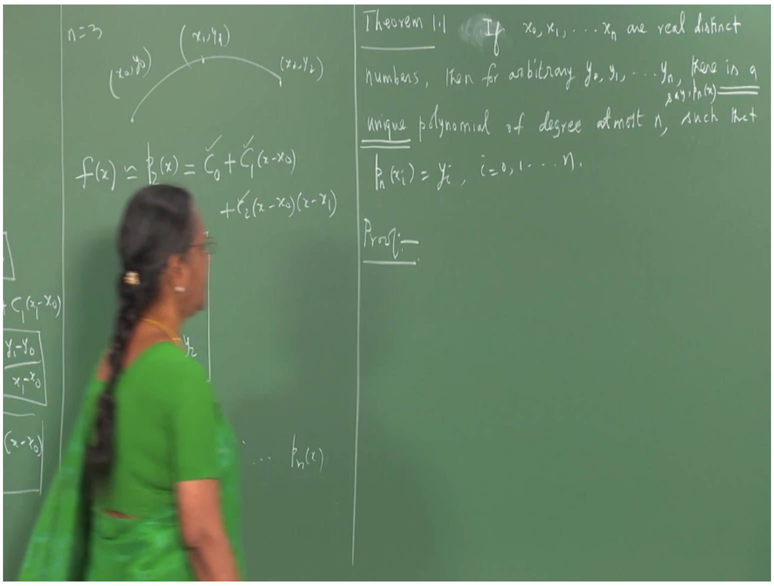

So we present this result in the form of a following theorem, so the result is as follows we call this theorem as 1.1 so it says if x 0 x 1 etc x n are real distinct numbers then for arbitrary y 0 y 1 y 2 etc y n there is a unique polynomial of degree at most n say p n(x) such that p n(x i) is equal to y i for i is equal to 0 1 2 3 up to n.

So given n plus 1 distinct points x 0 etc x n for any arbitrary values which are associated with these n plus 1 values mainly y 0 y 1 etc y n the result says there is so that it says there exists a polynomial and this polynomial is a unique polynomial what is its property it satisfies the condition that it passes through the points x i y i so p n(x i) equal to y i. That is proved the uniqueness part. So we want to show that there is a unique polynomial 1 and only 1 polynomial that has this property that interpolates the set of discrete data points.

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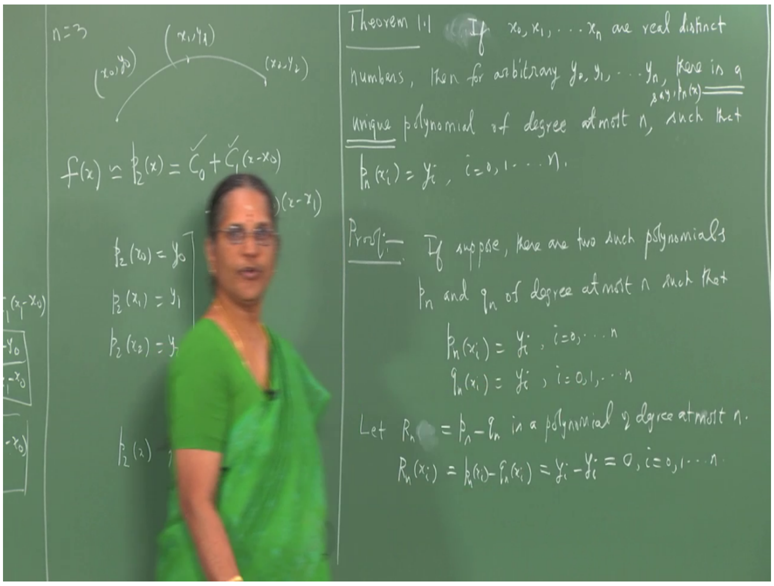

So let us assume on the contrary if suppose there are two such polynomials p n and q n of degree n such that p n satisfies the condition p n(x i) is y i and q n also satisfies the condition p n(x i) equal to y i for i is equal to 0 1 2 3 up to n. So let us call by R n is p n minus q n. So p n and q n are polynomials of degree at most n and so R n which is difference between these two polynomials is a polynomial of degree at most n.

Then what other property does R n satisfy. Let us see if I take evaluate R n at (x i) then R n (x i) is p n (x i) minus q n(x i) but I know p n(x i) is y i and q n (x i) is also y i. So R n (x i) is 0 for i is equal to 0,1,2,3 etc upto n. So we know that R n is polynomial of degree at most n and R n (x i) is 0 for i is equal to 0 to n. Now R n being a polynomial of degree at most n it can have have at most n zeroes. If it is not the 0 polynomial.

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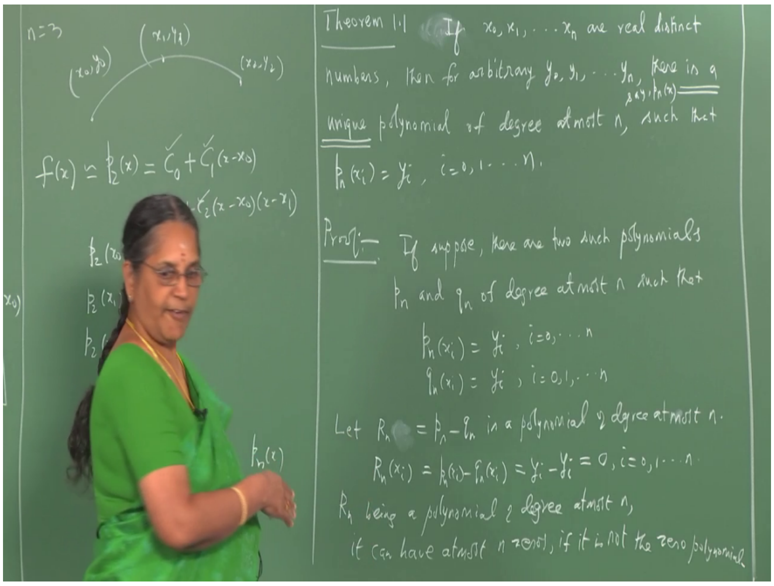

So let us write down the result. R n being a polynomial of degree at most n it can have at most n zeroes if it is not the zero polynomial. But what have we seen about R n what is the property that we know about R n.

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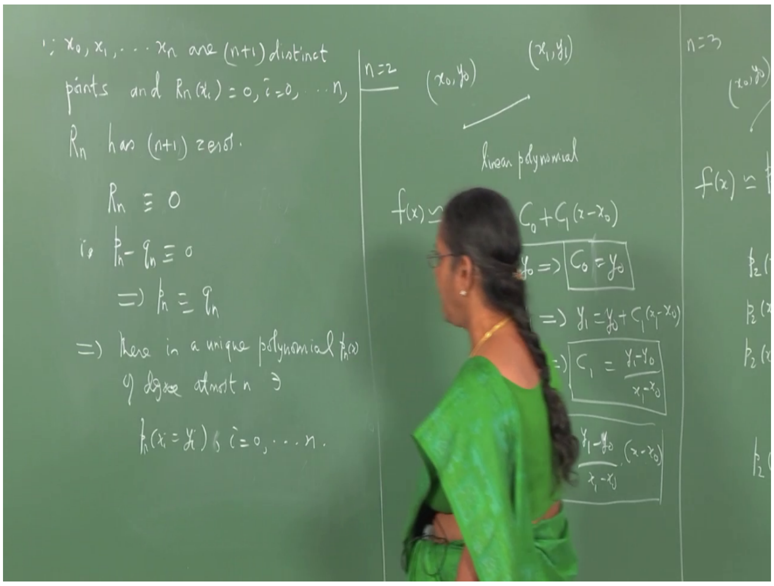

Since x 0 x 1 etc x n are n plus 1 distinct points and R n (x i) is 0 for i is equal to 0 1 2 3 etc n its clear that R n has n plus 1 zeroes what are its zeroes they are x 0 x 1 etc x n. It has n plus 1 zeroes. But what do we know about R n is a polynomial of degree at most n and it can have at most n zeroes if it is not up to 0 polynomial.

Here we show that R n has n plus 1 zeroes so R n has to be the 0 polynomial. So R n must be identically 0 that is P n minus q n must be 0 or this implies that P n must be identically the same as q n. So which shows that there is a unique polynomial of degree at most n.

Say unique polynomial p n(x) of degree at most n such that p n(x i) is equal to y i for i equal to 0 1 2 3 up to n. So you will be able to determine a unique polynomial that interpolates the given set of data points. So the uniqueness part of theorem has been proved. So we now move on to the existence part and the existence is through construction of such polynomials as we have already indicated.

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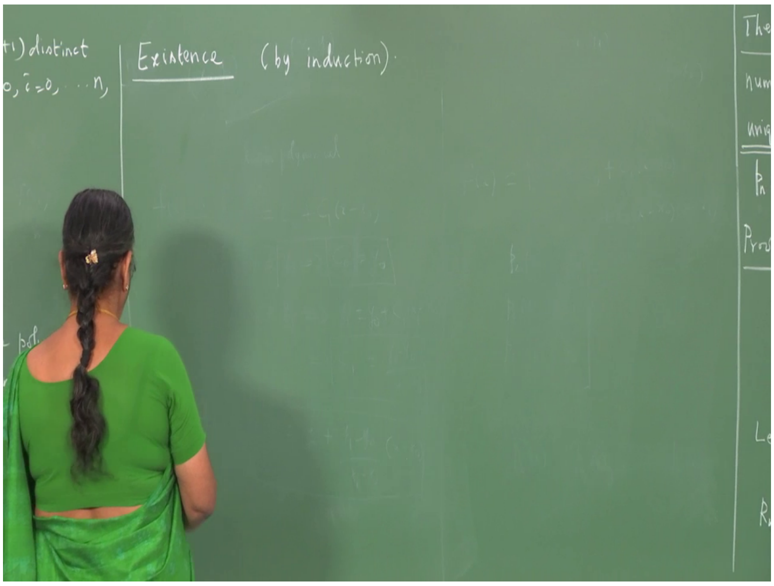

So we shall use induction to show the existence part of this proof, we first show that there exist a polynomial of degree 0 and if we have a polynomial of degree k that satisfies the given property then we show that there exist a polynomial of degree k plus 1 satisfying the property and hence by induction the result is true for all n and the proof is complete and thereby we would have shown the existence of such a polynomial that interpolates the function.

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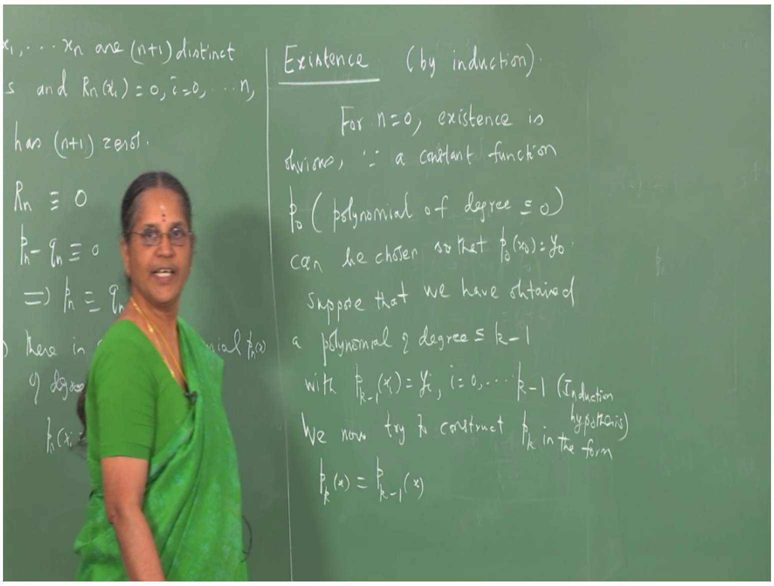

So let us start with n is equal to 0 so for n is equal to 0 the existence is obvious in the sense that since a constant function p 0 namely a polynomial of a degree less than or equal to 0 can be chosen such that p 0 (x 0) is y 0. Suppose that we have obtained a polynomial of degree less than or equal to k minus 1 with p k minus 1 (x i) equal to y i for i is equal to 0 1 2 3 upto k minus 1. So given information at k points you can find a polynomial of degree k minus 1 that interpolates at these points x i y i.

So suppose that we have already obtained a polynomial of degree less than or equal to k minus 1 satisfying this property. So this is our induction hypothesis. So what should we show? We should show that we would be able to obtain a polynomial of degree k less than or equal to k satisfying p k (x i) equal to y i. We now try to construct p k in the following form namely p k(x) is equal to p k minus 1 (x) I recall what we did in the beginning of this lecture. Having constructed p 0 we constructed a polynomial of degree 1 by taking p 1 (x) to be P 0 plus a constant times x minus x 0.

So how did we construct P 2 having constructed p 1 we can now construct p 2 such that p 2 is p 1 plus c 1 into x minus x 0 into x minus x 1. Given the information at three points x 0 y 0 x 1 y 1 x 2 y 2 simply extend their Idea having constructed P k minus 1 (x) degree less than or equal to k minus 1 take the polynomial p k in the form p k(x) is p k minus 1 (x) plus, A some constant arbitrary constant which we will have to determine multiplied by x minus x 0 into x minus x 1 etc x minus x k minus 1).

Let us see whether this construction works how do we ensure that this construction works we must show that p k satisfies the condition that p k at x i for y i for i is equal to 0 1 2 3 up to k.

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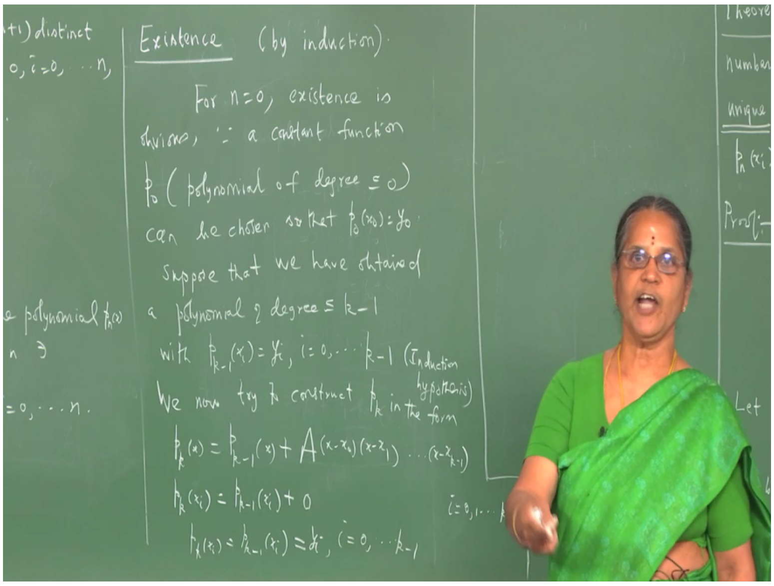

So let us first check what happens at the x i so it is p k minus 1(x i) plus I shall take my i to be any 1 of 0 1 2 3 up to k minus 1. So when x is x i for i between 0 to k minus 1 I observe that it has factors x minus x i for i is equal to 0 k minus 1. And therefore this will give you 0. So p k (x i) is equal to p k minus 1 (x i) but I know p k minus 1(x i) is y i for i is equal to 0 1 2 3upto k minus 1.

So my polynomial p k satisfies the condition that at points x 0 x 1 etc x k minus 1 it interpolates the function by satisfying the condition that p k (x i) is y i but what do we want we want this polynomial to be such that p k (xk) must be y k. So we impose that condition.

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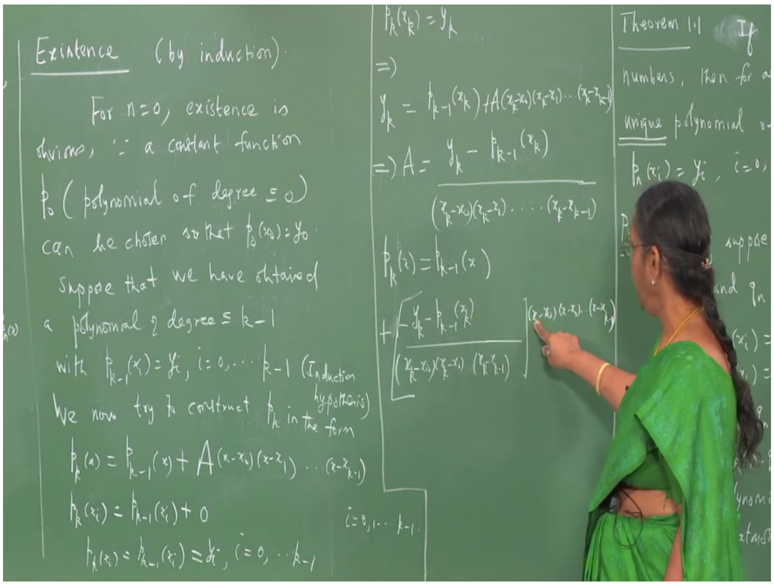

So p k (xk) must be equal to y k which implies that y k is equal to p k minus 1 (xk) plus A times x k minus x 0, x k minus x 1 and so on upt x k minus x k minus 1, which immediately tells you that a the arbitrary constant must be y k minus p k minus 1 at x k divided by x k minus x 0 into x k minus x 1 etc upto x k minus x k minus 1.

So the constant A is determined. And so we can substitute here and write down what the polynomial p k(x) is of degree k so that must be equal to p k minus 1(x) which is already known you have constructed plus A is y k minus p k minus 1 (xk) divided by xk minus x 0 into x k minus x 1 etc upto x k minus x k minus 1.

This multiplied by the factors namely x minus x 0 x minus x 1 etc x minus x k minus 1. And you observe k such factors and so the leading term will be x to the power of k. So your polynomial p k (x) is a polynomial of degree k minus 1 plus a term involving at this constant multiplied by some factors whose leading term is x to the power of k so you have constructed a polynomial of degree k. That interpolates the function at a set of points x 0 x 1 etc x k. So there are k plus 1 points. So p k interpolates the function at a set of k plus 1 points x 0 y 0 x 1 y 1 etc x k y k.

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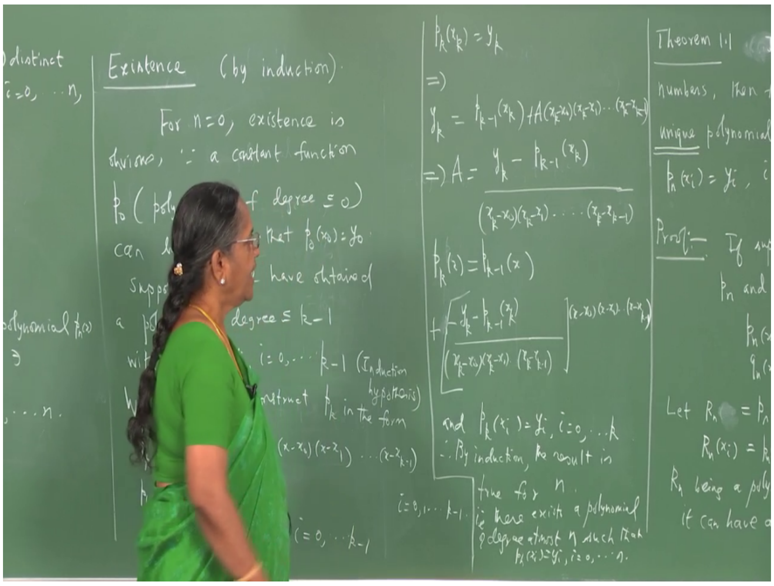

So assuming that p k minus 1 has been obtained we have been able to construct a polynomial of degree k. So the result is true for all n by induction. So p k (x) is this and we have shown p k (x i) is y i for i is equal to 0 1 2 3 up to k. So therefore by induction the result is true for n what does that mean? that means there exists a polynomial of degree at most n such that p n(x i) is equal to y i for i is equal to 0 1 2 3 up to n.

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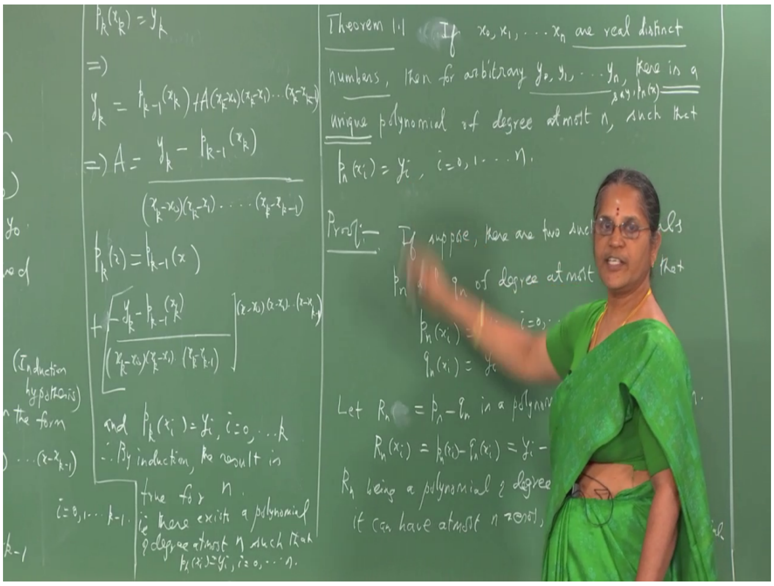

And that is what we wanted to show. Given n plus 1 real distinct numbers for arbitrary values y 0 y 1 etc y n associated with these x 0 etc x n. There exists a unique polynomial of degree at most n satisfying this condition. The polynomial so constructed is called as an interpolation polynomial that interpolates the function at a set of these n plus 1 discrete points. So now we have answered these question. Yes there exists a polynomial and this polynomial is a unique polynomial satisfying the condition so the next question is ok how are we going to get this polynomial if there are two points it is easy three points yes with some computation.

We can get this polynomial when the number of information given to us becomes more and more in practical applications this is what will happen we will have information about a function to be constructed by set of several points and going one by one taking a third degree polynomial then constructing the fourth degree polynomial etc is going to be a tedious process. And therefore we would like to get some simple methods by means of which we should be in a position to construct this unique polynomial for a given set of discrete data.

So let us see how we can do it? and finite differences is going to be an important concept that is going to help us in construction of interpolation polynomials and let us see what one understands by finite differences.