Numerical Analysis
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Lecture No 28
Numerical Solution of Ordinary Differential Equations–11
Boundary Value Problems Shooting Method

In the last lecture we understood the shooting technique for solving linear boundary value problem with Dirichlet boundary condition; we would demonstrate this method for some examples in this class.

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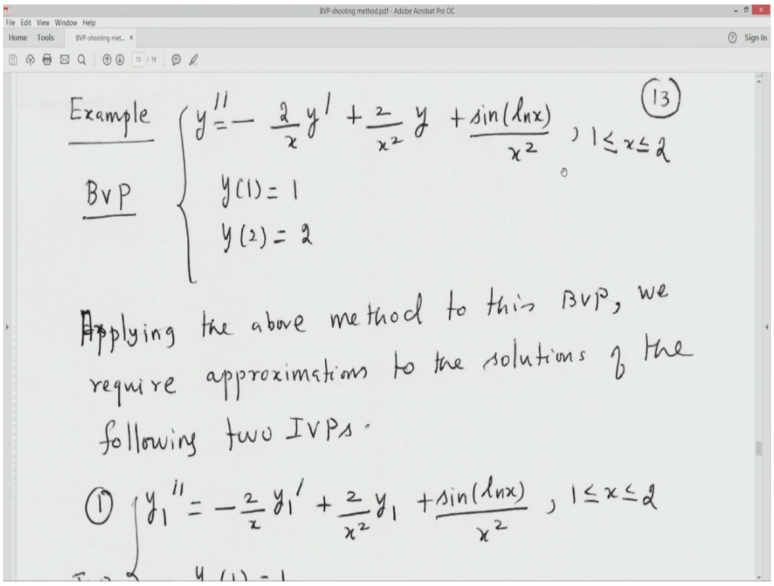

Let us consider a linear boundary value problem which is governed by this differential equation for x in the interval 1 to 2. The Dirichlet boundary conditions are y of 1 is 1, y of 2 is to 2, so we would like to apply shooting technique to solve this linear boundary value problem, which in turn requires approximation of solutions to two initial values problems, so we must define those 2 initial value problems let us do that. For the 1st initial value problem, the governing differential equation is the same as the governing differential equation for the linear boundary value problem namely a non–homogeneous differential equation, so we have written down the equation here in the interval 1. What should be the initial condition for the 1st initial value problem?

Take the left end condition that it arrived in the boundary value problem and use this as an initial condition for the 1st initial value problem, so it is y 1 at 1 equal to 1. Since it is the 1st initial value problem, I call the unknown as y 1, what should be y 1 prime at this initial point

1? It can be arbitrarily prescribed, so we prescribe it at 0 so that defines the 1st initial value problem so it is clear, let us now move to second initial value problem. What is it? The 2nd initial value problem corresponds to the homogeneous differential equation, which is obtained from the given linear boundary value problem, this is the non–homogeneous term so I must omit this term and consider the corresponding homogeneous differential equation that governs the boundary value problem.

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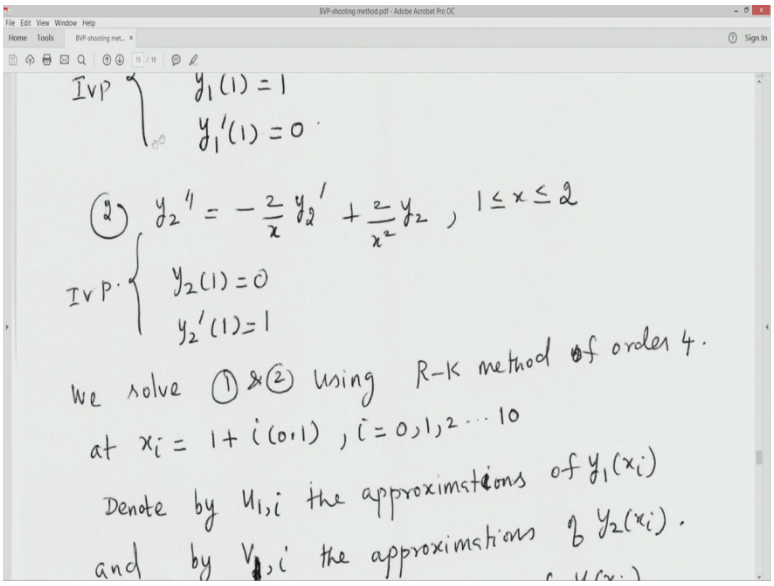

So equation that governs the 2nd initial value problem is going to be y 2 double prime = – 2 by x y 2 prime + 2 by x square y 2 for x in the interval 1. What about the initial conditions here? We have already seen that the solution y 2 for this initial value problem must satisfy the initial condition that y 2 at the initial point 1 must be 0 and y 2 dash at 1 can be arbitrarily prescribed but it should be nonzero value. So we prescribe y 2 dash at one to be equal to 1 so we have described the 2 initial value problems appropriately and denote the solutions of these problems respectively by y 1 and y 2. So let us solve these 2 initial value problems by Runga Kutta method of order 4 so what do we have to do? We have to divide the interval 1 to 2 into a number of equal sub intervals, so let us divide the interval into 10 equal sub intervals.

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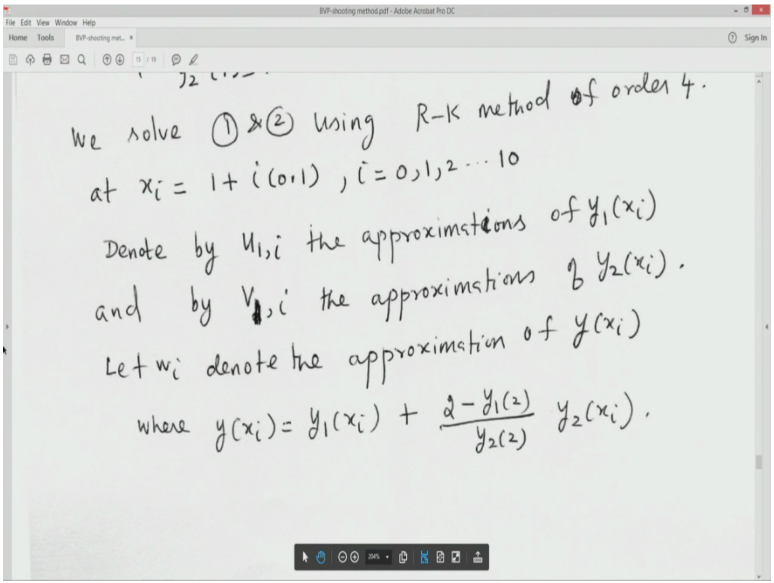

What is the width of each subinterval, it is going to be 2 – 1 by 10, so 1 by 10 will be the width of each of the subintervals and so the points in the interval 1 to 2 will be 1, then 1 + 1 by 10 that is 1.1, 1.2, 1.3, etc up to 1.9 and finally 2 these are the points in the interval 1 to 1 at which we seek an approximation to solution of each of these initial value problems, how do we do it, we have to use Runga Kutta method of order 4 to obtain the solution. So I shall denote by u 1 i their approximation of y 1 at x i and by v 1 i the approximation of y 2 at x i, and denote by w i the approximation of y at x i. What is y at x i, we have already seen we have to generate the solution of boundary value problem by a linear combination of the 2 solutions of the 2 initial value problems namely y of x is y 1 of x + k times y 2 of x, where k is given by beta – y 1 b by y 2 at b, so that is what I have written here.

So if y of x i is the solution at the point x i in the interval 1 to 2 then y of x i is y 1 of x i plus beta, what is beta? It is y at 2, just see what is prescribed as y at 2 in the boundary value problem, y at 2 is prescribed to be 2 so beta is 2 – y 1 at b, what is b, b is the other endpoint of the interval namely 2, by y 2 at b that is y 2 at 2. Do we have a knowledge of these values y 1 at 2 and y 2 at 2, that is what we are determining now by solving the initial value problem which we have described above, so the solution will give us y 1 of 2 and y 2 of 2, we can use here and find out what y at x i is. We said we are solving the 2 initial value problems by Runga Kutta method of order 4, so we have listed the solution of the 2 initial value problems in the following table.

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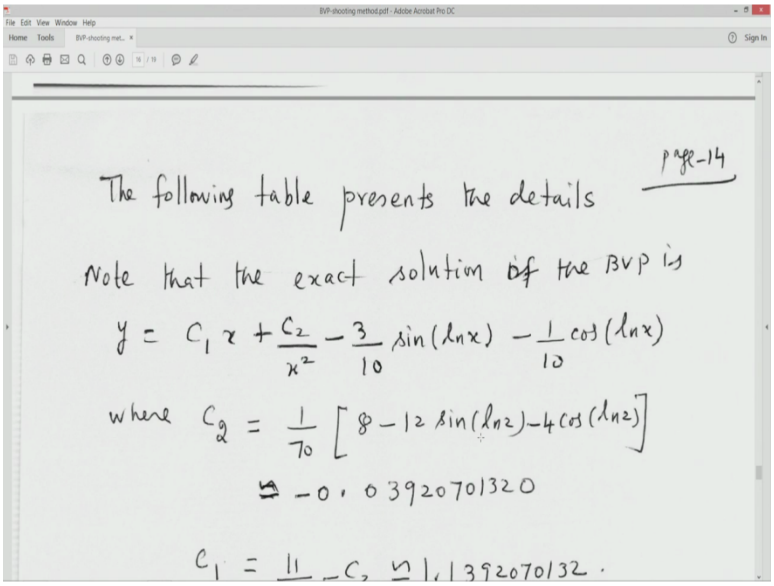

Before that let us see the exact solution of the boundary value problem, so it is easier to determine the exact solution of the linear boundary value problem, which is governed by the 2nd order differential equation so it turns out to be y is C 1 x plus C 2 by x square – 3 by 10 into sin of log x –1 by 10 cos of log X, where C 2 and C 1 are given.

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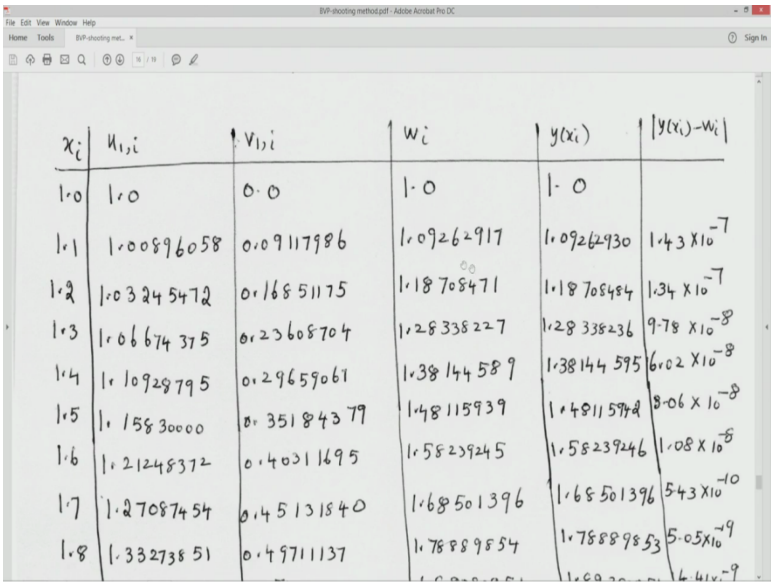

x_i	$u_{1,i}$	$v_{1,i}$	w_i	$y(x_i)$	$\lvert y(x_i) - w_i \rvert$
1.0	1.0	0.0	1.0	1.0	
1.1	1.00896058	0.09117986	1.09262917	1.09262930	1.43×10^{-7}
1.2	1.03245472	0.16851175	1.18708471	1.18708484	1.34×10^{-7}
1.3	1.06674375	0.23608704	1.28338227	1.28338236	9.78×10^{-8}
1.4	1.10928795	0.29659061	1.38144587	1.38144595	6.02×10^{-8}
1.5	1.15830000	0.35184379	1.48115939	1.48115942	8.06×10^{-8}
1.6	1.21248372	0.40311695	1.58239245	1.58239246	1.08×10^{-8}
1.7	1.27087454	0.45131840	1.68501396	1.68501396	5.43×10^{-10}
1.8	1.33273851	0.49711137	1.78889854	1.78889853	5.05×10^{-9}

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1.1	1.00896058	0.09117986	1.09262917	1.09262930	1.43×10^{-7}
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1.7	1.27087454	0.45131840	1.68501396	1.68501396	5.43×10^{-10}
1.8	1.33273851	0.49711137	1.78889854	1.78889853	5.05×10^{-9}
1.9	1.39750618	0.54098928	1.89392951	1.89392951	4.41×10^{-9}
2.0	1.46472815	0.58332538	2.000...0	2.00...0	

So we give the table in which the point x i at which the solutions are obtained are listed, so namely starting from 1 and moving in steps of h which is 0.1 we have these points x i at which the solution are being determined for the 1st initial value problem whose solution we have determined, we have denoted by y 1 i, the 2nd initial value problem v 1 i and y of x i namely w i and the exact solution is listed in this column and the last column gives you the absolute error namely the difference between the exact solution and the approximate solution. So this column gives you the solution of the 1st initial value problem at the x i, this gives you the solution of the 2nd initial value problem.

And we compute the linear combination of these 2 solutions and write down the solution w i in this column and we compare this with the exact solution and compute the absolute error and we see that our approximate solution matches very well with the exact solution and the absolute error is negligibly small. The reason for this greater accuracy in this problem is because we have used Runga Kutta method of order 4 where the error is of order of h to the power of 4, and this result in greater accuracy of the approximate solution that we have obtained for this boundary value problem.

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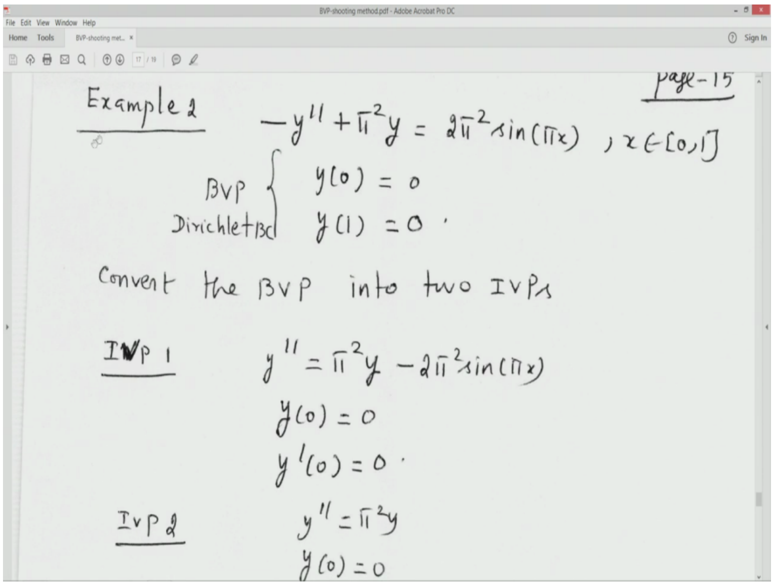

Let us consider another example, namely consider the boundary value problem which is – y double prime + Pie square y = 2 Pie square into sin Pie x, which is a linear boundary value problem where x belongs to the closed interval 0 1. And now what are the boundary conditions? They are Dirichlet boundary conditions and are prescribed as y of 0 = 0 and y at 1 is also 0, so we want to apply shooting technique to solve this problem so we need to write down 2 initial value problems so let us write them out. The 1st initial value problem has governing equations to be given by the non–homogeneous differential equation for the boundary value problem so I have written down the differential equation as it is. What about the initial condition?

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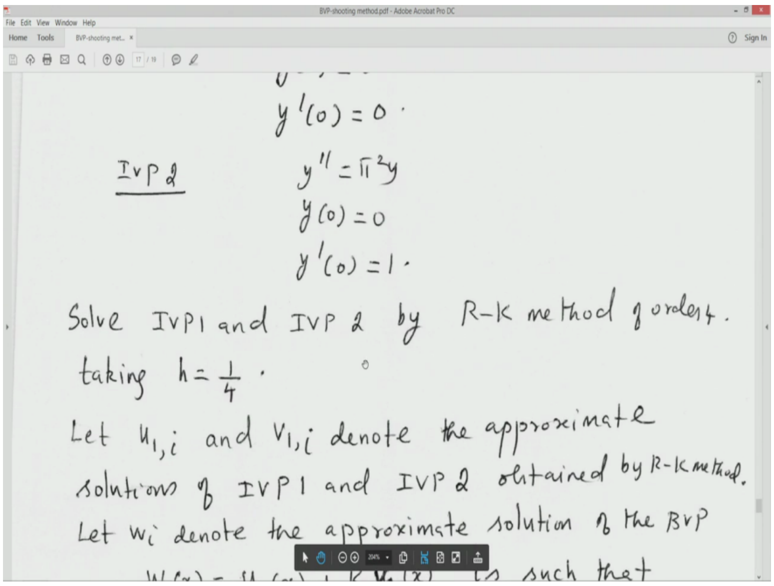

Y of 0 should be taken as 0 and it is the same as what is prescribed in the boundary value problem, so I have that condition written down here. What about y Prime 0, it can be arbitrarily prescribed so we prescribe it to be 0 that completes the definition of the 1st initial value problem, what about the 2nd initial value problem? We should consider the corresponding homogeneous differential equation for the boundary value problem, namely put R of x = 0 and take the corresponding homogeneous differential equation which is this. What about the initial condition for this initial value problem? y of 0 is 0 but y prime 0 must be prescribed as a non–zero value so let us take y prime 0 to be equal to 1, so the 2nd initial value problem is also defined.

Now that the 2 initial value problems are known to us, we again solve these 2 initial value problems by Runga Kutta method of order 4. So what should we do? We should divide the interval in which the differential equation is defined namely 0 to 1 into number of equal subintervals, so let us take the number of equal subintervals to be 4 and therefore, the step size will be 1 – 0 by 4, so h is 1 by 4. So as before we denote by u 1 I and v 1 i the approximate solutions of the 1st and 2nd initial value obtained using Runga Kutta method of order 4. Then if the approximate solution of boundary value problem denoted by w of x, then w of x is given by u 1 of x + K into v 1 of X. And how do you determine this K, it is determined by imposing the condition that w at the other endpoint 1 must be equal to whatever that is prescribed in boundary value problem.

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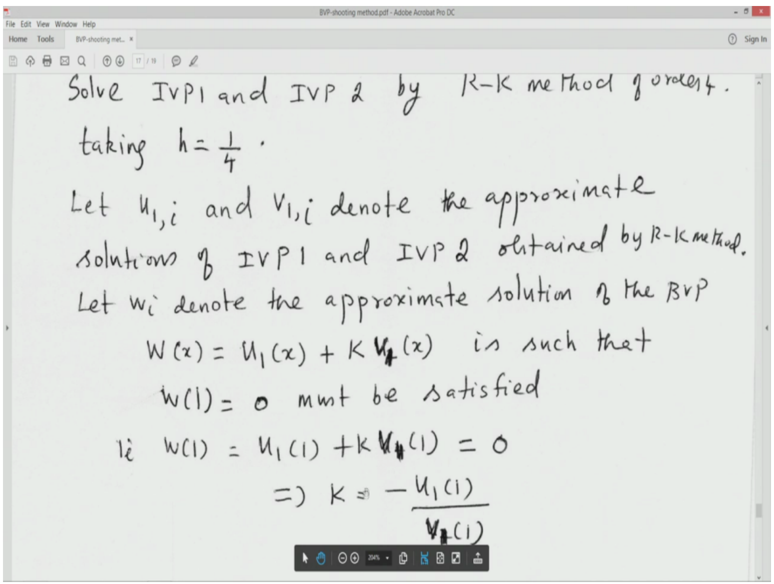

What is prescribed in the boundary value problem? It says y at 1 should be equal to 0, we want to find K such that w of 1 should be equal to 0 so we compute w of 1 that is u 1 at 1 + K into v 1 at 1, but this must be 0 that it immediately give us K to be – u 1 at 1 and v 1 at 1. Do we know these values, yes we solve the 2 initial value problems by Runga Kutta method starting from the initial condition prescribed at 0 and moved in steps of h which is 1 by 4 and go to the point x = 1 and obtain the approximations to the solution of the 2 initial value problems at that point namely x = 1, which we have denoted by u 1 of 1 and v 1 of 1.

So this will be known to us when we have obtained the solution of the initial value problem and therefore K can be immediately obtained and K can be substituted here and u 1 of x plus K times v 1 of x will give you an approximation to the solution of the boundary value problem at any x in the interval 0 to 1 in particular at the x i that we have chosen in the interval 0 to 1, the approximations to the solution of the boundary value problem can be obtained, so let us see what the solution is.

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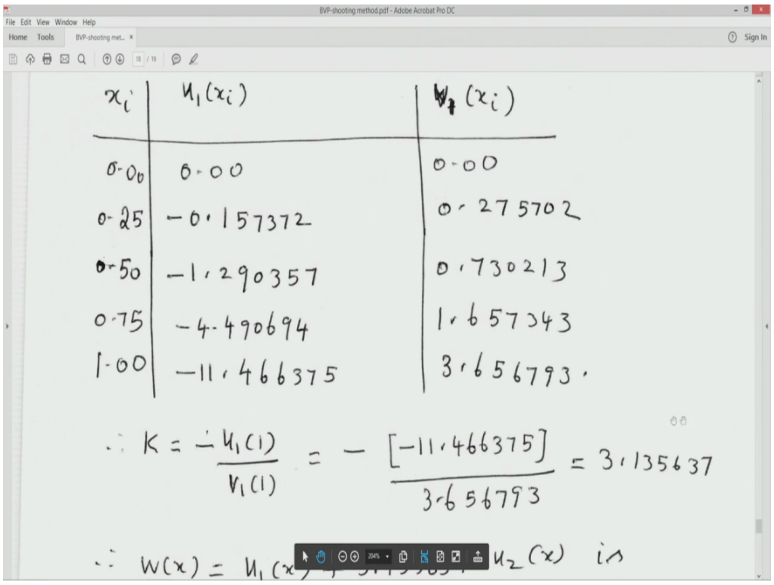

So the results are presented in the following table namely; the x i are such that they are equally spaced and the step size is 1 by 4, so the x i are 0, 1 by 4, half, 3 by 4 and 1, the solution of the 1st initial value problem denoted by u 1 of x i as specified at these x i and that of the 2nd initial value problem denoted by v 1 of x i are given here at these x I and we said that K is the value of u 1 at 1 by v 1 at 1 into – 1. So u 1 at 1 if this and v 1 at 1 is this and so this gives you the value of K. So having determined K, what is the approximate solution to the boundary value problem?

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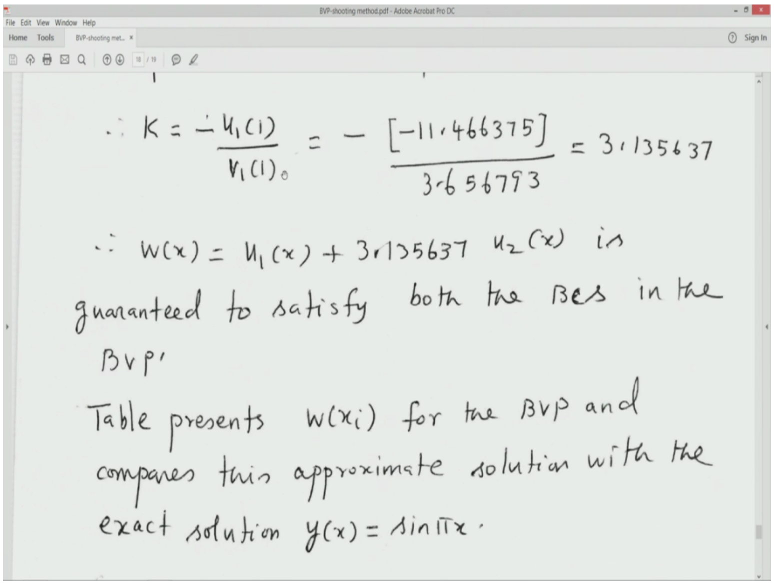

W of x is u 1 of x plus k times u 2 of x and it is guaranteed to satisfy both the party conditions in the boundary value problem because we have already discussed these details. So the following table presents the approximate solution of the boundary value problem and the exact solution y of x i and computes the absolute error.

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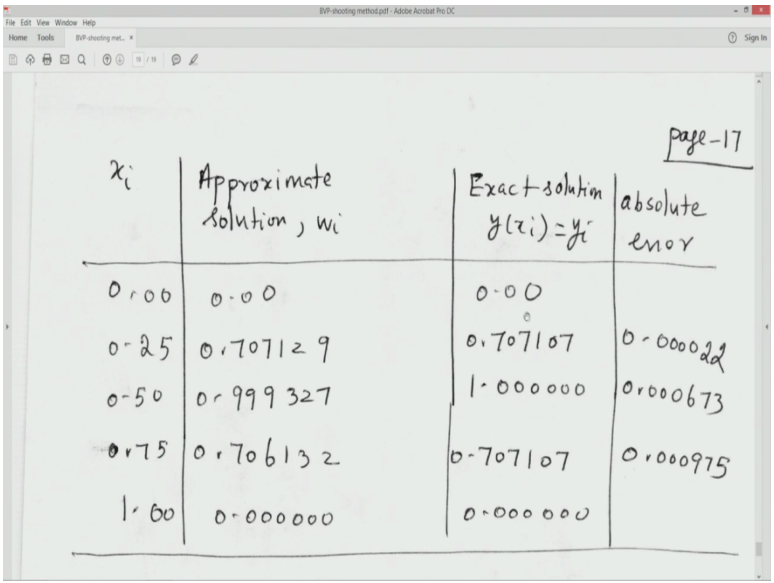
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x_i	Approximate solution, w_i	Exact solution $y(x_i) = y_i$	absolute error
0.00	0.00	0.00	
0.25	0.707129	0.707107	0.000022
0.50	0.999327	1.000000	0.000673
0.75	0.706132	0.707107	0.000975
1.00	0.000000	0.000000	

What is the exact solution of this problem? This y of x = Sin Pie x, so that column for the exact solution gives Sin Pie x i at these points x i, so they are written down here. So we now compare our approximate solution at each of the x i with the exact solution at that x i and we observe that the accuracy is quite good, there is an excellent accuracy or there is an excellent matching between the approximate solution and the exact solution and the absolute error is negligibly small. As mentioned above this is because of Runga Kutta method that we have used if the method is of order 4, and because the error is of order h to the power of 4 the solution that we have got for approximation matches well with the exact solution of the problem obtained any x i in the interval 1 to 2.

In spite of the fact that the step size h that is chosen by us will be 1 by 4 is not very small, we have been able to show that the accuracy is excellent. So by solving the 2 initial value problems, which are defined as described and taking a combination of these 2 solutions being are being able to generate the solution of the boundary value problem and this technique this shooting technique for solution of linear boundary value problem which is governed the Dirichlet boundary conditions. One can also solve the linear boundary value problem with Neumann or Robin boundary conditions similar to what we have done for Dirichlet boundary conditions; we shall see these details in assignment problems.

Coming to non–linear boundary value problems using shooting technique is out of scope of this course and therefore we stop our discussion on shooting technique for the linear boundary value problems and this completes our discussion on the numerical solution of ordinary differential equation and let us summarise the different techniques that we have used in developing or in obtaining a medical solution of ordinary differential equation. We described first what an initial value problem for the 1^{st} order equation subject to an initial condition and obtained the solution of initial value problem and we described a number of numerical methods namely Taylor series method, Eulers method, Runga Kutta method of different orders and any of these single step methods can be used to obtain solution of an initial value problem.

And then we moved on to multistep method, we developed Aden molten multistep method, Aden Bashforth multistep method and Milne multistep method, which requires information about the solution at a certain number of previous point in order to obtain the solution at the specified point. And we use these multistep methods in the form of predictor connector methods and predicted the required solution and then used the corrector to correct this predicted value and successively use this corrector obtain the solution correct to the desired degree of accuracy. So these are all different techniques for solving initial value problems and then we moved to boundary value problems and we focused our attention on linear boundary value problems and we looked into the linear boundary value problems with the Dirichlet or Neumann or Robin type of boundary conditions.

And described finite different techniques of solving the boundary value problems with any of these types of boundary conditions, and finally we have taken up a shooting technique for solving linear boundary value problem. And the basic idea in the shooting technique is to convert the given boundary value problem into 2 initial value problems with appropriate initial conditions and obtaining a solution of this initial value problems using any of the techniques that we have described earlier and then combine these 2 solutions and generate a solution of the boundary value problem. When conditions on P of x and Q of x that appears in the governing differential equation on the boundary value problem are satisfied as given the theorem, the body value problem has a unique solution and this can be determined using a shooting technique, this is what we have discussed in this class.

So these are all different techniques for solving either an initial value problem or a boundary value problem and with these discussions we will close the section on numerical solution of differential equations. In the next class we shall take up the topic on numerical solution of equation of the form f of x = 0, where f of x is either purely algebraic or transcendental and we shall develop a number of direct as well as iterative techniques for solving an equation of the form f of x = 0 and the form error analysis and see which of the methods that we described is going to score over the other methods and what is the advantage of considering different methods, which are either direct methods or iterative methods, we shall continue this discussion in the next class.