Numerical Analysis
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Lecture No 31
Root finding Method 3–Newton Raphson Method 1

Good morning we have been discussing about numerical methods, which helps us to solve equations of the form f of x = 0. In the previous class we discussed bisection method with the help of which one can generate a sequence of successive iterates, which are approximations to root of the equation f of x = 0, which lies in the interval of the form a b, where f of a into f of b is less than 0. Then we observe that although the sequence of iterates is guaranteed to converge to a root of the equation correct to the desired degree of accuracy, the procedure was a very slow procedure, so this demands that we have a method which has faster convergent rate, one such method is Newton Raphson Method. We shall discuss this methods which helps us to solve an equation of the form f of x = 0 numerically.

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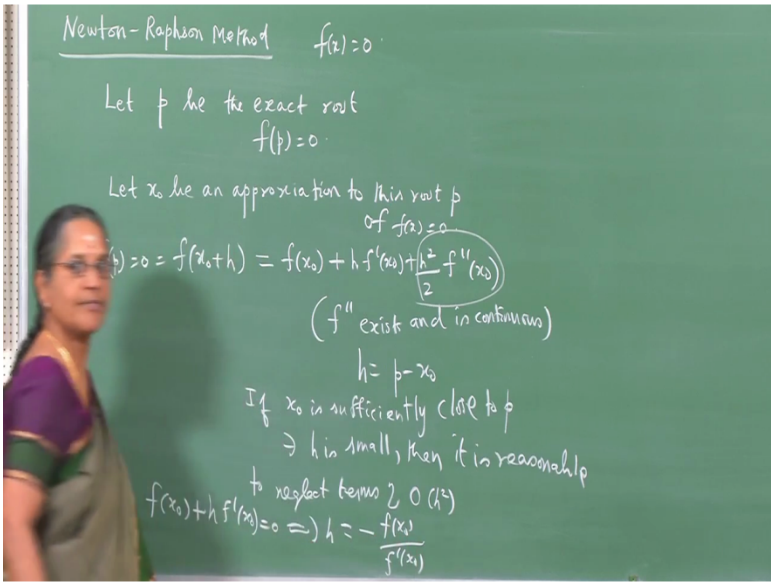

This method involves Linearisation of the function f of x about the point x 0. So let us assume that p is an exact root of the equation, what does that mean? It means f of p = 0 but our goal is to determine this p, so let us start with some initial approximation to this root p, so let x 0 be an approximation to this root p of the equation f of x = 0. So f of p is 0 and that is f of x 0 + h because this x 0 is an approximation, I have therefore made some error so if suppose I add that error to x 0 then f of x 0 + h will be such that it is 0 so I would like to now expand f of x + h about x 0, so f of x 0 + h into f dash of x 0 + h square by factorial 2 into f double dash of x 0.

So under the assumption that f double dash exists and is continuous and your h is p – x 0, so if h is small what does that mean, if your x 0 is sufficiently close to p such that h is small then it is reasonable to neglect this term which is of order of h square. So we shall assume that h is very small and that terms of order of h square can be neglected, so it is (())(4:09) terms of the order of h square in the above expansion and solve for h from the equation f of x 0 + h into f dash of x 0 is 0 and what does that give? That gives you h as – f of x 0 divided by f dash of x 0 so having determined h what do we have?

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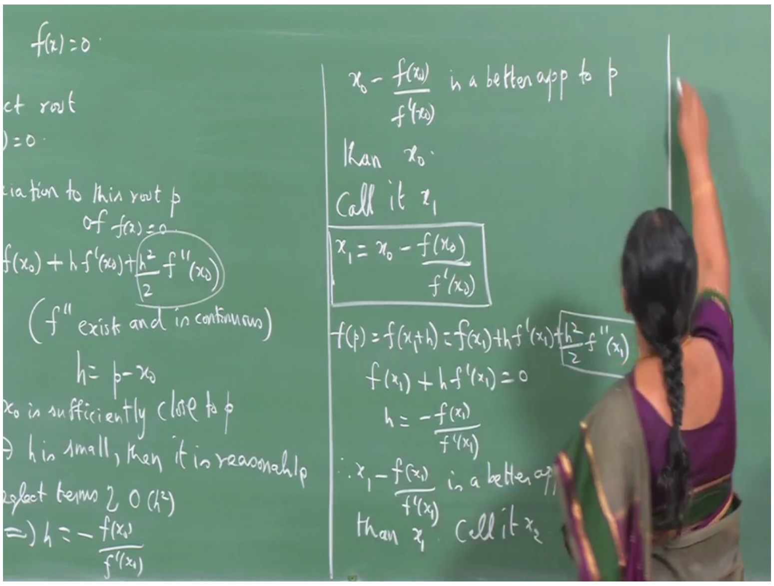

We have x 0 – h, what is h? f of x 0 divided by f dash of x 0 is a better approximation to this root p of the equation f of x = 0 than our initial approximation which is x 0. And therefore I call this new approximation as x 1 so that x 1 will be x 0 – f of x 0 by f dash of x 0, so I have now first approximation to a root of the equation f of x = 0. But now I would like to improve this approximation so my f of p is f of x 1 + h, what is it I expand about the point x 1 that will be f of x 1 + h into f dash of x 1 + h square by factorial 2 into f double dash of x 1. And when my h is very small such that letter x1 plus h = p then I am justified in omitting terms of order of h square.

The same argument and what we have given here and so I omit this term and solve for the new h which is f of x 1 + h into f dash of x 1 = 0 is the equation which is satisfied by h and that gives me h = – f of x 1 by f dash of x 1 and therefore, x 1 – f of x 1 by f dash of x 1 is my x 1 + h, so this is better approximation to the root p, which we are trying to determine numerically than the approximation with which we started here namely x 1, so I call this as x 2.

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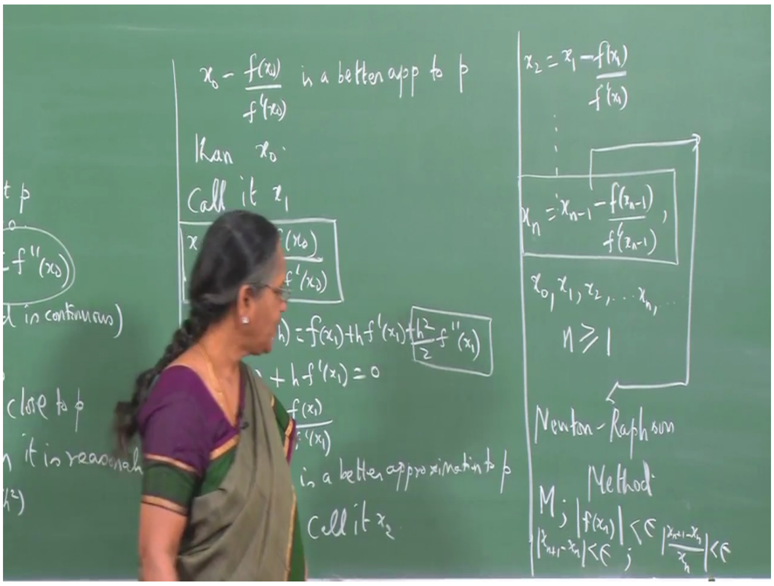

And hence I have x 2 to be given by x 1 – f of x 1 divided by f dash of x 1, so I continue in this way and then say at the nth step I have x n to be given by x n – 1 – f of x n – 1 divided by f dash of x n – 1, so I have a method which generates successive approximations x 0, x 1 x 2, et cetera, x n and I can continue right, so this is a method which gives me a way to generate these successive iterates so starting from n which is equal to 1, I can generate all these iterates values which are approximations to root of the equation and the method is what is called the Newton Raphson method. So Newton Raphson Method is essentially generates a sequence of successive approximations x 0, x 1, etc, x n starting with an initial approximation x 0 and uses this to generate the successive approximations.

So we have obtained say n such approximations and therefore we have a sequence of approximations where do we stop this we already have discussed. Namely you perform M number of iterations and stop there or having determined x n check whether the absolute value of f of x n is less than the prescribed tolerance namely epsilon or you check whether x n + 1 – x n is less than epsilon namely successive iterative values for the approximation values of a root of the equation are such that the difference between them is less than the given accuracy, then you have checked that your absolute error is less than the prescribed tolerance or you check whether x n + 1 – x n divided by x n is less than epsilon, where you have checked that the relative error is less than epsilon.

So if anyone of these or these 2 conditions are satisfied, then you stop your iterations using this method and find out that when you of x n + 1 at which the condition is satisfied, say if you have taken this as the stopping criterion and say that this is an approximation to the root of the equation namely p and so you have been able to determine a 0 of the function f or a root of the equation f of x = 0 correct to the desired degree of accuracy. So let us now try to illustrate this method by means of the following example.

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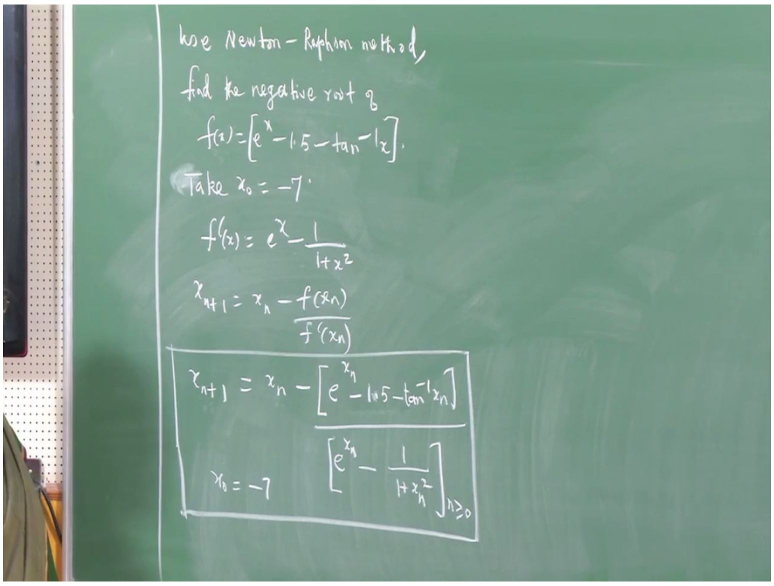

So the problem is, use Newton Raphson Method and find the negative root of f of x equal to this given that an initial approximation to this root is x 0 = – 7. So we are given f of x, so we need to compute the derivative and the method is given by x n + 1 is x n – f of x n divided by f dash of x n, so it is x n – what is f of x, it is this so f of x n is e power x n – 1.5 – tan inverse of x n divided by f dash of x n, so we have computed f dash of x so that will give e to the power of x n – 1 by 1 + x n square, so this is the method which we will have to use to generate the successive iterates given that x 0 is – 1, so (())(12:00) greater than or equal to 0. So when n is 0, this will give you x n in terms of x 0 – f of x 0 by f dash of x 0 and so on, so let us work out the successive iterates starting with an initial approximation which is x 0 = – 7.

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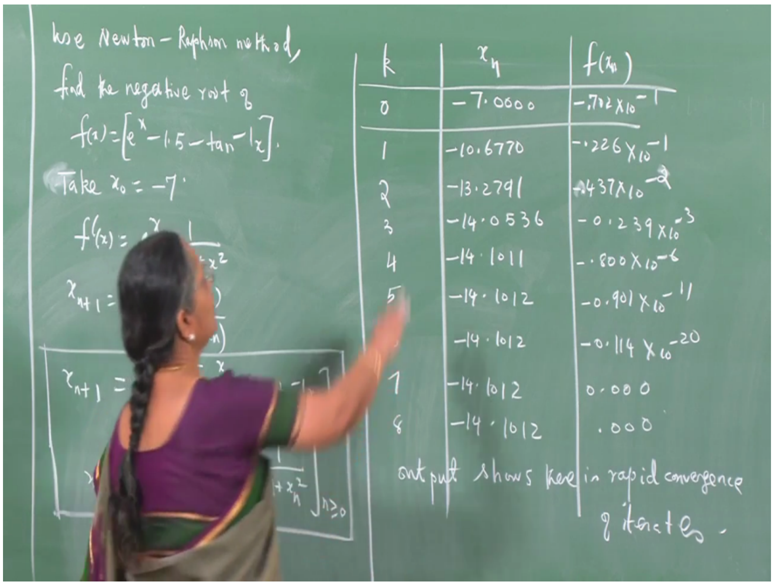

The successive iterates are as follows; x 0 is – 7, the value of the function is – 0.702 into 10 to the – 1 so I generate x 1 which is x 0 – f of x 0 by f of f dash of x 0 and that gives me the value of x 1 to be – 10.6770 at which the function value is this. Then x 2 is x 1 – f of x 1 by f dash of x 1 and the values are obtained and the corresponding function values are written here. So we use this iterative method and then generate the successive iterates and we see that the successive iterates begin to converge to a root of the equation, which is – 14.1012. And we observe that the output that we have given here, there is rapid conversion of these successive iterates. And we observe that at the 6th step our function value is almost close to 0 and therefore the Newton Raphson Method gives a sequence of iterates which converge rapidly to a root of the given equation f of x = 0, so this is how you numerically obtain a solution of the given equation f of x = 0 using Newton Raphson Method.

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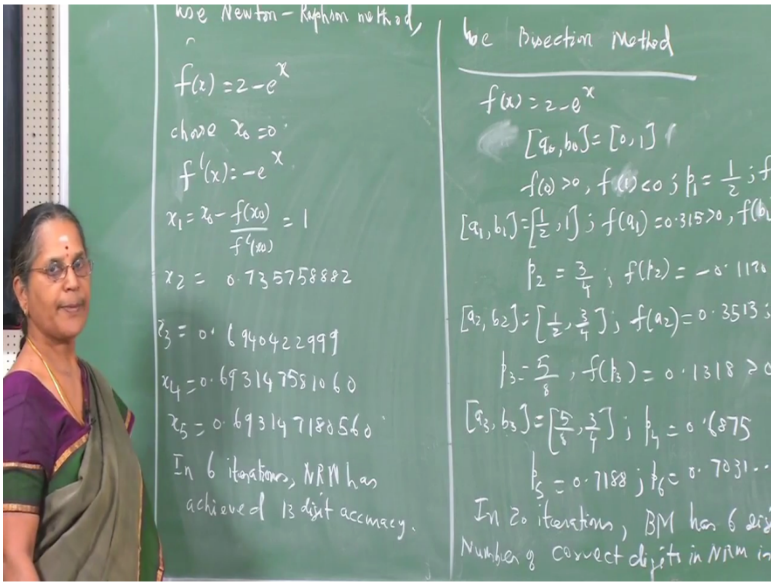

Let us now consider another example, so having understood Newton Raphson Method we would now like to compare this method with bisection method and see whether the sequence of iterates converge much rapidly as compared to those generated by bisection method, let us discuss this by taking the following example. Suppose I am interested in solving the equation f of x = 2 – e power x, I start with an initial approximation say x 0 = 0 so I require f dash of x and my Newton Raphson Method is given by x n + 1 is x n – f of x n by f dash of x n 4 n greater than or equal to 0.

So I take n to be 0 and obtain the 1st approximation x 1, which is x 0 – f of x 0 by f dash of x 0 and that turns out to be 1, then use x 1 and generate x 2 which is x 1 – f of x 1 by f dash of x 1 and that gives me x 2 and I continue to generate the sequence of iterations. And I observe that in 6th iteration, this Newton Raphson Method has achieved 13 digits accuracy, now I will like to solve the same problem using bisection method.

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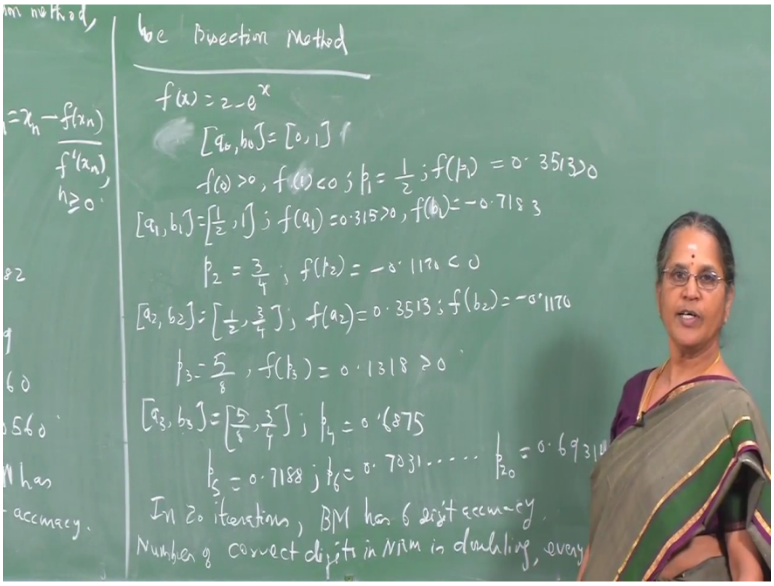

So given the equation f of x = 2 – e power x = 0, if I want to apply bisection method I would first like to have an interval that encloses a root of the equation. So I observe that f of 0 is positive and f of 1 is negative, so the interval 0 1 encloses the root of the equation so I called that interval as a 0 b 0. So I compute the midpoint namely 0 + 1 by 2, so p 1 will be half and I evaluate F of p 1 and that turns out to be positive and therefore, F of p 1 into f of 1 is negative so there is a root between half and one, so I called it a 1 b 1 is half into 1 this interval that encloses a root of this equation, so evaluate F of a 1 positive, F of b 1 negative, so p 2 is half + 1 by 2 and therefore I get an approximation of the 2nd step to the root of the equation, and I continue in this way and I observe that when I have performed 20 such iterations, the midpoint of that interval at iteration turns out to be this.

And I see that after 20 iterations the bisection method has only 6 digits accuracy whereas, Newton Raphson Method at the end of 6th iteration has achieved 13 digit accuracy and therefore, the number of correct digits in Newton Raphson Method is doubling in every iteration and Newton Raphson Method converges much faster than the bisection method. So we have a new numerical method by means of which we can generate a sequence of iterates which converges rapidly to root of the equation. But we require an initial approximation x 0 which is very close to the root of the equation, how do you get this approximation? One of the methods is bisection methods, so you given an equation f of x = 0, find out an interval in which a root lies by determining say a b such that f of a into f of b is negative, so this root lies within this interval.

So perform some few iterations by bisection method and get an approximation to a root of the equation call that as an initial approximation and start using Newton Raphson Method and generate a sequence of successive iterates, which will converge rapidly to a root of the equation. So we have developed Newton Raphson Method, we have seen how one can use this method to obtain a sequence of iterates converging to a root of the equation, we have also compared that to methods which we have learned namely the bisection method and the Newton Raphson Method and we observe that the Newton Raphson Method converges much faster as compared to the bisection method. So let us now try to see the geometrical interpretation of this Newton Raphson Method, essentially Newton Raphson Method involves linearisation of the function f of x, so let us see how it is done.

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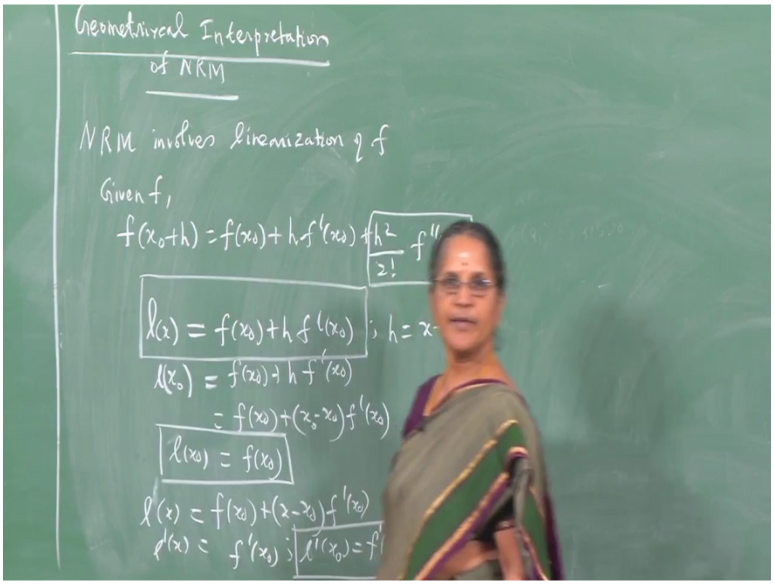

So we would like to give a geometrical interpretation of Newton Raphson Method, so we have already seen that Newton Raphson Method involves linearization of this function f. What did we do? Given this function f we said f of x 0 + h is f of x 0 + h into f dash of x 0 + h square by factorial 2 into f double dash of x 0 and so on. Then we said that if we linearise this function about the point x 0 by omitting terms of order h square, then the linearization of the function l of x about x 0 is a function of x, which is f of x 0 + h into f dash of x 0. What are the properties of this function l of x let us see. If I evaluate l of x at x 0 then it is f of x 0 + h into f dash of x 0, what is h here? h is x – x 0 so it is f of x 0 + x – x 0 into f dash of x 0. So I am evaluating at x 0 so this will be x 0 – x 0 so that will give me f of x 0, so here h is x – x 0.

So what do I observe? I observe that the function which linearises this f about the point x 0 has the property that it takes the same value as f at x 0; namely l of x 0 is f of x 0. Let us find out what is l dash of x, let me 1st write–down l of x as f of x 0 + x – x 0 into f dash of x 0. So what is l dash of x? It is simply f dash of x 0, so at x 0 what is l dash of x 0? It is this constant value f dash of x 0. So what is the property that the function l has? It takes the same value as the function at x 0 such that it takes the same value as the derivative of the function at x 0 and therefore I am justified in approximating this function f of x in a neighbourhood of the point x 0 by a linear function given by l of x equal to this or I am justified in approximating this function f in a neighbourhood of x 0 by a straight line.

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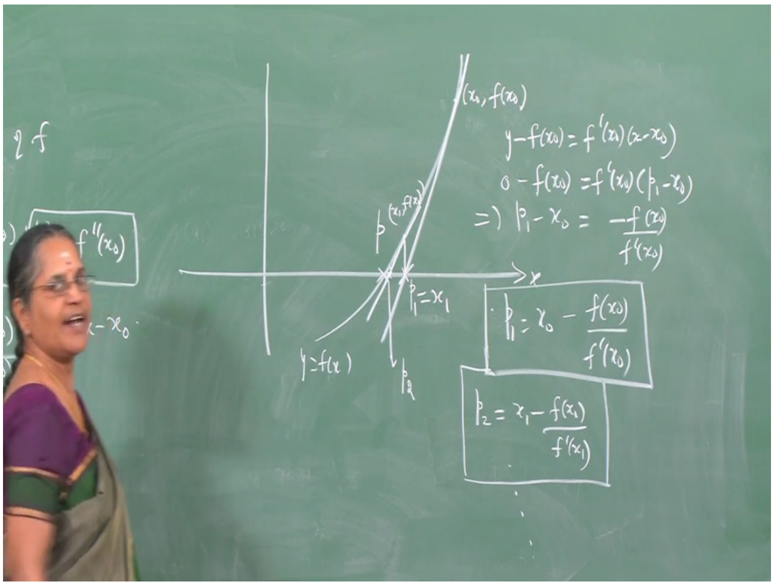

So if I am given the function f and I am interested in solving equation f of x = 0 and I draw the draft of f of x, geometrically the point where the graph of f crosses the x axis gives you the exact root of the equation p. So now I am choosing a point x 0 very close to p and take the corresponding point x 0, f of x 0 and the curve, I would like to approximate the function f in the neighbourhood of x 0 by a straight line whose slope is f dash of x 0. So I draw a tangent to the curve at the point x 0, f of x 0 and see where it crosses the x axis and I call that point as p 1. Let us see what we have done, I have a straight line which is tangent to the curve at the point x 0 f of x 0 passing through the point x 0, f of x 0 and having slope f dash of x 0.

So what is the equation of the line? Equation of the line is y – f of x 0 equal to its slope multiplied by x – x 0. Then I check where does this line cross the x axis, say at some point x = p 1 y is 0, so f of x 0 is f dash of x 0 into p 1 – x 0, which gives me p 1 – x 0 is – of x 0 by f dash of x 0 and therefore p 1 = x 0 – f of x 0 divided by f dash of x 0 and you observe that this is essentially your Newton Raphson Method, which is giving you the 1st approximation to a root of the equation when you have started with an initial approximation as x 0. So having got p 1 you call that point as x 1 and then draw a tangent to the curve of the point x 1, f of x 1. In the point where it crosses the x–axis you call that as p 2, then by the same argument you will end up with p 2 as x 1 – f of x 1 divided by f dash of x 1 so you have a 2nd approximation obtained using the point where the tangent to the curve at x 1, f of x 1 crosses the x axis.

And you observe that this is essentially the 2nd approximation generated by Newton Raphson Method, so you continue and you observe that every time you come closer and closer to the root of the equation namely p at which the graph of the function y = f of x crosses the x axis, so Newton Raphson Method essentially involves linearization of the function f about the point and approximating the function by means of a straight line, which is the tangent to the curve at that point say x I, f of x i and this generates a sequence of iterates which converts to a root of the equation f of x = 0. So it looks as though starting with some initial approximation and use Newton Raphson Method, generate a sequence of iterates, then the sequences (())(27:31) to converge to a root of the equation but this is not always true.

There are examples of functions whose shape is such that and the starting values are such that the Newton Raphson Method fails to generate iteration which converge to root of the equation, so we shall give an example to demonstrate this statement.

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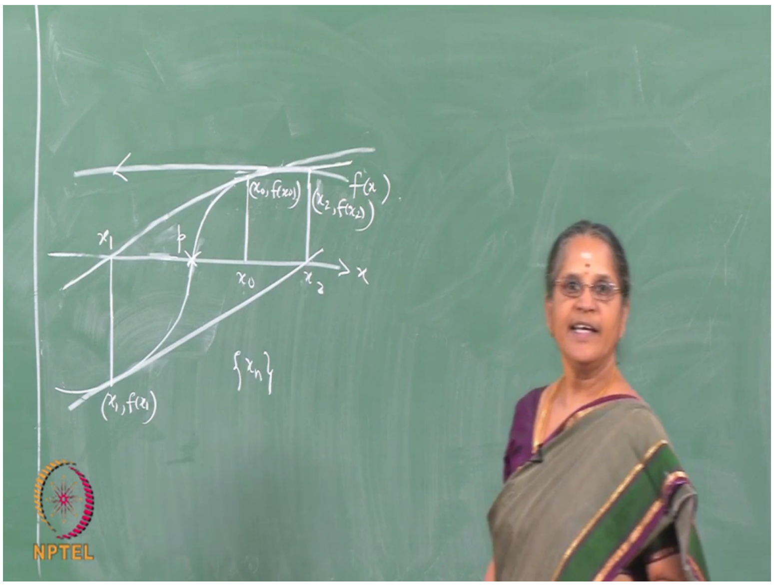

So let us consider this example, we are given y = f of x and we are asked to determine root of this equation using Newton Raphson Method so I have drawn the graph of y = f of x which is this, I start with where does it cross? It crosses the x axis at this point so this is the actual root of the equation. I start with an initial approximation x 0 so I take a point x 0, f of x 0 on this curve and then drop a tangent to the curve and I see that the tangent to the curve needs the x axis at x 1 so that is my next approximation to a root of the equation, so it falls on the other side of this root p. I now take the point x 1, f of x 1 on the curve y equal to f of x and draw a tangent to the curve at that point and see where it crosses the x–axis. So it is at x 2 where it crosses the x–axis, so I take a point x 2, f of x 2 on this curve and then draw for tangent to the curve and I see that the tangent is like this which is parallel to the x axis.

And therefore there are starting values x 0 and functions f of x whose shape is such that the successive iterates generated by Newton Raphson Method fails to converge and therefore, whenever we want to make use of Newton Raphson Method it is important to make some statements about how close our initial approximation x 0 is and we should take care of the shape of the function y = f of x whether it satisfies certain properties because this example clearly illustrates or demonstrates that the Newton Raphson Method may not always converge to root of the equation because it depends on how close our initial approximation is and what is the shape of the graph of the function y = f of x where we are interested in solving the equation f of x = 0. So we will continue with the error analysis and the convergence of Newton Raphson Method in the next class.