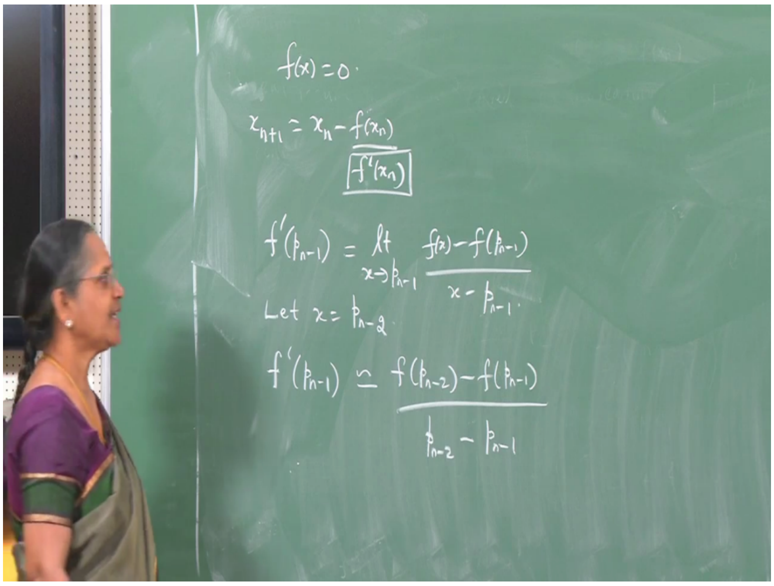
Numerical Analysis
Professor R. Usha
Department of Mathematics
Indian Institute of Technology Madras
Lecture No 33
Root finding Method 5–Secant Method, Method of False Position

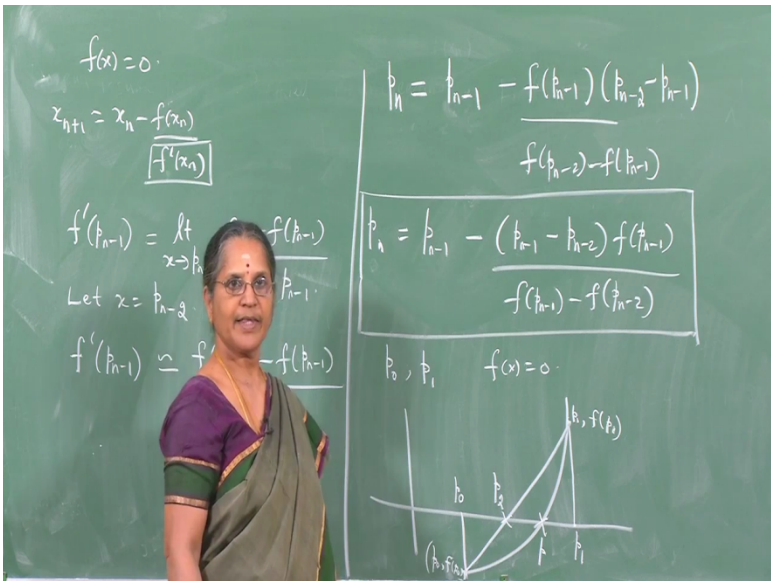
In the last class we discussed about Newton Raphson Method which helps to solve an equation of the form f of x = 0. It produces a sequence of iterates which converge to a root of the equation f of x = 0, the method is given by x n + 1 = x n – f of x n by f dash of x n. We observe that the method involves computation of the derivative of the function at each step of iteration that is the drawback in Newton Raphson Method. Namely at each iterative step we not only have to evaluate this function f of x n but also evaluate its derivative at x n so it involves 2 function evaluations at each step. So we would like to now develop a method which does not involve evaluation of derivative of a function at each step of iteration, so we make a slight variation as follows.

(Refer Slide Time: 1:46)



So we know my definition of derivative of a function that f dash at some p n – 1 is limit as x tending to p n – 1 as f of x – f of p n – 1 by x – p n – 1. If suppose I left x to be say p n – 2 then f dash at p n – 1 can be approximated by f at p n – 2 – f at p n – 1 by p n – 2 – p n – 1. So I would like to make use of this approximation to a derivative and then slightly modify Newton Raphson Method as follows.

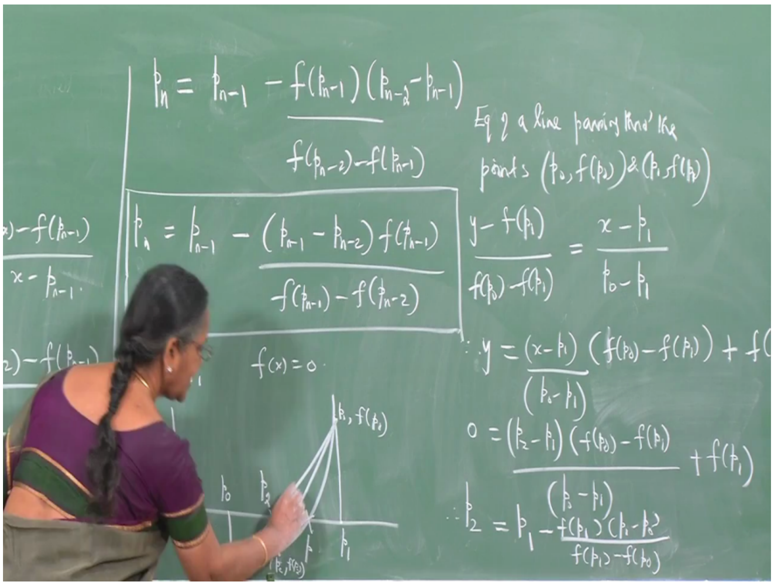
(Refer Slide Time: 2:47)



P n = p n – 1 – f at p n – 1 by f dash at p n – 1, but f dash at p n – 1 is f of p n – 2 – f of p n – 1 and numerator will have p n – 2 – p n – 1, so which I can write as p n – 1 – p n – 1 – p n – 2 into f at p n – 1 by f of p n – 1 – f of p n – 2. So if I want to compute an nth approximation to a root of the equation then I have the method given as p n is p n – 1 – p n – 1 – p n – 2 f of p n – 1 – f of p n – 2 multiply by f of p n – 1. So this tells you that at any step you require the knowledge of the successive approximations at the previous 2 steps namely to find what p n is, you must have a knowledge of p n – 2 and p n – 1. If you know that then use the right hand side to find what p n is at the nth step. Suppose say you start with 2 initial approximations p 0 and p 1 of a root of the equation f of x = 0.

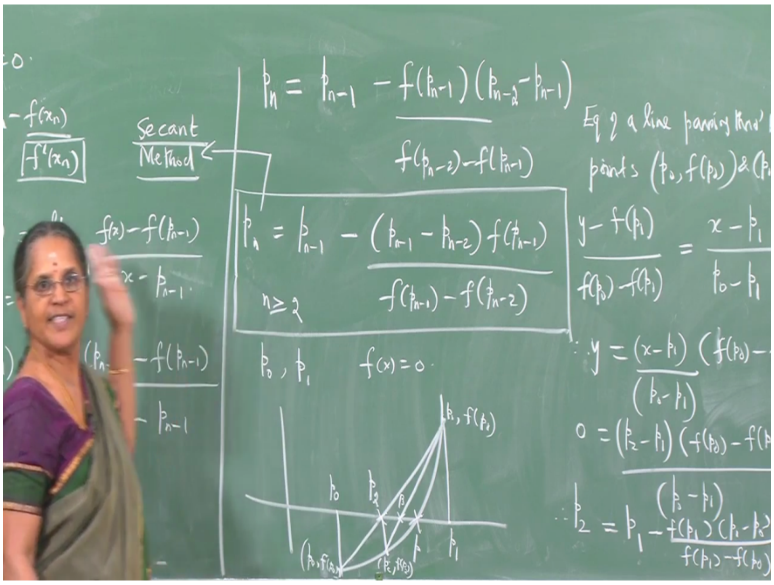
So this is the graph of the function say y = f of x, so I make 2 approximations p 0 and p 1 and then I join the line passing through the points p 0, f of p 0 and p 1, f of p 1 and I see where this line crosses the x axis and I call this as my next approximation namely p 2. The actual root is the point where the graph of the function crosses the x axis, so this is the actual root. Initially I take 2 values p 0 and p 1 which are close to this p and then consider a line passing through the points p 0 f of p 0 p 1 f of p 1 and then see where this line crosses the x axis and take that point as a next approximation to the root of the equation, let us see what is p 2.

(Refer Slide Time: 6:37)



So what do I need? I require the equation of a line passing through the point p 0, f of p 0 and p 1, f of p 1. It is given by y – f of p 1 by f of p 0 – f of p 1 = x – p 1 divided by p 0 – p 1, and therefore we have y to be given by x – p 1 by p 0 – p 1 into f of p 0 – f of p 1 + f at p 1. What do I want? I want y to be 0 and x is p 2, namely the point where the graph of the function crosses the x–axis. So that gives you p 2 – p 1 into f of p 0 – f of p 1 divided by p 0 – p 1 + f at p 1, this gives me p 2 as p 1 – f of p 1 into p 1 – p 0 by f of p 1 – f of p 0. What is this p 2? p 2 is the point where the line joining p 0 f of p 0 and p 1 f of p 1 crosses the x axis and that gives me the next approximation then what do I do? I join the line passing through the points p 2, f of p 2 and p 1, f of p 1 and see where it crosses the x–axis and call that as p 3.

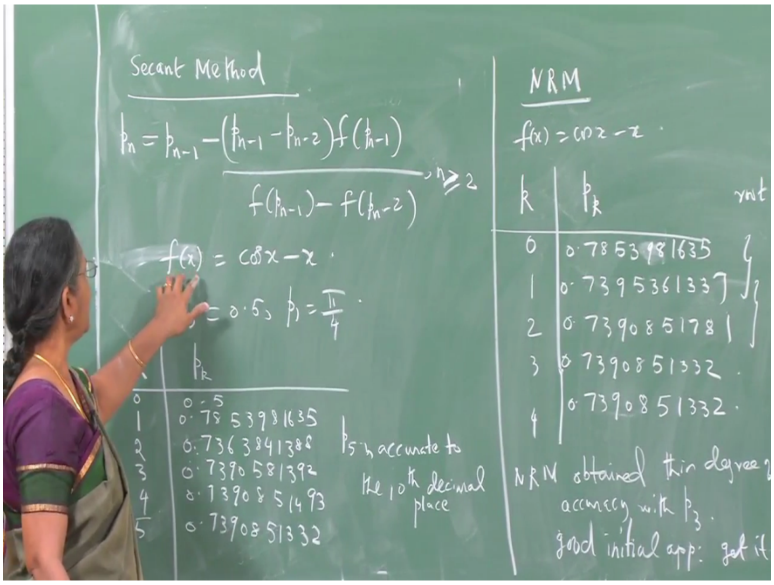
(Refer Slide Time: 10:10)



So that I can express p 3 now in terms of p 2, what do you do? p 3 will be p 2 – f of p 2 into p 2 – p 1 by f of p 2 – f of p 1, the computations are analogous to this and then I continue this computation and generate successive iterates and at the nth step my p n is given by p n – 1 – f of p n – 1 into p n – 1 – p n – 2 by f at p n – 1 – f at p n – 2 for n greater than or equal to 2. The method so derived is what is called the Secant method, why is it called the Secant method? The reason is, it approximates because in this interval by means of the cord or the Secant passing through the 2 points p 0, f of p 0 p 1, f of p 1 and so on, so it derives its name Secant.

And you observe that as we move closer and closer to the root, right this Secant in the limit will become a tangent to the curve at the point p and it is this tangent line which approximated the curve in Newton Raphson Method and that is why the method derives its name as Secant method and we observe that we do not have evaluation of derivative of a function, instead we have two function evaluations here again that both of them are evaluation of the given function at 2 different winds namely p n – 1 and p n – 2, so let us now compare the Secant method with Newton Raphson Method and see which converges faster. Let us now compare the performance of the Secant method and the Newton Raphson Method.

(Refer Slide Time: 11:53)

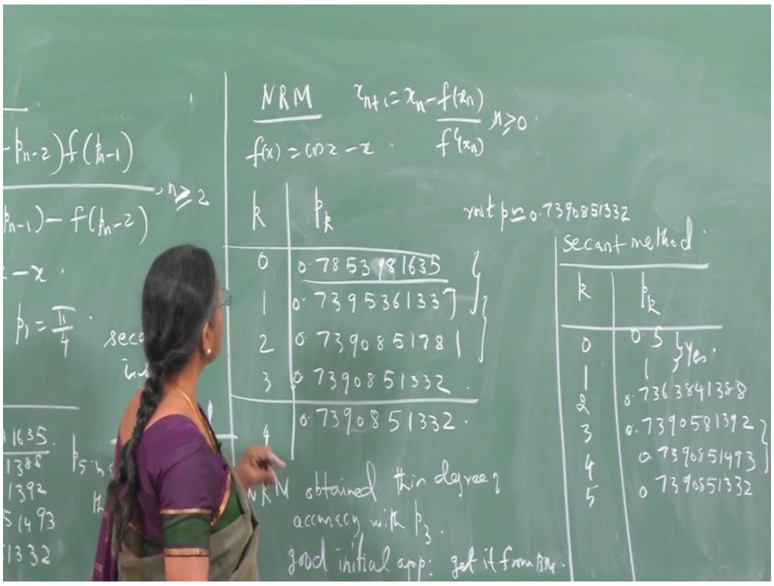


We will consider determining the root of the equation f of x = cos x – x = 0. By Secant method starting with 2 initial approximations p 0 as 0.5 and p 1 as Pie by 4, we observe that the successive approximations are given by this and the results at p 5 is accurate to 10 decimal place, the root of the equation is 0.7390851332 and that is obtained at the 5^{th} using Secant method correct to 10 decimal places. We solved the same equation using Newton

Raphson Method which is given by x n + 1 = x n – f of x n by f dash of x n for n greater than or equal to 0. Starting with an initial approximation which is this, it is the same as the p 1 here, we observe that Newton Raphson Method obtains this accuracy with p 3 so the 3rd iteration we are able to get the same accuracy for root of this equation as that was obtained by Secant method say at this step.

And we observe that both Newton Raphson Method and Secant method requires good initial approximations to a root of the equation. And comparing the 2 methods, we observe that Secant method is slower than Newton Raphson Method, so you may ask me why then you consider this method which is slower than Newton Raphson Method because you want to generate a sequence of iterates which converges faster. But there is an advantage when you Secant method, you do not have to compute the derivative of a function at each step of your iteration, you know what the function f of x is and you have to evaluate it at 2 different points, that is the advantage of the Secant method. Whereas, in Newton Raphson Method you need to compute two different function evaluations namely the function f and its derivatives at each step of iteration.

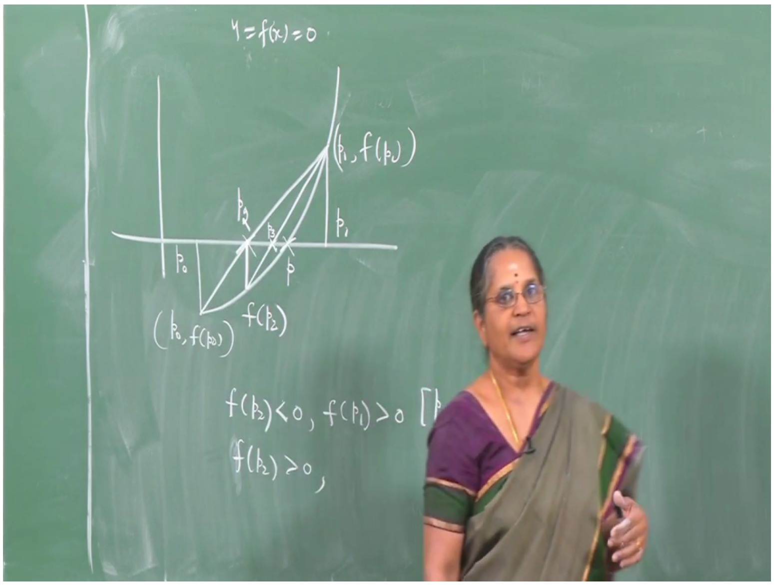
(Refer Slide Time: 15:13)



Another observation that we would like to make is the following. When we use Newton Raphson Method, we observe that we have successive iterates which are such that iterates P 0 and P 1 they do not enclose this root of the equation similarly, P 1 and P 2 if you look at these 2 values this interval does not enclose the given root of the equation so we come across in the computation of iterates using Newton Raphson Method that the interval of interest does not enclose a root of the equation, but when we consider the successive iterates, the iterates converge to the root of the equation. The same thing happens in case of Secant method also, when we look at Secant method we observe that p 0, P 1 encloses the root of the equation however, as we generate successive iterations we come across a step where P 3, p 4 does not enclose a root of the equation.

And therefore this suggests that we must develop a method which takes care to see that a root of the equation lies in the interval at each step of our iteration. So one of our observations from here is that Newton Raphson Method and Secant method are not enclosure methods, bisection method was a enclosure method because at each step of iteration we have an interval which encloses a root of the equation however, Newton Raphson Method and Secant method are not enclosure methods as evident from this example. So we would like to now modify Secant method in such a way that we also want to ensure that at each step of iteration we have an interval which encloses a root of the equation and this can be done very easily, what should we do?

(Refer Slide Time: 18:00)

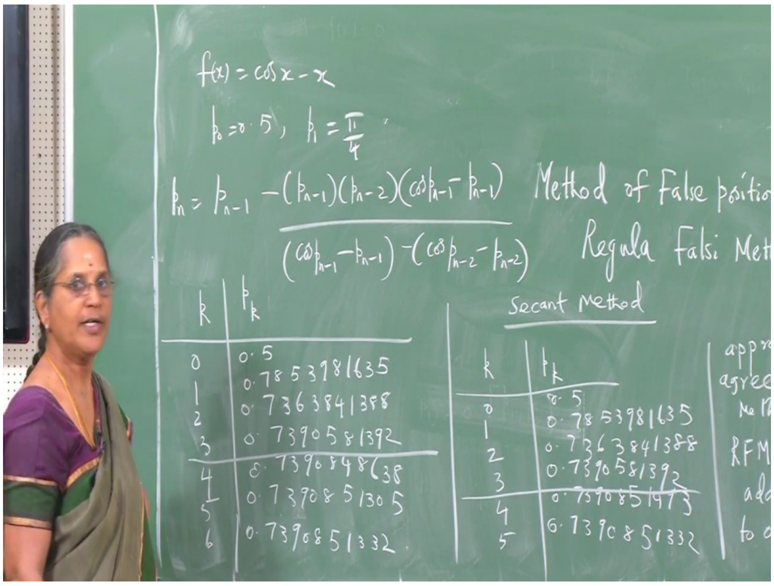


Let us consider the given equation y = f of x = 0, so we draw the graph of this function, the point where it crosses the x axis is the actual root. What did we do in Secant method? We started with 2 initial approximations namely P 0 and P 1, then we approximated this curve by means of a straight line passing through the points P 0, f of P 0 p 1 f of p 1 and then the point where it crosses the X–axis was taken to be p 2, but what do we want now we said that we should ensure that at any step of iteration the interval must enclose a root of the equation. So when we compute this P 2, we check the sign of f of P 2, if suppose f of P 2 is negative and f of p 1 is positive then we know that a root lies in the interval p 1 to p 2. Here that is what happens, f of p 2 is negative so a root lies in the interval p 2 to p 1 and so we join the straight line through the points p 2 f of p 2 and p 1 f of p 1 and the point where it crosses the X–axis is P 3.

Suppose say in the case which we discuss, f of p 2 turns out to be positive then we check whether a root lies between P 0 and P 2 or P 2 and P 1, accordingly we select that interval in which the root lies and then we take a straight line joining the 2 points appropriately in such a way that at one of the points the function value has a negative sign and the other one has a function value which is positive sign and therefore we ensure that or root lies in this interval and continue in our computations and determine the successive iterates.

If we do this namely if we start with some 2 initial approximations and continue to generate the successive approximations such that at each step we make sure that root lies in that interval and then generate the successive iterates, then we are essentially solving the problem by what is called as the Method of False Position namely it is a Regula Falsi Method. So we are solving by method of False Position and it is also known as Regula Flasi Method. Essentially it is a Secant method but it only takes care to check that at each step a root lies within the interval, so let us work out an example and see how Regula Falsi method is used in the solution of equation f of x = 0.

(Refer Slide Time: 22:31)

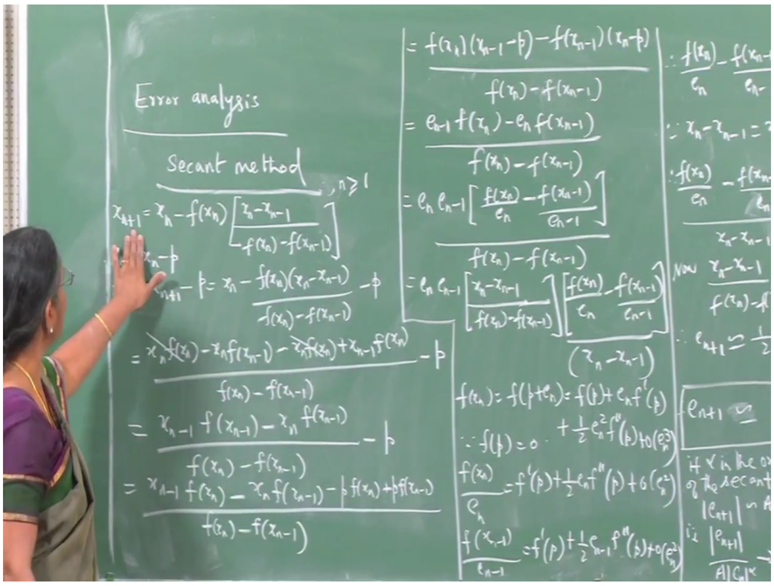


Let us now compare the performance of Regula Falsi method and Secant method. So let us solve the same problem f of x = 0 given by cos x – x = 0 and start with the same 2 initial approximations P 0 and p 1. Regula Falsi method is essentially Secant method but it only takes care that at each time an interval encloses a root of this equation and accordingly chooses that interval as the 2 successive approximation for the next step. So we observe that the successive approximations are given by this when you apply Regula Falsi method and the result that we have earlier obtained using Secant method is given by this and we observe that the approximations obtained by the Regula Falsi method and the Secant method, they agree through p 3 so these are the same as what we have obtained in Regula Falsi method also.

The Secant method continues further and root of the equation is octane as this correct to the desired degree of accuracy at the 5th iteration. On the other hand, Regula Falsi method requires one more iteration, namely result is obtained correct to the desired degree of accuracy only at the 6th iteration but with Secant method we could obtain it at the 5th iteration. So this is slower than the other one but it enforces an extra condition ensuring that at each step we have a root which lies within an interval and therefore, Regula Falsi method comes under enclosure method whereas, the Secant and the Newton Raphson Method do not fall under the class of enclosure method.

So we see how nicely each of these methods have been generated by seeing some drawbacks in the previous method that we have derived and then the question now comes to which class of methods do Newton Raphson Method and Secant method belong to if they are not in the class of enclosure methods. We are sure that Newton Raphson Method belongs to the class of fixed point iteration method; we now perform the error analysis of Secant method.

(Refer Slide Time: 25:38)



Secant method is given by x n + 1 = x n – f of x n into this and e n is x n – p, so we want to obtain relationship connecting e n + 1 and e n, so we start with e n + 1 which is x n + 1 – p, but x n + 1 is given by this because we are using Secant method now so it is x n – this expression – p. So we simplify this expression, so it is x n into f of x n – x n f of x n – 1 – f of x n into x n + f of x n into x n – 1 by the denominator – p, so x n, f of x n cancels so we have x n – 1 f of x n – 1 no we have x n – 1 f of x n – x n f of x n – 1 by f of x n – f of x n – 1 – p. So we simplify this step, so this term appears as it is, this also appears as it is so – p into f of x n + p f of x n – 1 by the denominator. So we collect terms involving f of x n , so we have this term and this term having f of x n so multiplied by p – or x n – 1 – p – terms involving f of x n – 1 are these and they give you – f of x n – 1 into x n – p by the denominator.

But we know that x n – 1 – p is e n – 1, x n – p is e n, so this step follows from here. Now I remove the (())(27:51) e n from the numerator and e n – 1 from the numerator so I have e n en – 1 into f of x n by e n – f of x n – 1 by e n – 1 divided by this. And now e n e n – 1 and I would like to multiply and divide by x n – x n – 1, you will understand why we do it when we proceed further, x n – x n – 1 is multiplied and I have divided otherwise there is no change in this step as compared to this step. At this stage I would like to find out an expression for the factor which appears here, so I make use of the factor f of x n is f of p + e n that is f of p + e n into f dash of p + e n square by 2 f double dash of p + order of e n cube.

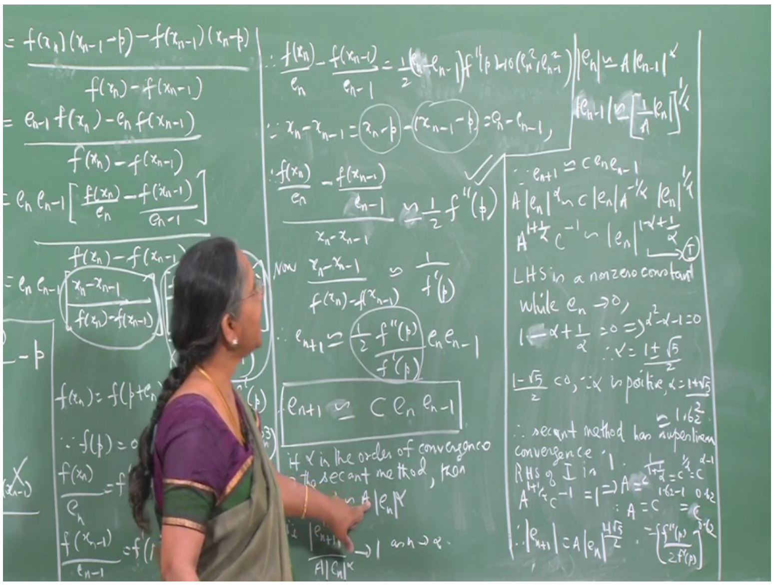
(Refer Slide Time: 29:24)

But I know that f of p is 0 because p is the root of the equation and therefore, I get f of x n by e n is f dash of p + this is divided by e n so half of e n f double dash of p + order of e n square because we have divided by e n. So similarly I can write down what is f of x n – 1 by e n – 1 that will again be f dash of p + half e n – 1 into f double dash of p + order of e n – 1 square. So I require this – this, so simply subtract this from here so f dash of p will cancel so you will be left with half f double dash of p into e n – e n – 1 so that is what we have written on the right–hand side. So half of e n – e n – 1 into f double dash of p + order of this term will involve e n square and e n – 1 square, so we now have an expression for the numerator, let us see what is the denominator.

So consider x n x n – 1, so I subtract p and add p to this, so I can write this as x n – p – of x n – 1 – p so that will be e n – e n – 1. So I can substitute for x n – x n – 1 as e n – e n – 1, so the expression here now is half of f double dash of p because I omit the higher–order terms because e n – e n – 1 cancels with e n – e n – 1, so I have an expression for this. Now I look at this factor, I know this is approximately 1 by f dash of p right, f dash of p is this by this approximately, so 1 by f dash of p is this expression so I also have replaced this by 1 by f dash of p. So I write down, where did we start, we started with e n + 1, so e n + 1 is approximately half of f double dash of p by f dash of p into 1 by f dash of p into e n into e n – 1 that is what appears as here.

Anyway, this is a constant, so I called that as C so e n + 1 is C times e n e n – 1. But in order to determine the order of convergence of a method we must have a relationship in the form e n + 1 is a constant times e n power Alpha, Alpha will determine the order of convergence. But here we have C into e n into e n – 1, so I must make further computations on this so that I obtain the form as e n + 1 = C into e n power some Alpha. So if Alpha is the order of convergence of the Secant method then I must have mod e n + 1 to be approximately some constant a times mod e n power Alpha since already a constant C appears I have taken constant to be A in this case. So what does this tell, mod e n + 1 by A times mod e n power Alpha must approach 1 as n tends to infinity.

(Refer Slide Time: 33:13)



And therefore from here mod e n + 1 is A mod e n power Alpha then mod e n will be approximately A into mod e n – 1 power Alpha, so from here I obtained mod e n – 1 is approximately 1 by A into mod e n raise to the power of 1 by Alpha. And I know that e n + 1 is C e n e n – 1, what is e n + 1? That is A mod e n power Alpha that is the left–hand side that must be approximately C into mod e n into what is mod e n – 1, it is 1 by A mod e n power 1 by Alpha so it is A power – 1 by Alpha into mod e 1 power 1 by Alpha. So I have A power 1 here – 1 by Alpha on this side so I have A power 1 + 1 by Alpha into C to the power of – 1 taking it to side that is approximately mod e n I have a power 1 here – Alpha , it comes to this side and power 1 by Alpha so mod e n power 1 – Alpha + 1 by Alpha.

I observed that the left–hand side A and C are constants, Alpha is a constant, so left–hand side is a nonzero constant while e n tends to zero as n tends to infinity, so I must have the power to which e n is raise to be 0 in order that this is satisfied. So 1 – Alpha + 1 by Alpha must be 0, so we has Alpha square – Alpha = 1 to be 0, so solve this quadratic equation that gives you Alpha to be 1 + R – root 5 by 2. We observe that 1 – root 5 by 2 is negative, I want Alpha to be a positive constant that is the order of convergence of the method so this is not possible, so Alpha is positive and so I take Alpha to be 1 + root 5 by 2 and that is approximately 1.62.

What is Alpha? Alpha is the order of convergence of Secant method and it turns out to be 1.62. What is the order of convergence of bisection method? 1, what is the order of convergence of Newton Raphson Method? 2, and Secant method has order of convergence which is 1.62 and so we say that the Secant method has super linear convergence right, it is better than bisection method, but it is slower than Newton Raphson Method which converges quadratically, so we say that this has super linear convergence. We still have to obtain asymptotic error constant, so we look at the right–hand side of 1 and that is equal to 1 because as e n tends to 0 as n tends to infinity and therefore, right–hand side of 1 will be 1 and so we have A power 1 + 1 by Alpha into C power – 1 must be 1, so A must C to the power of 1 by 1 + Alpha.

But I know that Alpha satisfies the equation 1 + 1 by Alpha – Alpha is 0, so I can replace 1 + 1 by Alpha by Alpha so I have C power 1 by Alpha, but 1 by Alpha is what? Alpha – 1 so C power Alpha – 1 but what is Alpha, 1.62 so A is C power 1.62 – 1 so C to the power of 0.62. Mod e n + 1 is A mod e n power Alpha, Alpha is 1 + root 5 by 2 and A is given by f double dash of p by twice f dash of p raise the power of 0.62. So our error analysis shows that Secant method is superior to bisection method, but it is inferior to Newton Raphson Method in the sense that the order of convergence of Secant method is super linear, it is 1.62 whereas that of bisection method is 1 and order of convergence of Newton Raphson Method is 2 as compared to the Secant method which is 1.62.

And these were all reflected in our examples when we solved equations f of x = cos x – x = 0, and we obtained the result correct to the desired degree of accuracy at the 3rd iteration by Newton Raphson Method whereas, at the 20th iteration by bisection method where as by Secant method we were able to obtain the solution correct to the desired degree of accuracy at the 5th iteration and it is already because of the order of convergence of these methods. As I had already pointed out, Newton Raphson Method and Secant method do not fall in the category of enclosure methods, we are sure that they belong to the class of fixed point iteration methods, so we shall discuss what we mean by fixed point iteration methods and then show Newton Raphson Method belongs to this class and we will continue with this in the next class.