Numerical Analysis
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Lecture No 27
Numerical Solution of Ordinary Differential Equations-10-Boundary Value Problems
Shooting Method

In the last class we saw how we can solve a boundary value problem with Dirichlet or Robin or Neumann boundary conditions using a finite difference method. In this class we shall consider another technique for solving boundary value problems namely a shooting technique. What is the basic idea in shooting technique? It converts the given boundary value problem into 2 initial value problems and solve the 2 initial value problems by the techniques which we have already studied, namely you can solve the initial value problems by Taylor series method or Runge Kutta method or Eulers method or Predictor corrector method and obtain the solution of the 2 initial value problems. Then combine the solution of these 2 initial value problems to generate the solution of the original boundary value problem, this is the basic idea behind the Shooting technique for solving a boundary value problem.

We shall focus our attention on 2 point boundary value problem, governed by 2^{nd} order ordinary differential equation Dirichlet boundary conditions in this class. Let us see why we call this method as shooting Method. Suppose say I want to hit a stationary target at a distance which is far away from me and I fire an artillery to hit this target, so depending upon the angle of the barrel the projectile will land at different locations on the target. Suppose say I start with a barrel angle which is m 1 and then fire the 1^{st} round and note down the location where it hits this stationary target and I again change the barrel angle to say m 2 and then fire the 2^{nd} round and note where the projectile lands and now I interpolate between these 2 targets and it would be possible for me to now find out that angle for the barrel at which I will be able to hit the target at the prescribed position.

And so there is an analogy between this problem and the problem that we have in hand namely the solution of the boundary value problem by a shooting technique, what is it? We want to convert the boundary value problem into 2 initial value problems and so the barrel angle corresponds to the slope dy by dx at say x = a namely the left endpoint, a, b of the interval in which the differential equation is defined. And then hitting the target corresponds to solving the initial value problem and finally combining the 2 solutions of the initial value problem corresponds to determining the target at which I should end up namely the solution of the boundary value problem.

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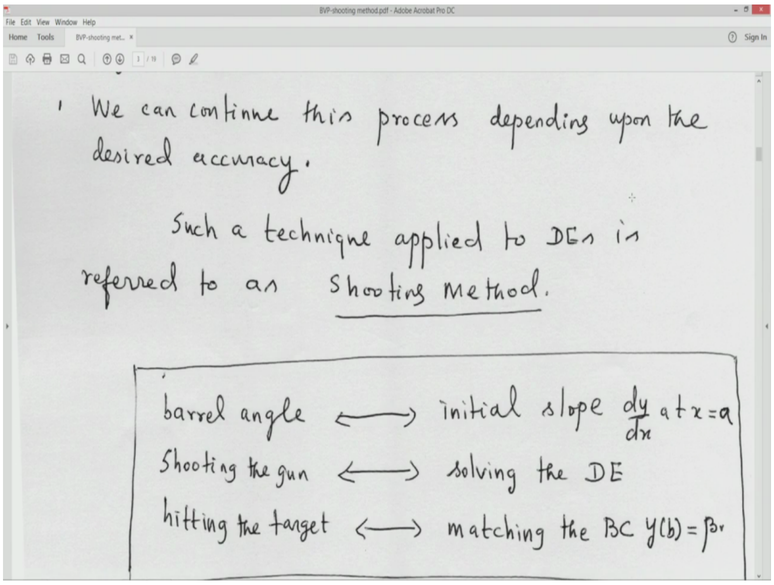

So the analogy that I have explained just now is displayed here. As I said earlier, the barrel angle corresponds to the initial slope dy by dx at x = a and shooting the gun corresponds to solving the differential equation and finally hitting the target corresponds to matching the boundary condition where up the other endpoint x = b namely we want y at b to be equal to Beta, so we consider the 2nd order a boundary value problem.

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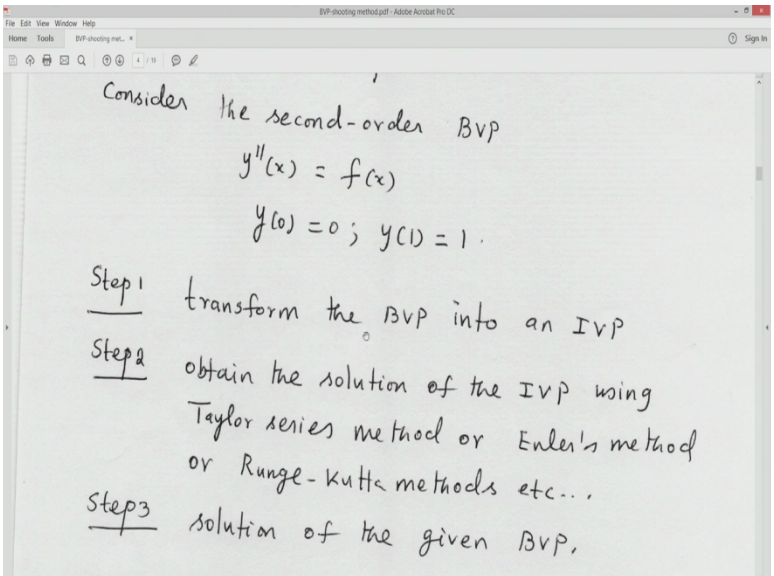

Y double prime x = f of x, y 0 is 0 and y of 1 is 1, so the differential equation is defined in the interval 0 to 1 it is a closed interval. What is our step 1? We have to transform the boundary value problem into 2 initial value problems, and in step 2 we should obtain the solution of these 2 initial value problems using any of the techniques that we have already learned. And step 3 is, combine the solutions of the 2 initial value problems and then generate the solution of the given boundary value problem and we know that this is possible in the case of linear boundary value problems.

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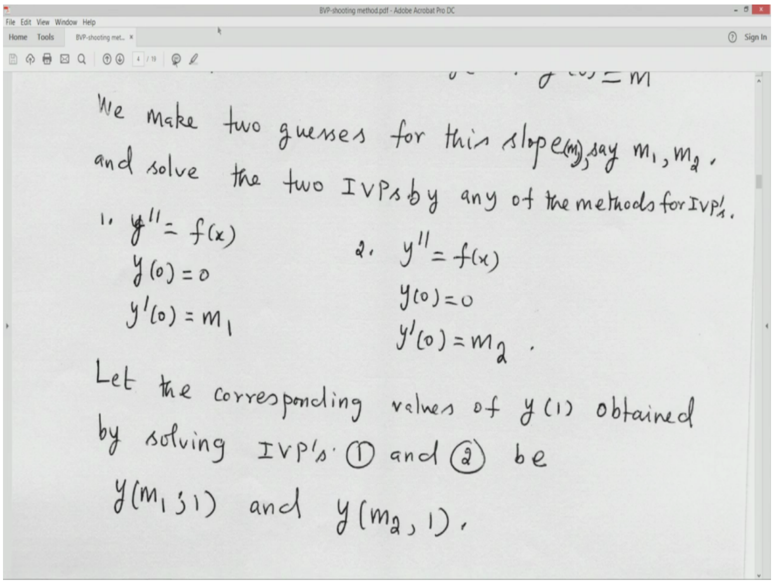

And it is clear that the Stepper requires the knowledge of the slope y dash at the initial point 0. We are given y of 0 as 0, but we do not know what y dash of 0 is, so we have to guess this value of y dash at 0, so we make two such guesses and consider 2 initial value problems say 1 and 2. In problem one we have y double prime = f of x, y of 0 = 0 which is prescribed and y dash of 0 is some m 1, which we have guessed because we have no knowledge of what y dash of 0 is that is our 1st problem, and what is our 2nd problem? It is again y double dash = f of x with y of 0 = 0 and y dash of 0 is m 2 which is again guessed by us and we observe that these are linear problems so we can apply any of the techniques that we have learned to solve these 2 initial value problems.

And let us denote the solution with the guess m 1 with y of m 1 and with the guess m 2, y of m 2, 1. So we have 2 solutions; y of m 1, 1 and y of m 2, 1 so we now use linear interpolation and write down the linear leg ranch interpolation polynomial, so what is it that I want? Corresponding to m 1 my solution is y of m 1, 1, corresponding to m 2 my solution is y of m 2, 1. What is it that I want, for some angle m, I want y of m, to be equal to 1 this is the condition which is prescribed in the boundary value problem at the other endpoint namely one, so we perform linear interpolation and write down the linear interpolating polynomial.

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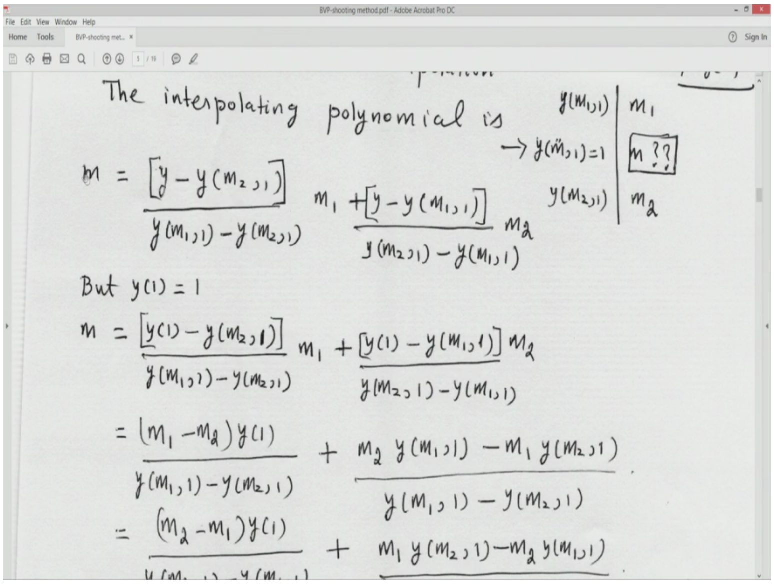

So we know that m is given by y – y of m 2 1 by difference y of m 1 1 – y of m 2 1 multiplied by m 1 + y – y m 1 1 by y of m 2, 1 – y of m 1 1 into m 2. But what do you want? We want that m for which y at 1 = 1, so I substitute y of 1 here and then determine what that m is.

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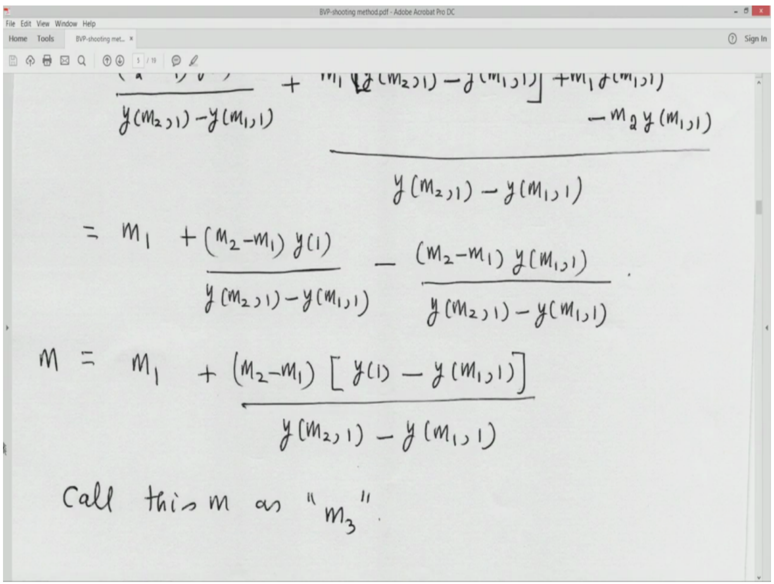

So we perform some simple computations and then we end up with the result that m = m 1 + m 2 – m 1 into y of 1 – y of m 1, 1 divided by the difference between 2 solutions of the 2 initial value problems that we have considered. We observe that the right–hand side term are all known to us since we have guessed m 1 and m 2 and we have solved the initial value problems and obtain y of m 1 1 and y of m 2 1 so terms are all known to us and there were we can determine what m is, and recall this m as m 3. And we now solve the initial value problem.

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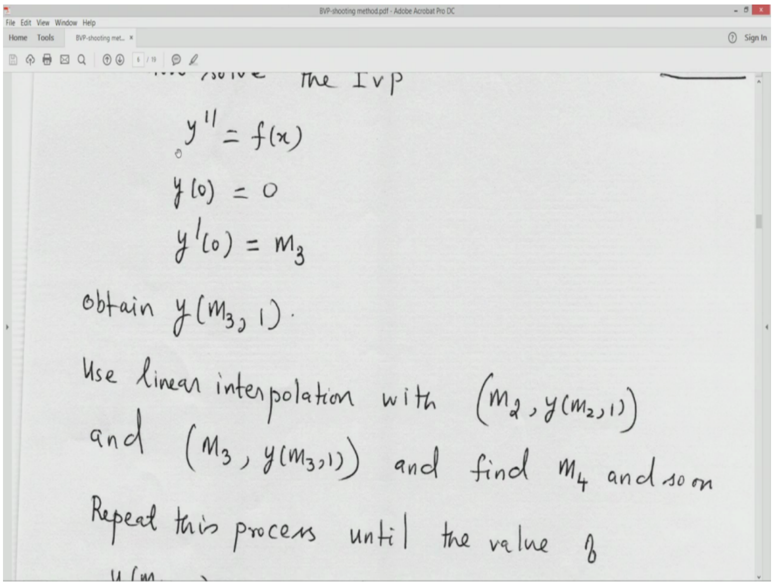

Y double dash is f of x with y of 0 equal to 0 and y dash of 0 is m 3 that we have computed just now using linear interpolation. And obtain solution by any of the techniques that we know for solving initial value problem and say the solution denoted by y of m 3, 1. Now use linear interpolation with m 2 y of m 2 1 and m 3 y of m 3 1 and find the new m 4 and continue this process till the solution y of m k, 1 agrees with y of 1 that is prescribed correct to the desired degree of accuracy. So it is clear that the speed of convergence of this method is highly dependent on how good our initial guesses are, so let us try to demonstrate this by the following example.

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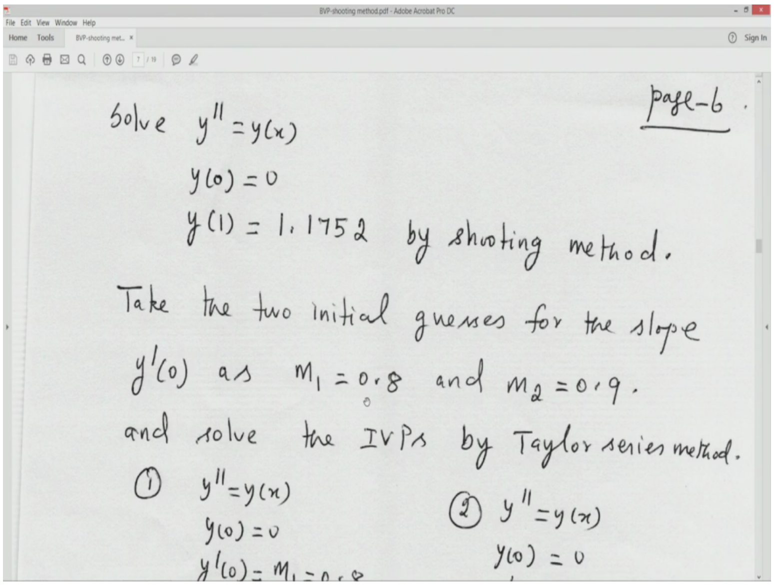

So the boundary value problem that is given to me is y double dash = y of x and y of 0 = 0 and y of 1 is say 1.1752 and we are asked to solve this problem by shooting method. So what is step one? Step one is we must solve 2 initial value problems which requires the knowledge of guesses about the slope y prime at the point 0, so we take the 2 initial guesses for the slope namely y prime 0 as m 1 so let us take it to be 0.8 and m 2, which is 0.9 and write down the 2 initial value problems.

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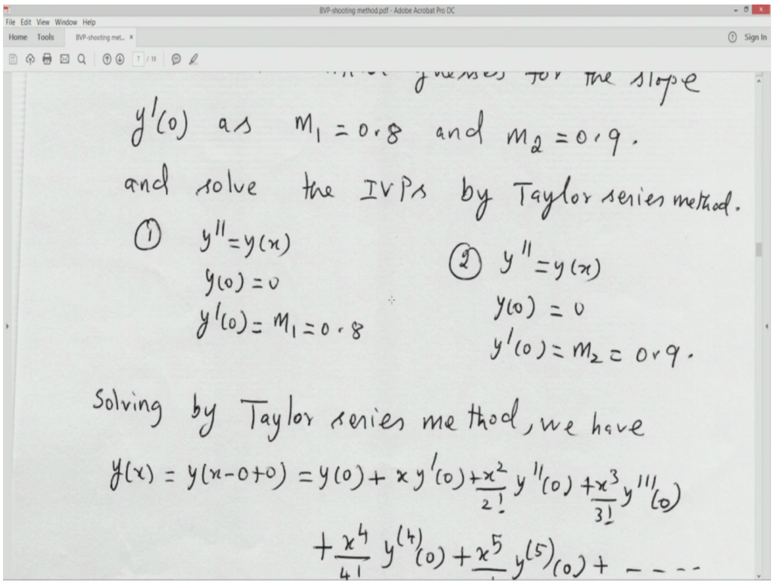

What are they? y double dash is y of x and y of 0 is 0 and y prime 0 is our 1st initial guess which is 0.8. What is the 2nd initial value problem, that is again y double prime = y of x, y or 0 equal to 0 and y prime 0 is m 2 and that is 0.9 that is what we have guessed. So I should use

Taylor series method in solving these two initial value problems. So what is Taylor series method? I have given an interval 0 to 1 and I have an initial value problem, two initial conditions are given to me and I have to solve this problem using Taylor series method, so I need to divide the interval 0 to 1 into a number of sub intervals of step size say h when I use equally spaced points and starting with the solution which is prescribed at the initial point I should match to the next step and then continue this process till I reach the point which I need to get the solution namely some point say x N which will be given to us.

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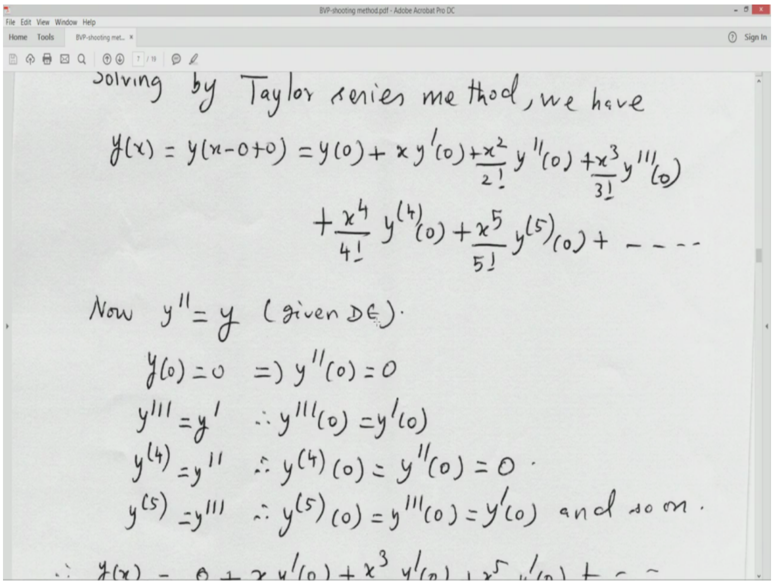

So in this case let us write down the solution by Taylor series method. So y at x is y of x – 0 + 0 as the initial point is 0, so I expand y about the point x = 0 so that is y of 0 + x into y dash of 0 + x square by factorial 2 y double dash of 0 and so on… + etc + x to the power of 5 by factorial 5 into fifth derivative of 0 and so on. But I know that y double prime is y that is the given differential equation, so I shall arrive at the values of these derivatives at 0 by making use of the given differential equation. So y double prime is y and y of 0 is 0, so y double prime of 0 is 0, from here I compute y triple prime, y triple prime is y prime so y triple prime at 0 is y prime and 0 and y prime at 0 depending upon what my guess is, I will substitute.

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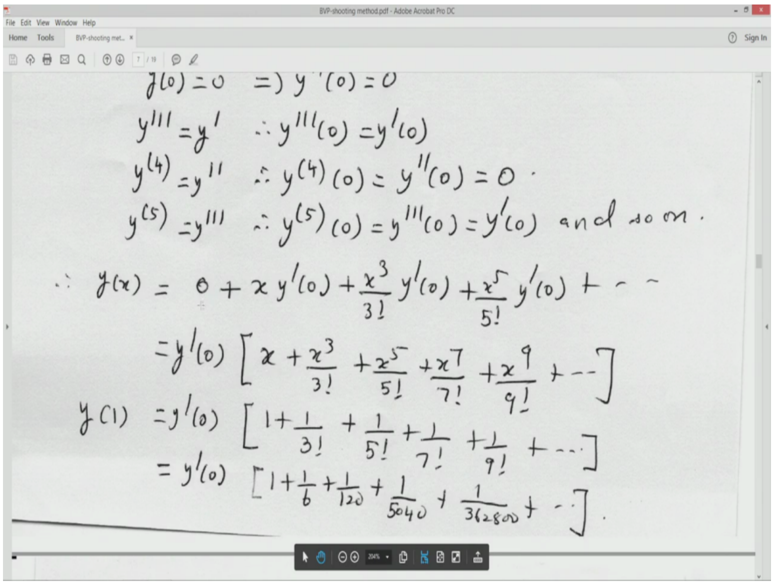

Let us work out the derivatives which we have to compute so that we can substitute for these derivatives in this solution. So the fourth derivative of y is y double prime and 4[th] derivative at 0 is y double prime at 0, which is 0. The 5[th] derivative is the 3[rd] derivative of who Y, so the 5[th] derivative at 0 is third derivative at 0 that is the 1[st] derivative at 0 and so on, so we write down the solution y of x as say y of 0 is 0 + x into y prime 0 + x cube by 3 factorial into again y prime 0 + x power 5 by factorial 5 into y prime 0 and so on. So this will be y prime 0 into x plus x cube by factorial 3 + x to the power of 5 by factorial 5 and so on, so it is y prime 0 into, when x is 1, 1 + 1 by factorial 3 + 1 by 5 factorial + 1 by 7 factorial + 1 by 9 factorial and so on and that is what is given by y at 1.

But what do we want? For some choice of y prime 0 we want y at 1 to be equal to 1 that is what the boundary value problem is. I do not know what my prime 0 is that will give me y of 1 to be equal to 1, which is prescribed as the boundary condition at the other endpoint x = 1 in the boundary value problem and that is why we want to use a shooting technique namely; make 2 different guesses for this y prime 0 namely m 1, which is 0.8 and m 2 which is 0.9 and obtain the solution y of m 1 1 and y of m 2 1.

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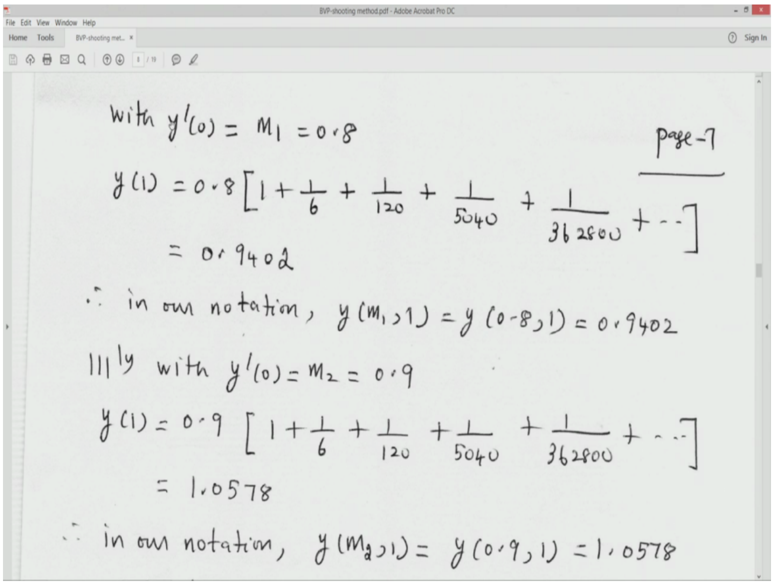

So we find using the expression above what is the y at 1 that is 0.8 multiplied by these terms, when you calculate it turns out to be this. In our notations what we have computed is y of m 1 1 and that turned out to be 0.9402. Similarly, the 2nd guess m 2 as 0.9 we work to do what y of 1 is namely, we are solving the 2nd initial value problem and that gives us 0.9 into these terms that will give you 1.0578 and this in our notation is y of m 2 1 and that turns out to be this. So now that we have solutions of the 2 initial value problems obtained using Taylor series method we use linear interpolation determine the new m, which we have called as m 3.

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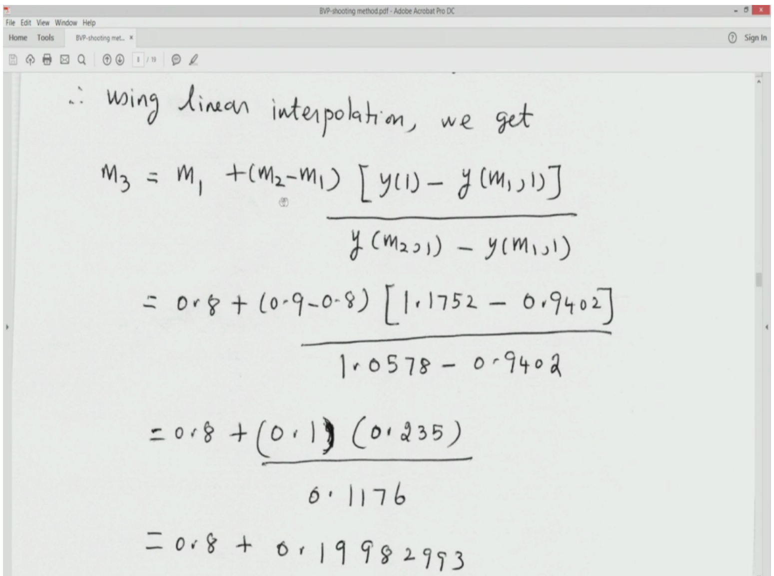

And that is m 1 which is 0.8 + m 2 – m 1 which is 0.9 – 0.8 multiplied by y of 1 – y of m 1 1 by the difference between the 2 solutions. But what is y of 1 is prescribed in the boundary value problem and that is given by 1.1752.

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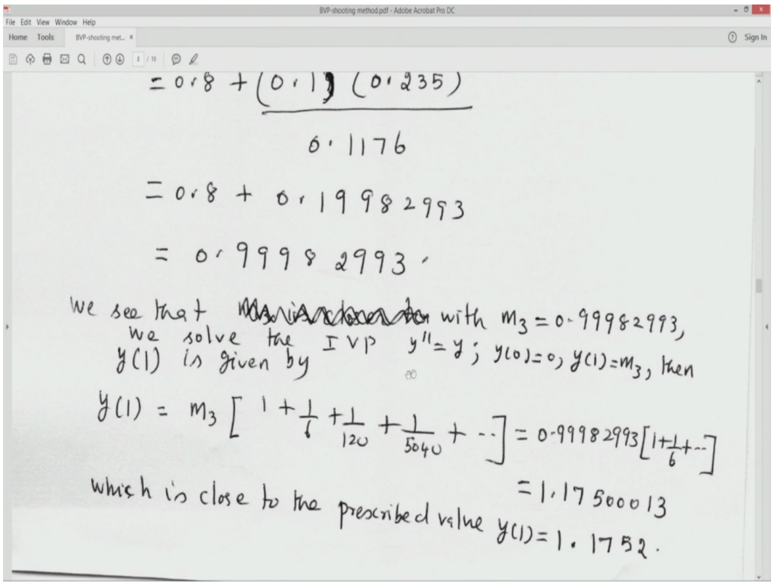

So I substitute that and then do the computation and I end up with the value of m 3 to be 0.99982993, so I have the value of m 3 which is the slope y dash of 0. So I would now like to consider a new initial value problem what is that? It is y double prime = y, y of 0 is 0 and y of one is this m 3 which we have computed. And so again I solve this initial value problem by Taylor series method, which gives us y at 1 to be m 3 times these terms, m 3 is available to us, so we substitute for m 3 and compute this and that gives us 1.17500013. And we observe that this y of 1 which has been computed to be this is close o the prescribed value which is 1.1752, so our initial guesses about m 1 and m 2 are correct and this has led to a value of the slope namely m 3, which gives us a solution y at 1 to be very close to the prescribed value which is 1.1752.

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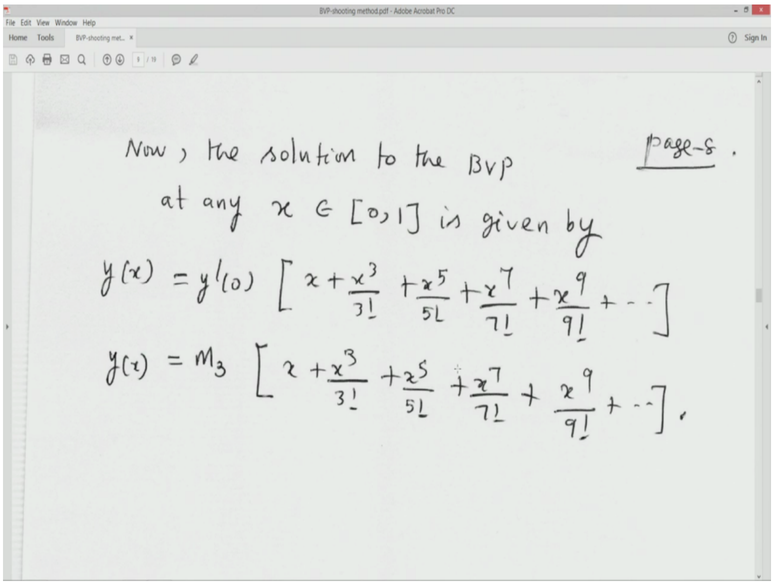

So having solved the initial value problem and determined the m 3 which should be the slope that I should use so that my solution will be close to the desired degree of accuracy, I write down the solution to the boundary value problem at any x in the interval 0 to 1. So I have already divided it into a number of subintervals by means of points, x not which is 0, x 1 which is say 1 – 0 by N if I divide the interval into N equal intervals, x 2 is 2 by capital N and so on. I called these points as x i, then the solution at any x which is x i is given by m 3, which is y prime 0 which we have computed multiply by x plus x cube by factorial 3 + x power 5 by factorial 5 and so on, where y at x i is given by m 3 into this with x replaced by the value of x i so that we obtain the solution, which is approximate solution to the boundary value problem at the point x i which is done using shooting method.

Namely we started with two initial value problems, solve these initial value problems by any one of the techniques that are known to us and then we arrived at a value of the slope y prime of 0 such that that is going to give us the value of y at the endpoint to be as close as possible to the prescribed value. Once this is done, we determined the solution of the boundary value problem, so this is what is that shooting technique. Let us now consider a result which gives us general conditions that ensure that the solution to a linear boundary value problem is unique.

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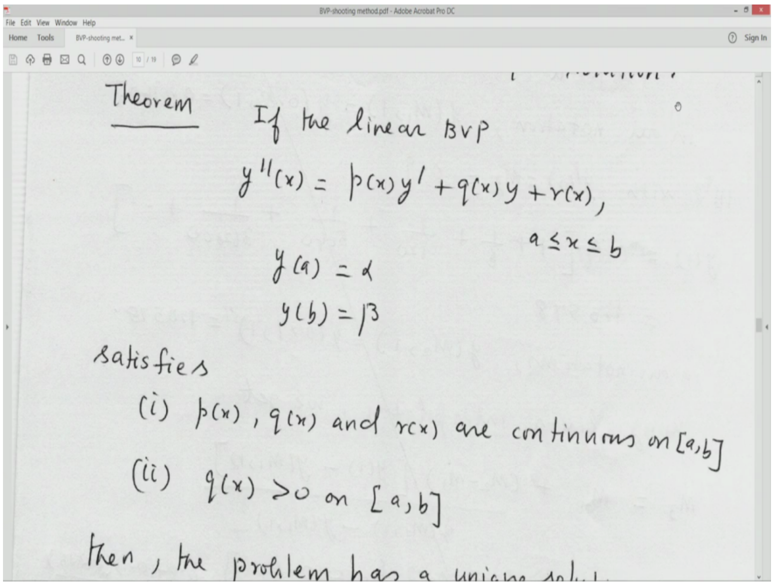

The result says, if the linear boundary value problem described by 2nd order ordinary differential equation, y double prime x = p of x into y prime + q of x into y plus r of x for x in the close interval a, b subject to Dirichlet type of boundary condition, y of a = Alpha and y of B is Beta satisfies the following condition; namely p of x and q of x and r of x are continuous functions on the closed interval a, b and that q of x is greater than 0 on the interval a, b then the problem has a unique solution. So when we work out the solution of the boundary value problem using a shooting technique before carrying out the computations, we try to ensure that the conditions of this theorem are satisfied from that particular problem that would ensure that the solution that we obtained is a unique solution. So our goal is to get an approximation to this unique solution because we are trying to solve the problem numerically, how do we do this?

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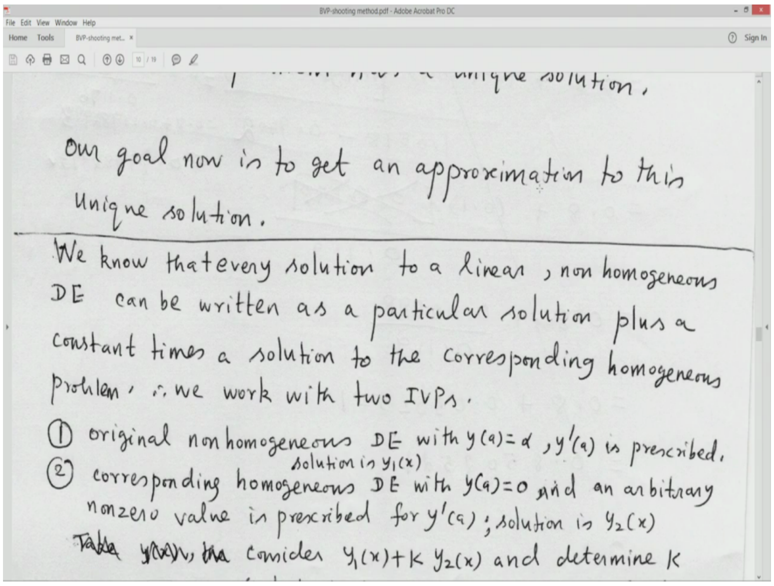

We know that every solution to a linear non–homogeneous differential equation can be written as the particular solution + constant times solution to the corresponding homogeneous problem therefore, we work with 2 initial problems. What are the 2 initial problems? The 1st problem is such that it is governed by the differential equation which is the same as the differential equation which is a non–homogeneous differential equation given in the boundary value problem. Then what are the initial conditions for this initial value problem? They are y of a = Alpha, this is one of the conditions prescribed in the boundary value problem at the end point x = a.

And we take y prime at a to be some prescribed value, it can be arbitrarily prescribed and we determine the solution of this initial value problem and call that as y 1 of x, then we consider the 2nd initial value problem and the 2nd initial value problem is governed by the corresponding homogeneous second–order differential equation which is given in the boundary value problem along with the initial conditions, what are they? y at a is 0 and an arbitrary non–zero value is prescribed for y dash of a and determine the solution of this 2nd initial value problem, call this solution as y 2 of x.

Then consider a combination of these 2 solutions of the 2 linear initial value problems namely y 1 of x plus K times y 2 of x, what is this constant K? In order that this combination is a solution of the boundary value problem, I must have this constant K to be such that the body condition at x = b must be satisfied. So we try to impose this condition and find what K is and finally we obtain the solution of the boundary value problem. So this is the procedure that we would like to adopt to obtain the solution of the given boundary value problem and this technique is what is referred to as the shooting technique, so let us see whatever that we have described here in the next few slides.

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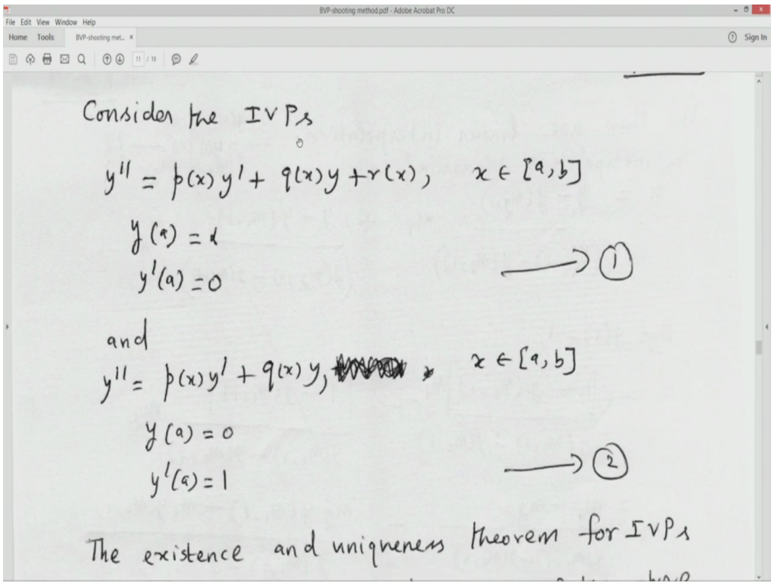

So what is my 1st initial value problem? The 1st initial value problem is such that it is governed by the 2nd order differential equation which is a non–homogeneous 2nd order differential equation that is given in the linear boundary value problem, which we have been asked to solve what should be the initial conditions that I should take, y of a = Alpha so this is essentially the condition that is prescribed at the left end point a in the linear boundary value problem. What is y dash of a? This I prescribe, I can prescribe it arbitrarily so I prescribe it to be such that y dash of a 0, so this is my 1st initial value problem, I hope it is clear to you.

Let us write down the 2nd initial value problem, what is the governing second–order equation in this case? It says, take the differential equation to be corresponding homogeneous differential equation for the boundary value problem, what do you mean by that? You have to put r of x to be 0 that will give you the corresponding homogeneous differential equation given in the boundary value problem so I can say that the 2nd initial value problem to be y double dash p of x y prime plus q of x into Y, so I have put r of x to be 0 and it is defined is the interval a, b. So what should be my initial condition in this case, we have already described. Take y at the point a to be 0 and you can prescribe y dash at a to be any nonzero value so let us take y dash at a to be equal to 1, I hope you have understood what the 2 initial value problems are.

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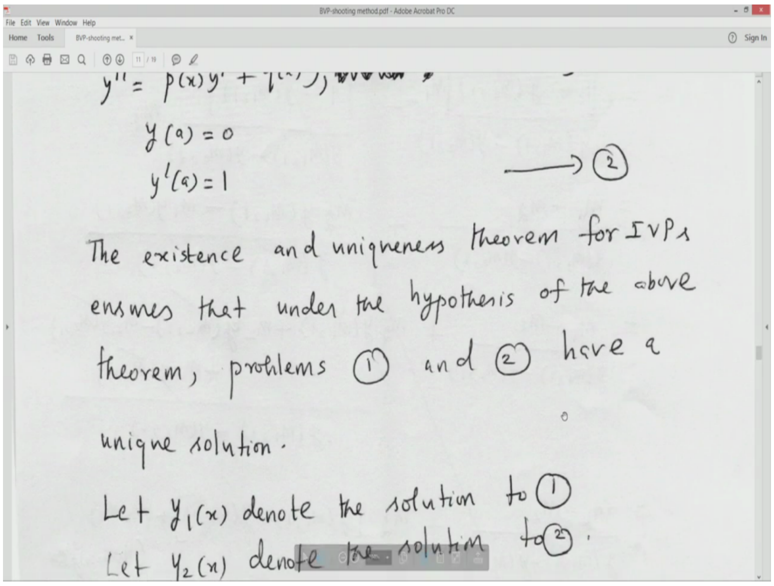

Now the existence and the uniqueness theorem for initial value problem ensures that under the hypothesis of the theorem that we have stated in the beginning, problems 1 and 2 which are initial value problem have a unique solution, so we move on to determining this unique solution. Suppose we denote them by y 1 of x and y 2 of x, then take a linear combination of these 2 solutions namely y of x = y 1 of x + Beta – y 1 at b by y 2 at b into y 2 of x.

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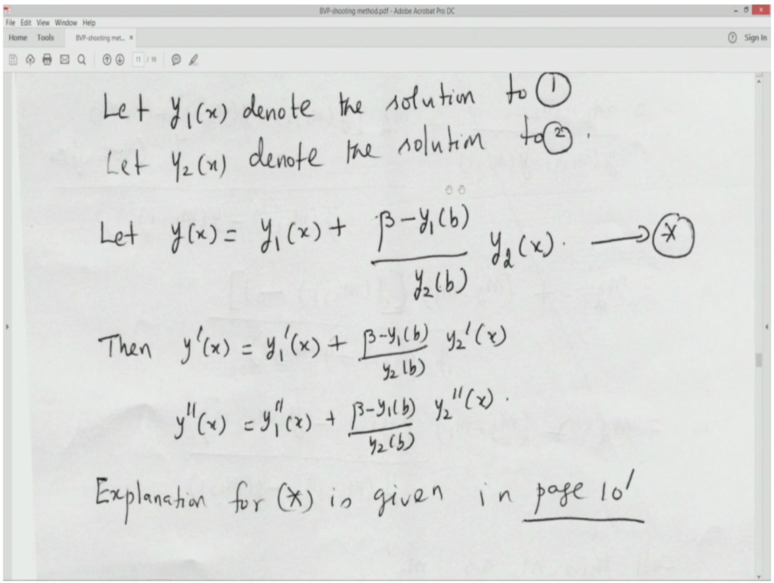

So you may ask me how suddenly you have written down a constant in this form namely Beta – y 1 of b by y 2 of b. Please recall we have written the solution in the form y 1 of x of x plus K times y 2 of x and we said that we determine K in such a way that the prescribed boundary condition at the other end point namely b is satisfied. What is the other endpoint condition? y at b must be equal to Beta, so we have imposed that condition we have arrived at the constant as this, so I have given the explanation here. So let us look into how we have arrived at the constant K as what is written there.

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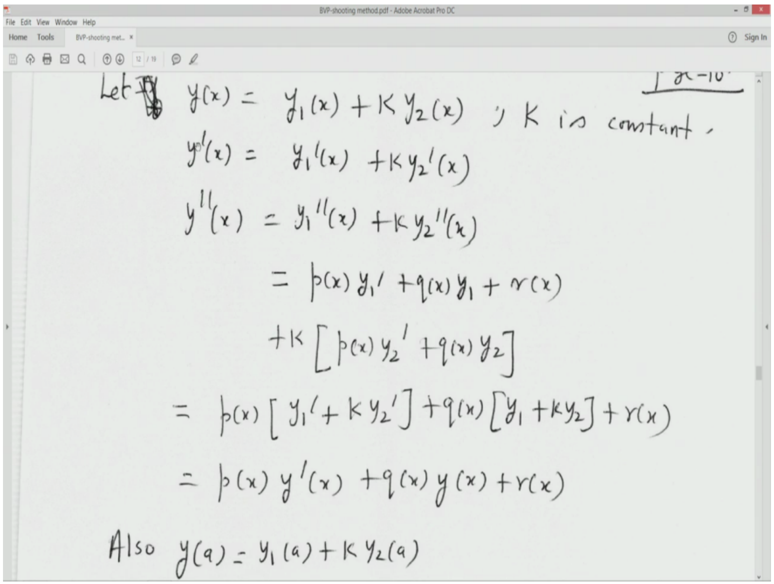

So we have taken y of x to be y 1 of x plus K y 2 of x, so we compute y dash of x, y double dash of x and so substitute for this y 1 double dash of x and y 2 double dash of x. What is y 1 double dash of x? It can be obtained from the 1^{st} initial value problem so y 1 double dash of x is p of x plus y 1 prime + q of x y 1 + r of x and y 2 double dash of x is p of x y 2 dash of x + q of x into y 2 because that corresponding to the homogeneous differential equation for the boundary value problem, so we write down the coefficient of p of x, which is y 1 prime plus k y 2 times + coefficient of q of x which is y 1 plus + K y 2 + this r of x. But what is y 1 prime plus K y 2 prime? You observe that it is y prime. And double prime + K y 2 double prime is y double prime, so we substitute from y 1 prime + K y 2 prime as y prime and y 1 + K y 2 as Y, so we get y double prime of x is equal to p of x into y prime x + q of x into y of x + r of x, what is this? This is the 2^{nd} order linear differential equation that governs the boundary value problem.

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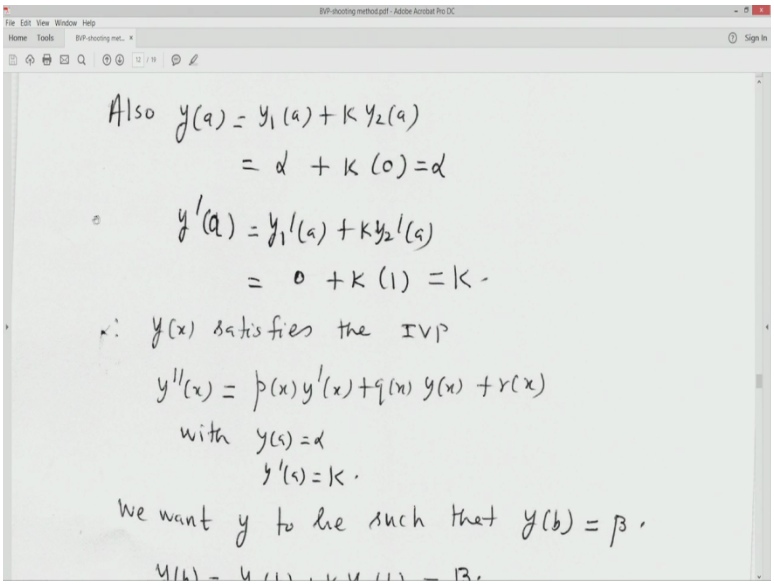

So what is it that we have shown that y of is given by y 1 of x plus K times y 2 of x is a solution of the non–homogeneous differential equation, which governs the boundary value problem that is what we have shown. Now let us see about the conditions, what do we want? We want y at a to be equal to Alpha which is prescribed, let us check what happens here. What is y at a? y at a is y 1 of a + K times y 2 of a, how have we chosen y 1 of a that is Alpha that is y 2 of a we have chosen to be 0, so it is Alpha plus K times 0 so it is equal to Alpha, so our y satisfies that one of the conditions at the endpoint x = a namely y of is Alpha. What about y dash of a? It is y 1 dash of a + K times y 2 dash of a, but y 1 dash of a has been taken to be 0 and y 2 dash of a is prescribed to be 1 and therefore it gives you what it is K is.

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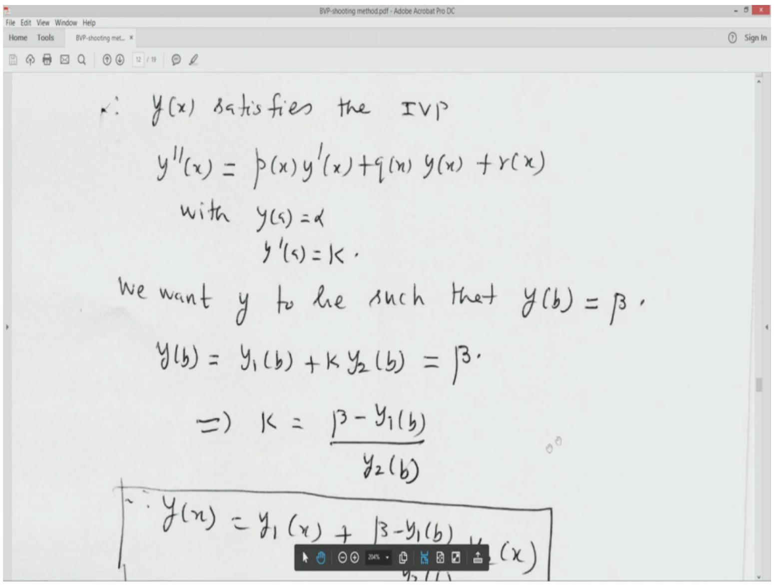

So our y of x which is y 1 + K y 2 satisfies the governing non–homogeneous differential equation for the boundary value problem along with the condition that y of a is Alpha and y dash of a is K, so we have a new initial value problem. But what do I want, I want the solution of this initial value problem such that when I compute y at b, it should give me Beta which is the prescribed value in the boundary value problem so I impose that condition namely y at b is y 1 at b + K time y 2 at n but what do I want, I want it to be Beta. So this immediately tells K is Beta – y 1 of b y 2 of b and it is for this reason that we have written y of x as y 1 of x + Beta – y 1 of b + y 2 of b into y 2 of x.

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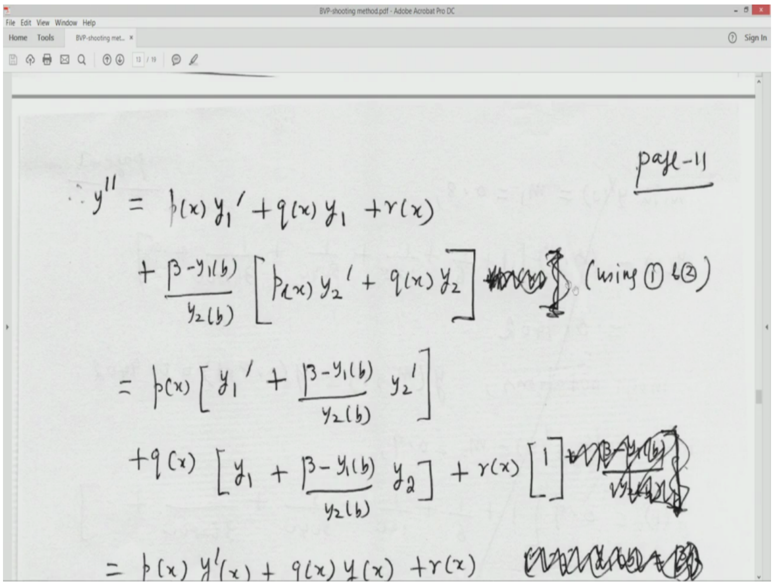

So we have with that we should now be able to show that our y double dash of x is equal to the right–hand side namely p of x into y dash of x + q of x into y of x + r of x. So instead of K, I have substituted that and so we have been able to show that y of x satisfies the differential equation, which governs the boundary value problem.

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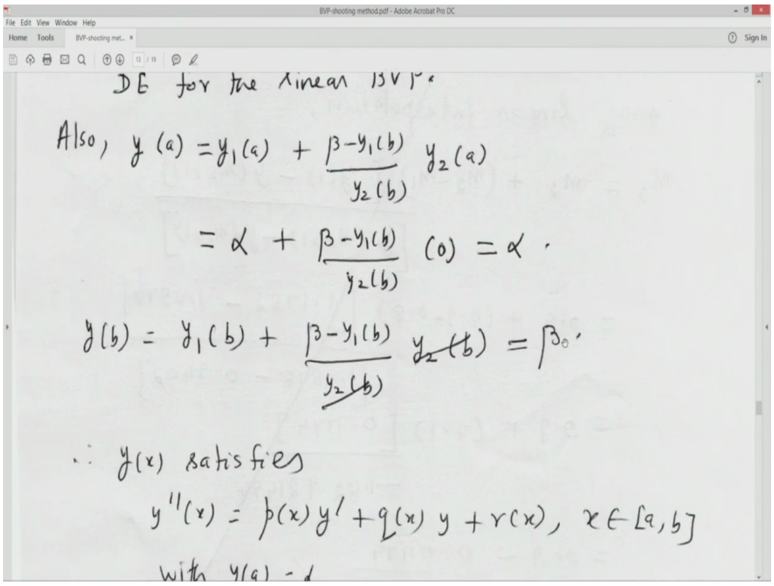

In addition, what is y of a? It is y 1 at a + this multiply by y 2 at a, which is 0 and this simplifies to Alpha so the boundary condition at the endpoint namely the left endpoint x = a is satisfied. Now what about the boundary condition at the right endpoint? y at b is y 1 of b + this multiplied by y 2 of b and that gives you Beta. So our y of x satisfies the governing differential equation for the boundary value problem along with the conditions which are Dirichlet boundary conditions and according to the condition of the theorem, if p of x and q of x satisfies the conditions specified in the theorem, the solution that we obtain is a unique solution of the linear boundary value problem.

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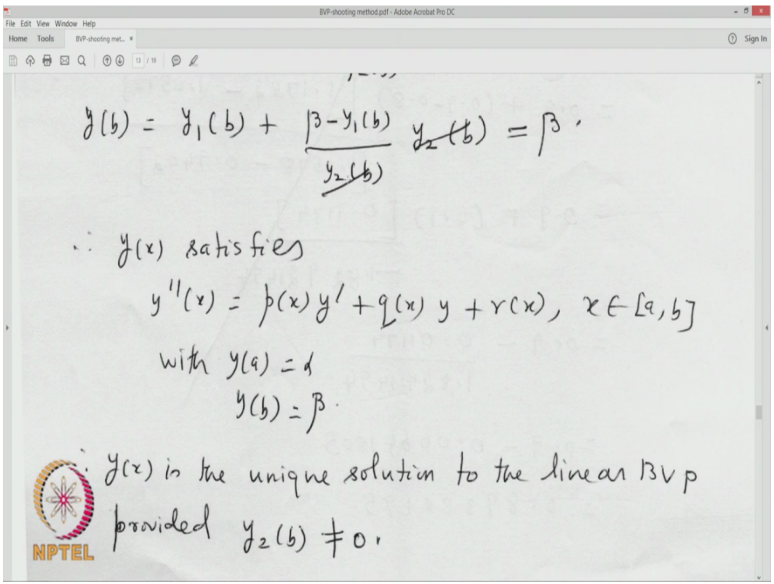

And one thing you must note here is that we must have y 2 at b to be different from 0, then in that case if the conditions in the theorem are satisfied for p of x and q of x then y of x = y 1 of x + K into y 2 of x will be a unique solution of the linear boundary value problem with Dirichlet boundary condition y at a = Alpha and y at b = Beta. So this describes the shooting technique for solving a linear boundary value problem with Dirichlet boundary condition. We shall try to demonstrate this method by considering some examples and we will continue with that in the next class.