Numerical Analysis
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Lecture No 30
Root Finding Methods– The Bisection Method–2

We have discussed bisection method which is an enclosure method for obtaining an approximate solution to a root of the equation f of x = 0. We shall first demonstrate this method by taking an example and then answer the questions that we had posed earlier regarding the convergence of the successive iterates that we get by using bisection method and what should be the starting criteria as we proceed with the bisection method in solving the problem f of x = 0 so let us solve the following problem.

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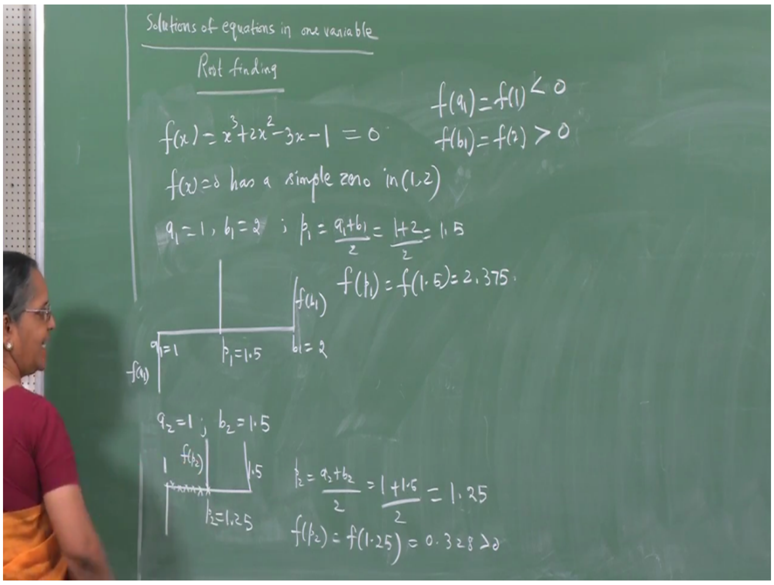

f of x = x cube + 2 x square plus – 3 x – 1 = 0, it is given that f of x = 0 has a simple 0 in the interval 1 to 2; namely this root of equation is of multiplicity 1, so we call that as a simple root or it is a simple 0 of the function f and we are required to determine that using bisection method and we already have the information that this 0 lies in the interval 1 to 2. Here a 1 is 1 b 1 is 2, we determine p 1, which is a 1 plus b 1 by 2 which is 1.5. p 1 is 1.5, we find what is f of p 1 namely f at 1.5 and that turns out to be 2.375. Now we have to determine which part of this interval a 1 to p 1 or p 1 to b 1 what encloses the root of the equation.

That requires the knowledge of the sign of f of a 1 and f of b 1 so I require f of 1 and f of 2 and we observe that f of 1 is negative and f of 2 is positive, so f at 1 is negative f at 2 is positive, so it is clear that a root lies in the interval a 1 to p 1 because f of a 1 into f of p 1 is negative so I take my a 2 to be 1 and b 2 to be 1.5, so my interval is 1 to 1.5 and it encloses the root of the equation and so I compute what p 2 is, p 2 will be a 2 + b 2 divided by 2, so 1 + 1.5 by 2 and that will give me 1.25. I know f of 1 is negative and f of 1.5 is positive, so I compute f at p 2 namely f at 1.25 and that turns out to be 0.328 which is positive and therefore a root lies in this interval 1 to p 2 because f of 1 into f of P2 is negative.

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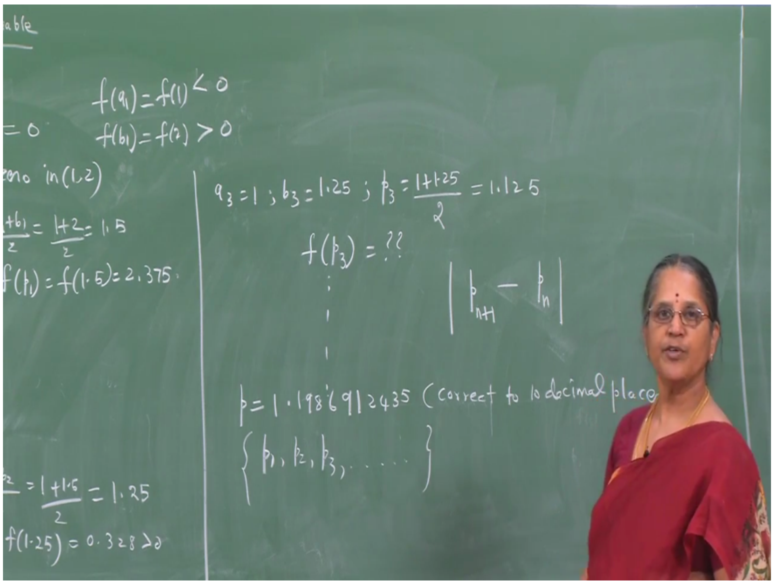

So I call my a 3 as 1 and b 3 as 1.25 and therefore we compute p 3, which is 1 + 1.25 divided by 2 and that turns out to be 1.125. And we find what letter f of P3 is, whether it is positive or negative and so on we continue determine each time where this root lies in which part of the interval this root lies and continue these steps by bisection method and when we continue for some more iterations, we end up with the root to be p = 1.1986912435 correct to 10 decimal places, I want you to work out these steps and then arrive at this solution. So what is it that we have generated? We obtained p 1, p 2, p 3 and so on, so we have performed a number of steps every time having the interval in which a root lies and determining a new interval whose length is half of the length of the previous interval which encloses a root and proceeding in this manner we have obtained a sequence of iterates.

And we have stopped at some point and have said that this is the root correct to 10 decimal places, what does that mean? We have computed the root at n + 1 step and the root at the end step and check that the difference between these 2 satisfies this accuracy and so we have stopped our computations and we have said that this is an approximation to a root of the equation which lies in the interval 1 to 2 of that equation. So now the question is, what is the guarantee that the sequence of iterates converges to a root of the equation, and if it converges where is it that we have to stop, what are the conditions under which we can stop the number of steps that we are using by bisection method so that the solution can be obtained, so we looked into these questions now.

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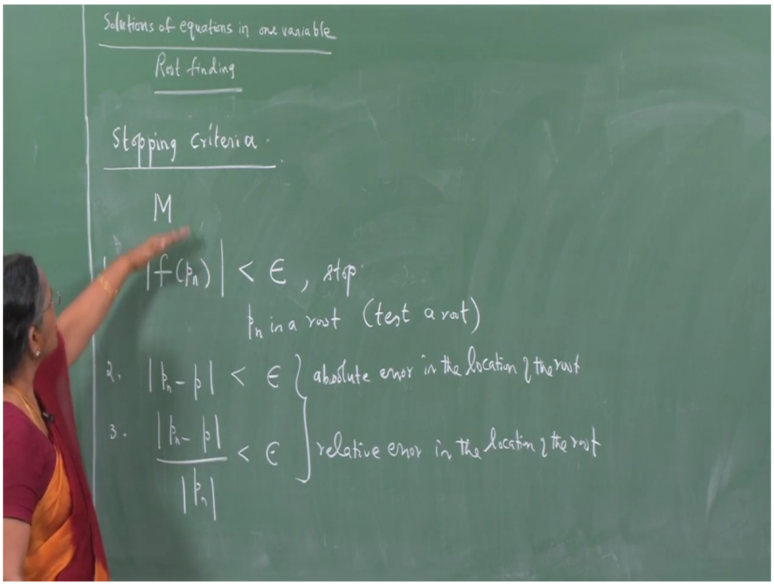

We now indicate the stopping criteria, specify the number of iterations say M iterations that you will have to perform. And therefore perform M iterations of bisection method and stop, there you do not worry about the accuracy of the solution that you have obtained, you just require a crude value of a root of the equation because you are performing some n number of iteration by bisection method that is 1 way of stopping. Is it going to be useful to you, can you do that? Yes, you can use the root that you obtained after M titration by bisection method as an initial approximation to some other numerical technique, which is going to be an iterative techniques for solving the problem f of x = 0. So it is useful if you stop your procedure by bisection method say after M iterations.

Even otherwise when you specify the number of iterations as m, it avoids the computations being repeated again and again and going into a loop that is also an advantage when you specify the number of iterations. The other stopping criteria are as follows namely you stop your computations when you observe that absolute value of f evaluated at p n is less than the prescribed convergence tolerance namely Epsilon, so every time you compute iterate p 1, p 2, p 3, etc say p 1 you check, anyway you compute what is the value of f at p n or f at p i to find out the sign of f of p i so that you can determine an interval which encloses a root of the equation at that step. Check whether in absolute value f of p n is less than Epsilon, if so then stop the number of iterations and write down that p n is a root of this equation, p n is an approximation to root of the equation f of x = 0.

The 2nd criterion that can say is check whether p n – p is less than Epsilon, namely the absolute error is less than Epsilon or thirdly you can check whether the relative error p n – p divided by p n is less than Epsilon. You can use any one of the 3 stopping criteria to stop the iterations or the computations in bisection method, what does this condition indicates? This indicates whether you are closer to a root of the equation. What about these 2 conditions? These 2 conditions check whether the absolute error in the location of the root is less than Epsilon, and this test whether the relative error in the location of the root is less than Epsilon, so these 2 conditions focus on the absolute and the relative error in the location of the root, whereas this test whether p n is the root of the equation because what do we want, if p is the root of the equation, f of p must be 0.

So by demanding that absolute value of f of p n is less than Epsilon I check whether p n qualifies to be the root of the equation. So, in addition to any one of these stopping criteria also specify the number of iterations M that will avoid going into a loop during the computations. Now that we know how to stop or where to stop when we perform bisection method for solving the equations, we will now discuss whether the sequence of iterates that we generate using bisection method converges.

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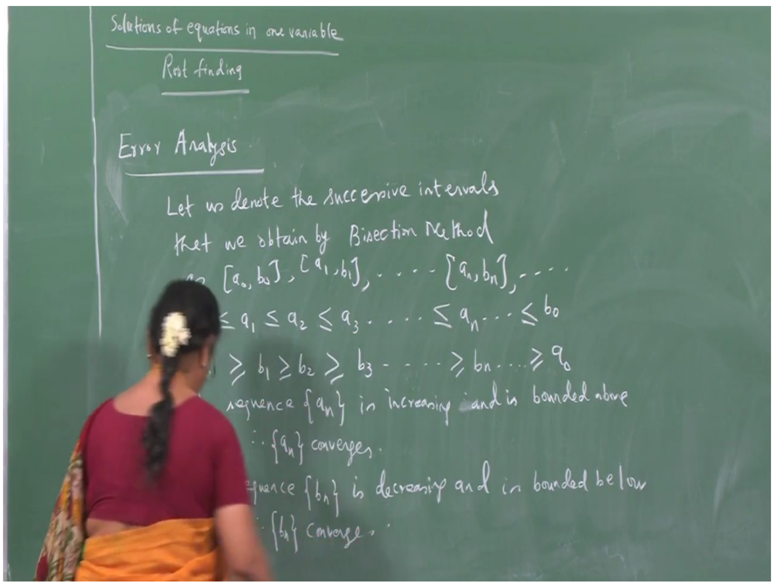

We perform error analysis to check that the sequence of iterates that we generate by bisection method converges to a root of the equation. So let us denote the successive intervals that we obtained a 0, b 0, a 1, b 1, etc, a n, b n and so on. So we observe that a 0 is less than or equal to a 1 is less than or equal to a 2 and so on is less than or equal to return a n and it is less than or equal to b 0. On the other hand, b 0 is such that it is greater than or equal to b 1, greater than or equal to b 2 and so on greater than or equal to b n and this will be greater than or equal to a 0, so the endpoints of the successive intervals that we determine satisfies this condition. So what does this indicates, this indicates that the sequence a n is increasing and is bounded above and therefore what can you can close the sequence, a n converges. What do you observe about the sequence b n? The sequence b n if you look at this we see that it is decreasing and is bounded below and therefore we conclude that the sequence b n also converges.

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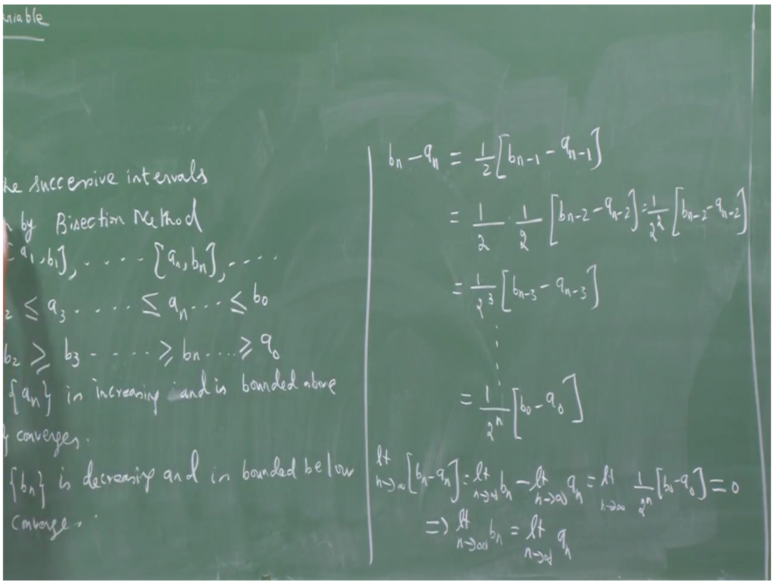

So let us write down what is b n – a n, namely the length of the interval of the nth step. This is half of b n – 1 – a n – 1, so in the next step b n – 1, a n – 1 will be half of b n – 2 – a n – 2, so it is 1 by 2 square into b n – 2 – a n – 2 and that will be 1 by 2 cube times b n – 3 – a n – 3. So if you continue this, then it is going to be 1 by 2 to the power of n times b 0 – a 0 and therefore, when I take the limit as n tending to infinity of b n – a n then it is limit n tending to infinity of b n – limit n tending to infinity of a n. And we know that a n and b n converge right, what is it that is equal to limit n tending to infinity of 1 by 2 power n into b 0 – a 0, so what happens to 1 by 2 power n as n tends to infinity that is 0. So what does this gives, this gives limit as n tending to infinity of b n is the same as limit n tending to infinity of a n.

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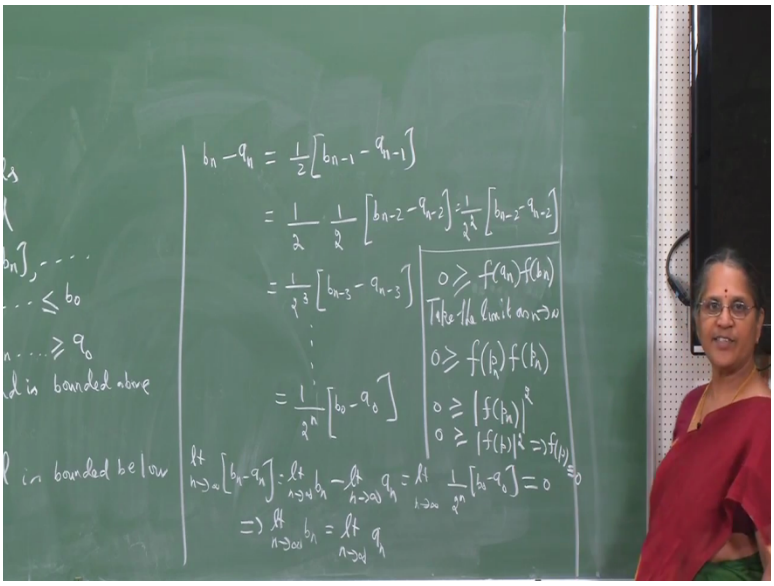

So here we have shown that the sequences a n and b n converge and here we have shown that they converge to the same limit, so let us now apply the limit the inequality, which inequality? We are applying bisection method for solving f of x = 0 and we determine in the beginning an interval within which a root lies and this is because of the consequence of Intermediate Value Theorem namely f of a into f of b is less than 0 and at each step of our iterations we ensure that we have an interval which is such that f of a n into f of b n is negative and therefore, 0 is greater than or equal to f of a n into f of b n, this is so at each step n = 0, n =1, et cetera et cetera, so we take the limit as n tends to infinity here then what will we have? 0 is greater than or equal to limit n tending to infinity of f of a n into limit n tending to infinity f of b n.

And we know that they converge to the same limit suppose say they converge to f of p n f of p n and this converges to f of p, so we have 0 greater than or equal to modulus of f of p n the whole square and so as n tends to infinity we have 0 greater than or equal to modulus of f of p the whole square from which it follows that f of p is identically 0 and therefore the sequence of iterates that we get namely p 1, p 2, etc p n converge to p which satisfies the equation f of p = 0 and so the sequence of iterates converge to a root of the equation f of x = 0 when we apply bisection method in solving f of x = 0. So the error are now, this clearly states that the sequence of iterates that we generate converge to a root of the equation f of x = 0.

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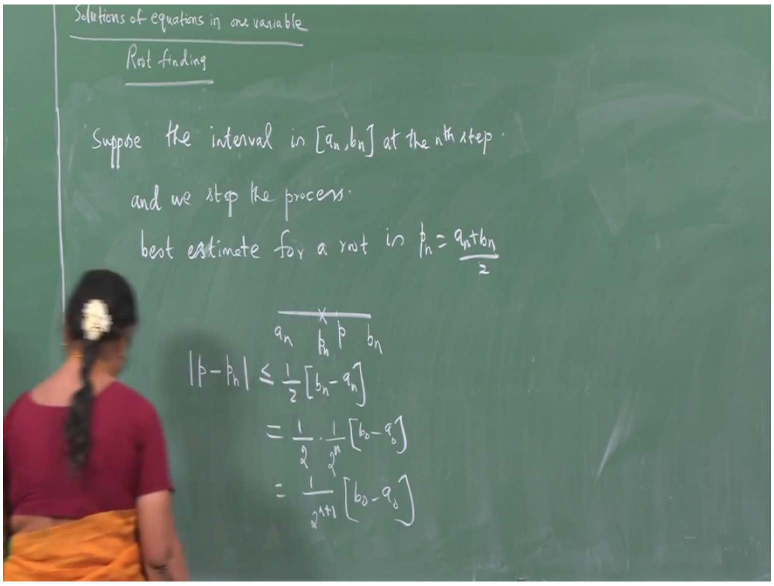

Let us now find out the theoretical bound on the error that we connect at each step. So to do that we consider the interval a n b n at the end step and we stop the process at this step then the best estimate for root is what, the midpoint of this interval namely a n + b n by 2, so there is an interval a n, b n in which a root lies and I do not want to continue beyond this, I stop the iterations and then I estimate what is an approximation to the actual root and that is p n that is a n + b n by 2. I clearly know that a root of the equation lies in this interval suppose say this were p that is the actual root. What I have determined is an approximation to this root p so I know that p – p n the distance between them is surely less than or equal to half of b n – a n that is clear, this distance is surely less than or equal to half of this interval length namely b n and – a n.

And therefore this is equal to 1 by 2 into I know that b n – a n is 1 by 2 to the power of n times b 0 – a 0, we started with a 0 b 0 and every time I reduce the length of the interval to be such that it is half the length of the interval of the previous time and therefore, this gives me 1 by 2 power n + 1 into b 0 – a 0 thus we have shown the following result.

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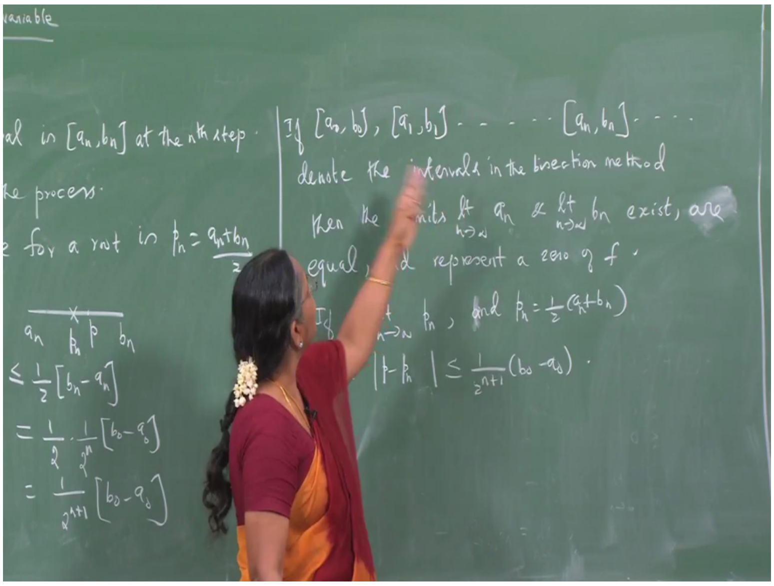

If a i b i are the intervals that we have generated during our computation of solution of f of x = 0 by bisection method, then the limit a n and limit a n as n tends to infinity exists and they are equal we have already shown that and it represents a 0 of f that has also been proved. In addition, if limit n tending to infinity of p n is p, where p n is the midpoint of the interval a n b n at the end step then we have shown that mod p – p n is less than or equal to 1 by 2 power n + 1 into b 0 – a 0, this is the interval with which we have started. So this gives you a theoretical error bound on the approximation that we have obtained to the exact root p of the equation f of x = 0. So let us see how this theoretical bound can be effectively used, when we know what is the accuracy with which we have to perform our computations, so let us consider the following problem.

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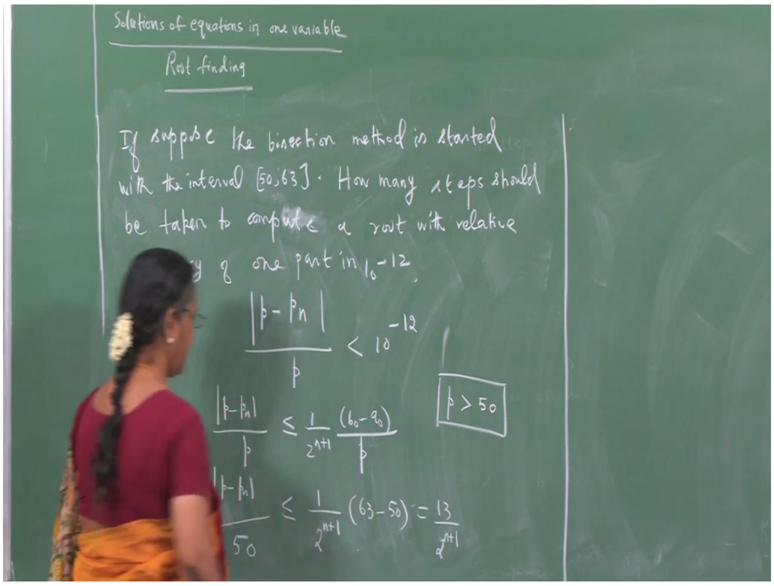

Suppose the bisection method is started with the interval 50 to 63, you are required to determine the number of steps that you should use to compute a root of the equation f of x = 0 with relative accuracy of 1 part in 10 to the – 12 namely how many steps of bisection method that you need to use so that when you compute at the end an approximation p n to be root of this equation, you end up with a result p n which is such that the relative error is less than 10 to the – 12, so you need to find out what this error is. And we know that p is a root of the equation and it is given that it lies in the interval 50 to 63 and therefore p is greater than 50.

So I want the relative error to be less than or equal to 1 by 2 power n + 1 into b 0 – a 0 divided by p that is mod p – p n by 50 to be less than or equal to 1 by 2 power n + 1 into 63 – 50 by 50 which is 13 by 50 into 2 to the power of n + 1. But what is it that we want? We want to find the number of steps n such that the root that we compute is accurate with relative accuracy of 1 part in 10 to the – 12.

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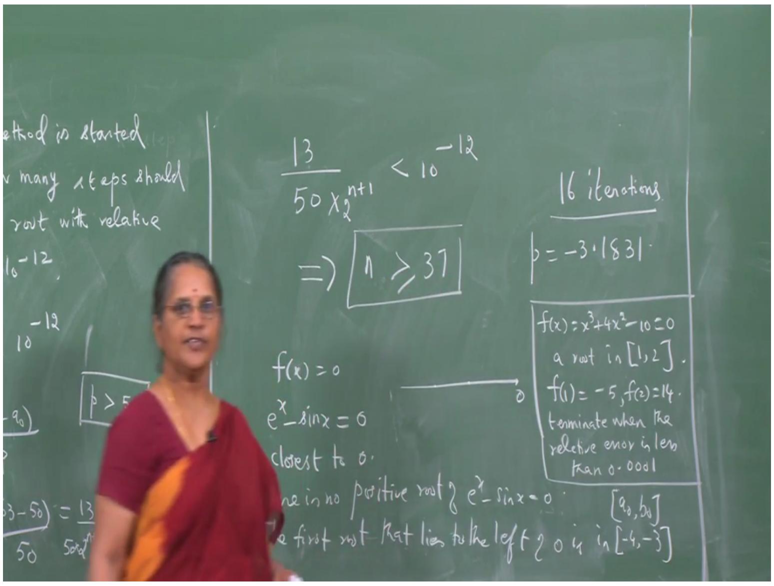

So we want that n which satisfies the condition that 13 by 50 to 2 to the power of n + 1 is less than 10 to the – 12, so when you determine this n you can show that it is greater than or equal to 37 so you require 37 iterations to compute root of the equation with relative accuracy of 1 part in 10 to the – 12, so the theoretical bound on the error that we have obtained helps us to find out the number of iterations that are required to get a result with relative accuracy which is specified. I shall give you some more problems which you can try later and then obtain the solution using bisection method. We will try to work out the solution of the equation f of x = 0, find the root of this equation e power x – Sin x = 0 closest to 0.

You can observe that when no positive root of the equation e power x – sin x = 0, the 1st root that lies to the left 0 is in the interval – 4 to – 3. You can check that f of – 4 into f of – 3 is negative that is the root which lies closest to the origin and it lies in the interval – 4 to – 3, there is no positive root of this equation. So start with this interval, call it as a 0 b 0 and work out the details and confirm say about 16 iterations and then obtain approximation to root of the equation show that an approximation to a root is given by – 3.1831. Let us consider another problem, also solve the equation f of x = x cube + 4 x square – 10 = 0 given that it has a root in the interval 1 to 2 since f of 1 is – 5 and f of 2 is 14 and you are asked to compute the solution such that you stop your iterations or terminate when the relative error is less than 0.0001.

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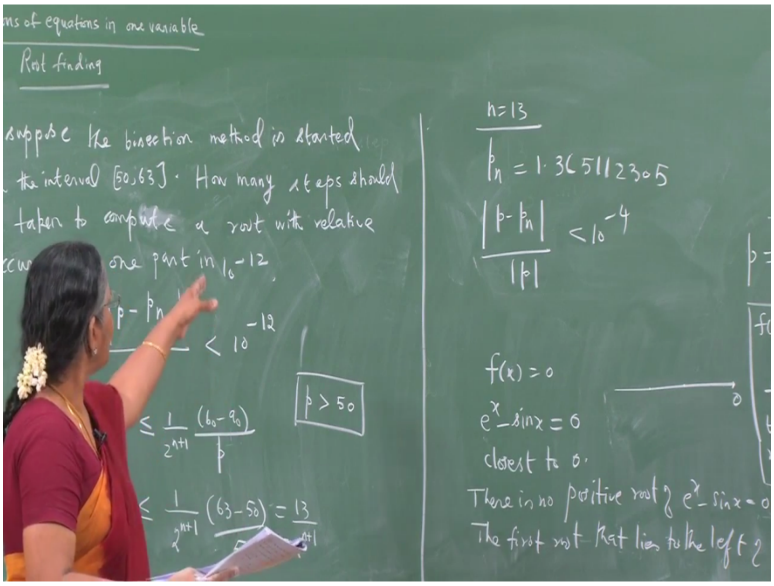

P n at the 13th step comes out to be 1.365112305 such that modulus of p – p n by mod p is less than 10 to the – 4 satisfying the requirement, please try to work out the details of solutions of both these problems. Let us look into the bisection method and observe certain details of this method, Bisection method 0 of the equation f of x = 0 and not the 0 of the equation f of x = 0, namely when you find an interval in which a root lies because f of a into f of p is negative, there may be more than one 0 of f of x = 0. Bisection method determines a 0 in that interval but a prior we do not know which 0 that its computes. Also, there are cases where bisection method chooses either the left interval in which a root lies or the right interval in which the root lies.

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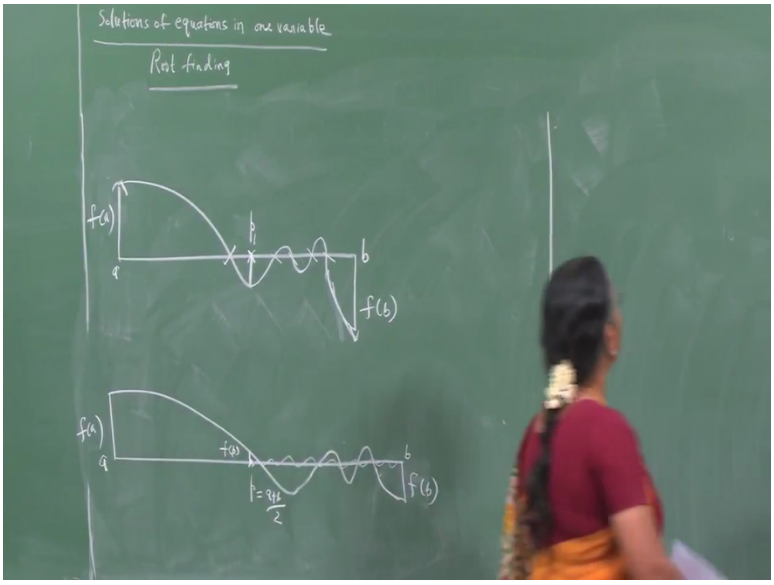

Say Suppose I consider the graph of f to be given by the following, where f of a is positive and f of b is negative. If suppose the graph is like this, then since f of a into f of b is negative, I determine the midpoint a + b by 2 and call it as say p 1 and I evaluate f at p 1 and see the sign of f of p 1. The graph is something like this and so I observe that f of p 1 is negative and therefore although there are 1, 2, 3, 4, 5 roots of this equation f of x in this interval a b, the bisection method chooses the left interval and computes that root which lies between in the left interval. Suppose I consider the graph of a function to be like this such that f of a is positive and f of b is negative and I compute therefore the midpoint of this interval say it is this which I call as a + b by 2 and I look at the sign of f of p, the graph tells me that f of p is positive and therefore the bisection method will choose the right interval and determine a 0 of the equation f of x = 0.

We observing this case that there are 5 zeros in that interval, it has chosen this interval in which a 0 lies and we start applying bisection method, we do not know at the priory to which root our iterations will converge, it will converge to one of the roots of the equation. Then if we look at the stopping criteria that we have already given, not all of the stopping criteria will be satisfied in all the cases, one of the conditions may be satisfied; the other one may not be satisfied and so on, we just illustrate this by the following examples.

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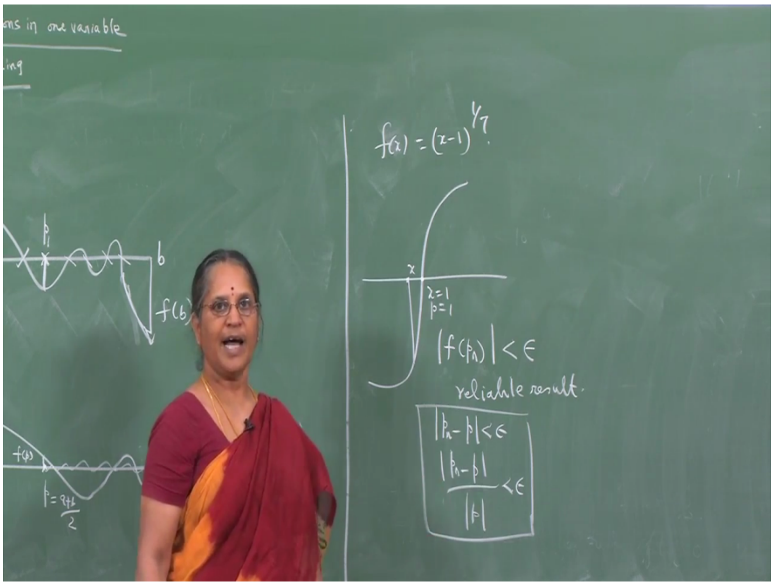

Suppose I consider the equation f of x = x – 1 to the power of 1 by 7, so it is clear that this equation has a root at x = 1. So if I draw the graph of this function, then I observe that the graph of the functions is almost vertical in the neighbourhood of root of this equation. So for this, modulus of f of p and less than Epsilon is chosen as the stopping criteria then that will give you a reliable result rather than choosing any of the other 2 stopping conditions namely the absolute error p n – p is less than Epsilon or the relative error p n – p at p is less than Epsilon, what is the reason? Suppose say I choose some x which is very close to 1, what happens all the x is very close to root of this equation namely p = 1.

What about f of x, f of x here absolute value is not close to 0 so the other two conditions may not give us a reliable result if we use the stopping condition. So in this case, this when used as a stopping criteria will give us a reliable result, so we have a case where one of the conditions produces a reliable result whereas, the other two conditions cannot be taken as giving us a reliable answer.

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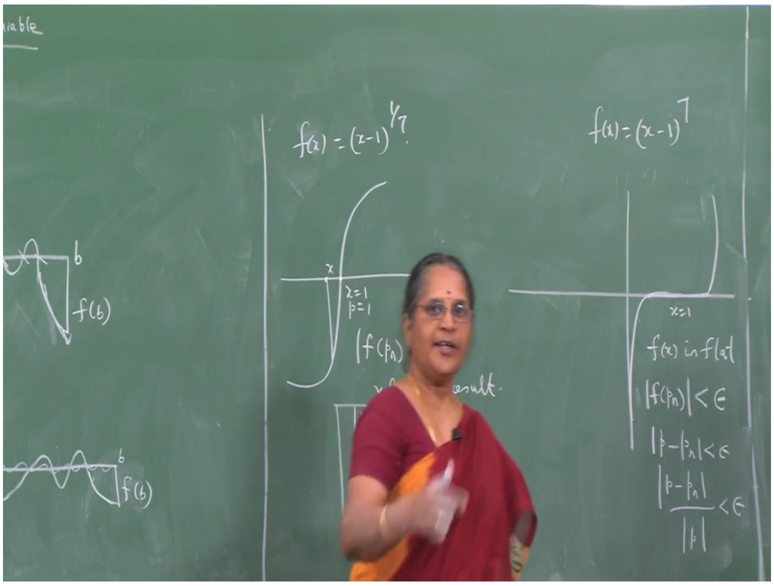

Let us now consider another example, f of x = x – 1 to the power of 7 in this case if we draw we again see that x = 1 is a root of that equation. If we draw the graph of this function, we observe that f of x is flat, in the near the root of the equation namely x = 1. So in this case terminate the iteration by satisfying the conditions that f of p n is less than Epsilon in absolute value will not guarantee reliable a solution because you have a region or an interval where the draft of f is flat and therefore, this cannot be used as a starting criterion. On the other hand, any one of the other 2 conditions namely the absolute error is less than Epsilon or the relative error less than Epsilon, which are based on the correctness of location of the root of the equation will give you the final answer, which is going to be reliable.

And therefore not all cases are such that all the 3 stopping conditions will be satisfied, some of them may fail in some cases. So these are some observations when we find root of the equation f of x = 0, so care must be taken to find out what the stopping criterion is that is going to give us a reliable result and then the computations will have to be carried out. So let us now look at the advantages and disadvantages of bisection method, the advantages are as follows; it is a very simple method, it is easy to apply and secondly it always encloses a root of the equation in an interval and the successive iterates that you generate converge to a root of the equation all these have been established in our discussion in this class.

And it is an enclosure method always enclosing a root of the equation, so we are guaranteed of the final result that we have an approximate very close to the root of the equation correct to the desired degree of accuracy. What are the disadvantages? The procedure is a slow procedure, secondly you may have to do a large number of iterations before any of the stopping criteria that you want to use is satisfied, so the number of iterations required to satisfy your demand may become very very large.

Thirdly, a good approximation to a root of the equation has been missed out by you in the middle during your computations and therefore, we need to look for some more numerical methods with the help of which we can determine an approximation to a root of the equation such that it is a faster method namely it converges to a root of the equation very quickly as compared to the convergence rate of this bisection method, so we will continue with such method in the next few classes.