Numerical Analysis
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Lecture No 26
Numerical Solution of Ordinary Differential Equations-9-Linear Boundary Value Problems (Finite difference method)

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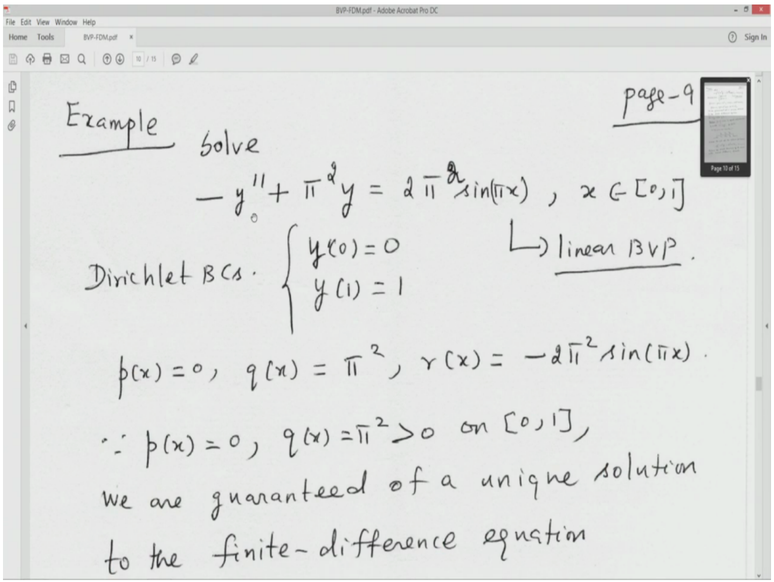

So let us now see how we apply a finite difference technique for solving a linear boundary value problem governed by Dirichlet boundary conditions, let us take the following example. So we have a second–order equation – y double prime + Pie square y = 2 Pie square into Sine Pie x in the interval 0 to 1. And there are Dirichlet boundary conditions, which are specified at the endpoints 0 and 1 of this interval 0 to 1, so y of 0 is 0 and y of 1 is 1. So we observe that p of x is 0 and q of x is Pie square and r of x is 2 Pie square into Sine Pie x with a negative sign. And since p of x is 0, q of x is positive on the interval 0 to 1. We have guaranteed of a unique solution to the finite difference equation for any value of h, this we have already established, h must satisfy the condition that h is less than 2 by L.

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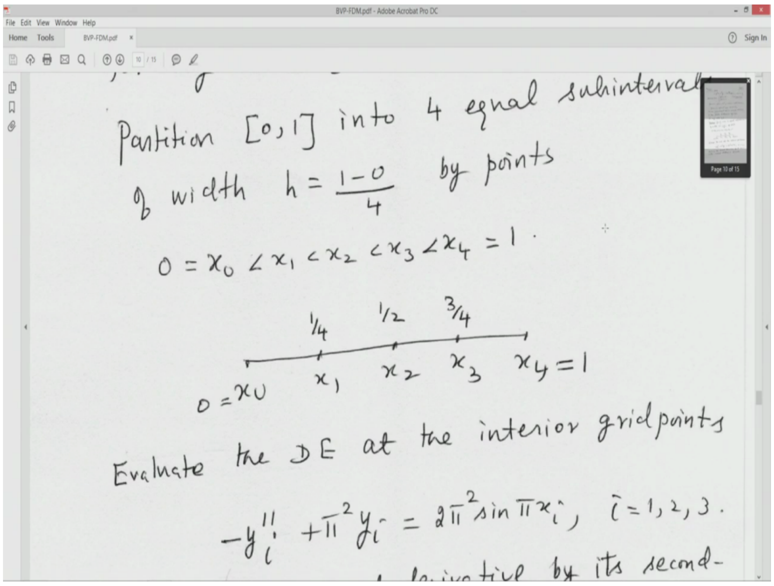

What is L? Modulus of p of x must be less than or equal to L for x in the interval a, b. So since p of x is 0 in this case, we are guaranteed of a unique solution to this finite difference equation for any value of x. So we start applying all the steps that we have discussed for the general second–order differential equation governed by boundary conditions, so in that step one is to partition the intervals 0 to 1. So let us divide this interval into 4 equal sub intervals of width h, which is 1 – 0 by 4 by means of points x 0, x 1, x 2, x 3, x 4 such that x 0 is 0, x 1 is 1 by 4, x 2 will be half, x 3 is 3 by 4 and x 4 will be 1. What do we have to do next, we will have to evaluate the differential equation at each of the interior points namely x 1, x 2, x 3, so it is – y double prime at i + Pie square y at i that is equal to 2 Pie square into sine pie x i for i = 1, 2, 3.

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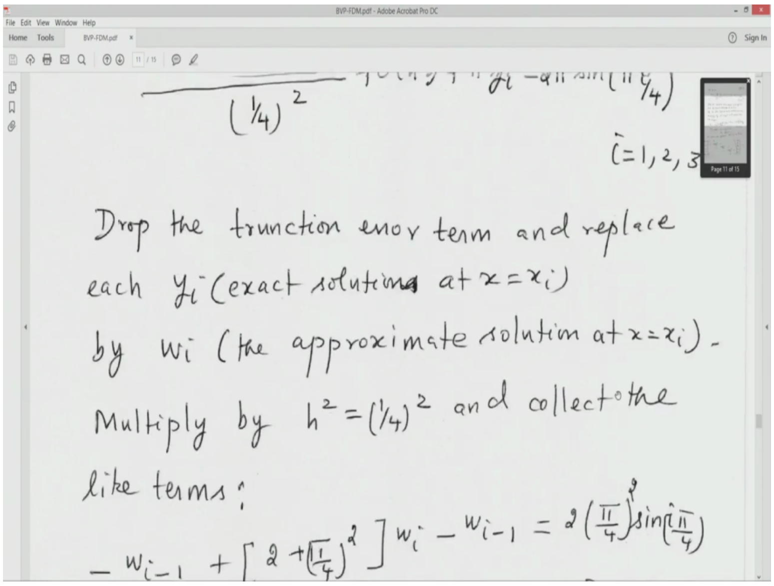

What is the next step? We should replace the derivatives that appear in the differential equation by appropriate finite difference scheme. So let us again replace the second order derivative in the differential equation by second order accurate central difference scheme. If we do that then we get – y i – 1 + twice y I – y i + 1 by h square, h is 1 by 4 + order of h square because the error in this approximation is of order of h square + Pie square into y i and that is equal to 2 Pie square into sine i Pie by 4 for i = 1, 2, 3. What should we do then? Next step is to drop these truncation error terms and replace each of the exact solution y i at these interior points by the corresponding approximate solution w i.

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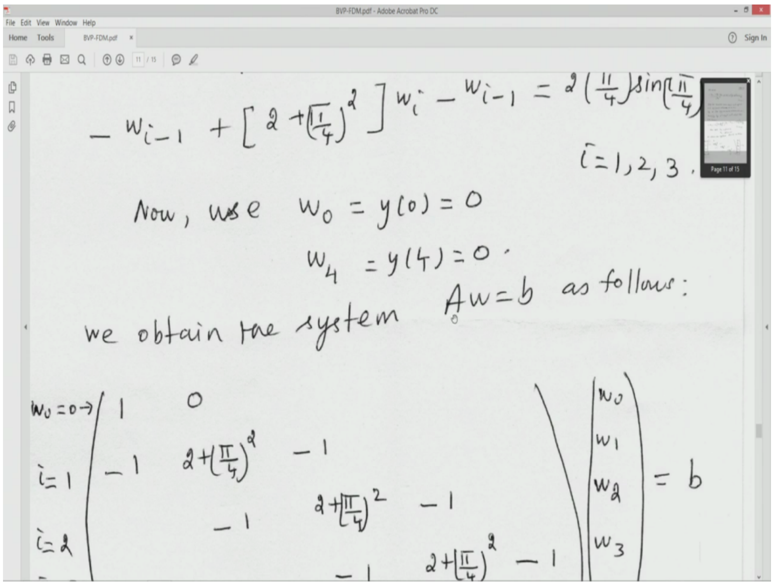

So when we do that, we get – w i – 1 + this coefficient into w i – w i – 1 and that is equal to the right–hand side, i = 1, 2, 3 and this equation has been obtained after multiplication by h square, which is 1 by 4 the whole square. Now we look at the boundary conditions, the boundary conditions are that y of 0 is 0 and therefore, w 0 is 0 and y at 4 is 0 and hence W4 will be 0, so we obtain the system A w = b as follows. So we take w 0 = 0 as the first equation, the next 3 equations are obtained by applying this at i = 1, 2, 3 and our last equation is given by w 4 = 0. So we combine all these namely I have i = 0 equation given by this, for i = 1, 2, 3, equations will be obtained from here and finally when i = 4, I have my equation given by w 4 equal to 0.

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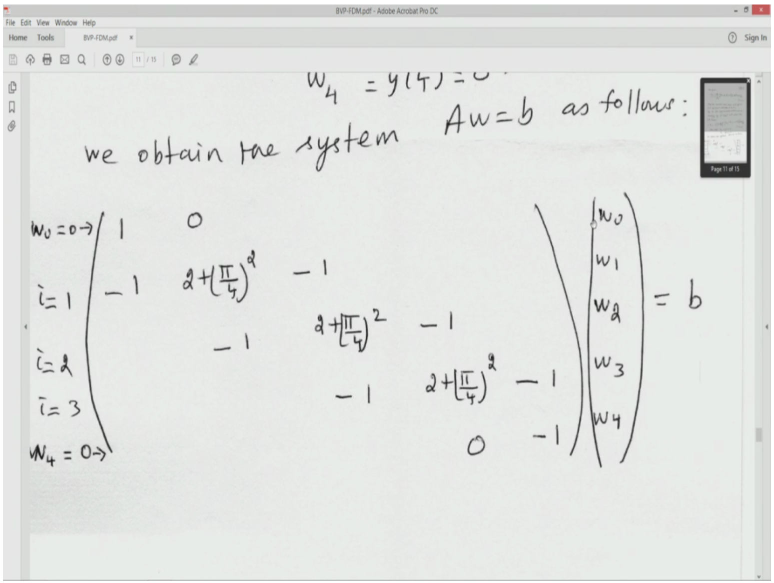

So I take my unknown vector as 5 cross 1 vector which has components w 0 to w 4 and I write down the system A w = b, with coefficient matrix a to b a 5 cross 5 matrix, so equations are expressed here. Corresponding to w 0 = 0 how do I express this, so 1 into w 0 + 0 into w 1 + 0 into w 2 plus 0 into w 3 + 0 into w 4 is equal to the right–hand side b and w 0 = 0, so b 0 will be equal to 0.

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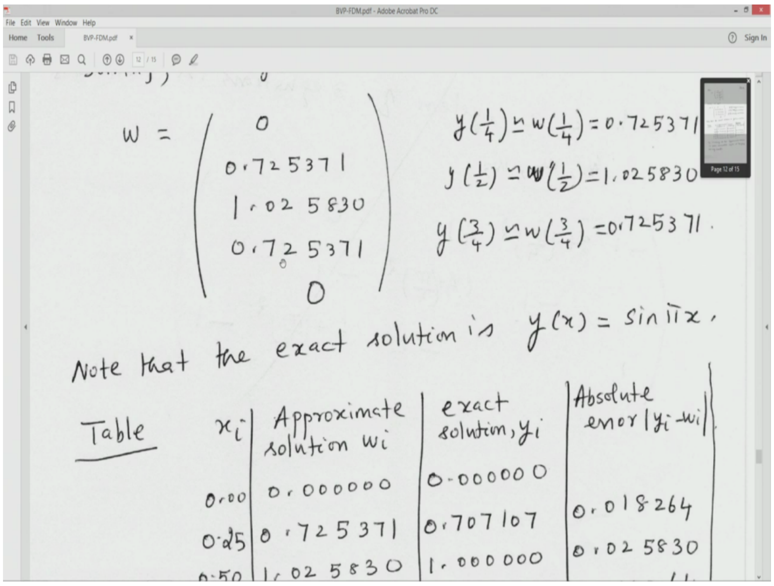

So I apply these equations one by one and write the system A w = b and we solve the system with the right–hand side vector b given by this and we observe that the solution vector w is given by this, which tells us w 0 is 0, w 1 which is an approximate solution at x 1 which is 1 by 4 namely y at 1 by 4 is approximated by w at 1 by 4 and that is given by this, y at half is this and that gives you an approximation namely 1.025830 and so on, so the solutions of the interior points which are approximations to the exact solutions y at the interior points are given by these. And we know that the exact solution of the differential equation is given by y of x = Sine Pie x, so it is natural to see whether theapproximate solution that we have obtained by taking the step size h to be 1 by 4 is reasonably a good approximation to the exact solution of the problem.

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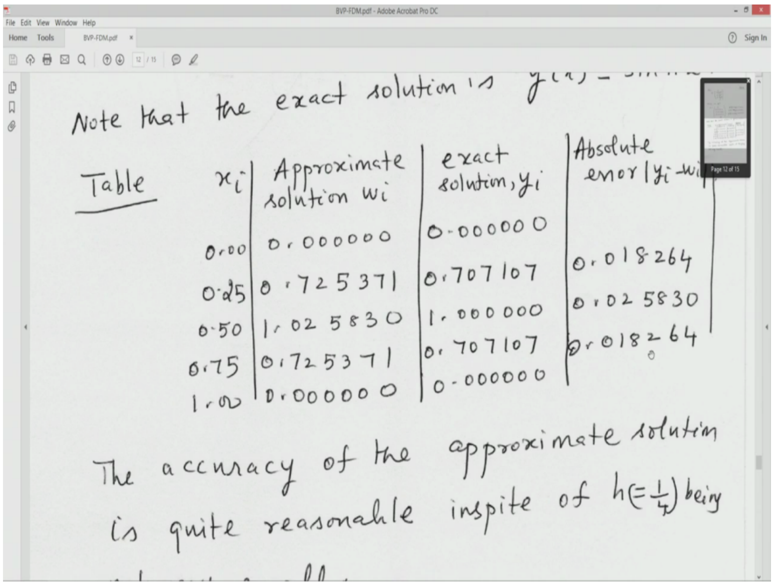

So we list the exact solution at each of the x i and compare it with the approximate solution at the corresponding letter x i given by w i, so we have a table of values where the letter x i are specified namely 0, 1 by 4, half, 3 by 4 and 1 at which the approximate solution and the exact solution are written down and we compute the absolute error namely Mod y i – w i, the exact – the approximate solution. And this column tells us the accuracy of the approximate solution is quite reasonable in spite of the fact that the step size h is which is taken to be 1 by 4 is not very small.

So we know that the system possesses a unique solution for any value of h because the conditions that we had stated were satisfied, which are sufficient conditions and therefore, solution possesses a unique solution and we had taken h to be equal to 1 by 4 and we observe that our approximate solution is such that it is reasonably accurate solution, which compare very well with the exact solution for this problem. So we have illustrated the finite difference method of solving a boundary value problem governed by the second order equations subjects to Dirichlet type of boundary conditions. What happens to the maximum absolute error in the approximate solution as a function of the number of subinterval?

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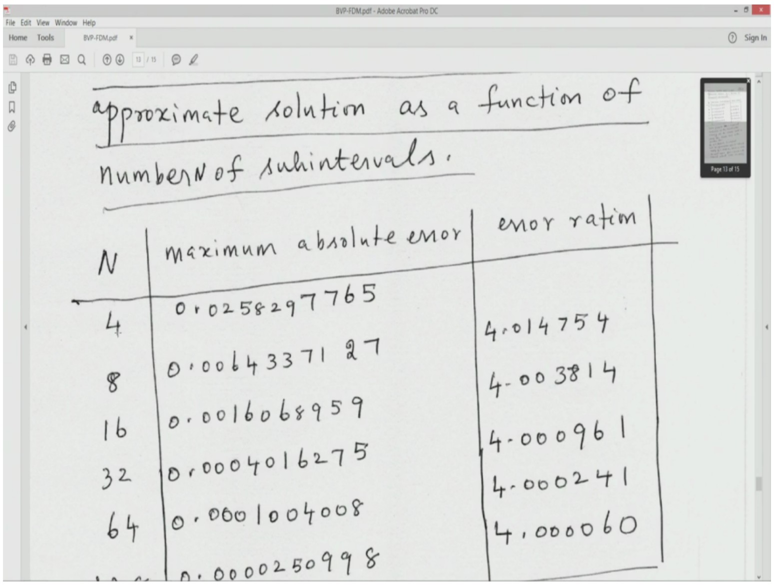

So when I take N to be 4, my h is 1 by 4 so I would like to increase the number of subintervals namely double the number of subintervals say 8. In this case h will be halved namely h will be 1 by 8 and I continue to double the number of subintervals, which I divide the intervals 0 to 1, by doing this every time the step size is halved. So I would like to now see what is the maximum absolute error and the corresponding error ratio, namely when I take 8 subintervals this is the maximum absolute error, and when I take 4 subintervals this is the maximum absolute error, so what is the ratio of this to this. And I would like to compute the error ratios by doubling the number of subintervals each time, so let us just look at this table and focus our attention on this column namely the column which specifies the error ratio.

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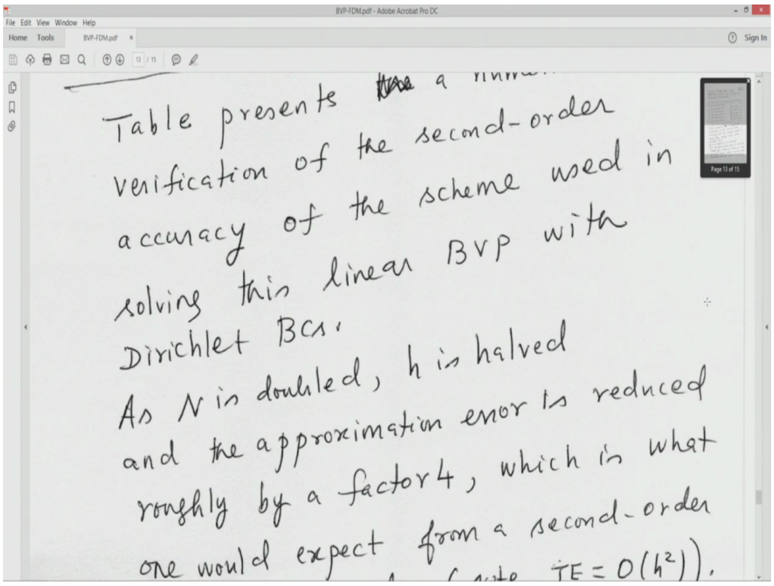

So the table actually presents the numerical verification of the second order accuracy of the finite difference scheme that we have used in the solution of the linear boundary value problem with Dirichlet boundary conditions. So we observe that as N is doubled, h is halved and the approximation error is reduced roughly by a factor of 4, let us just see the error ratio so we observe that each time when N is double and h is halved, this approximation error is roughly 4 times reduced that is what happens throughout. What does it indicates? It indicates that we have used second order accurate difference scheme to replace the derivative which appear in the differential equation and this is reflected in the fact that the approximation error is reduced by a factor of 4 and so the table clearly shows the numerical verification of the second order accuracy of the finite difference scheme that we have used in solving this boundary value problem with Dirichlet boundary condition.

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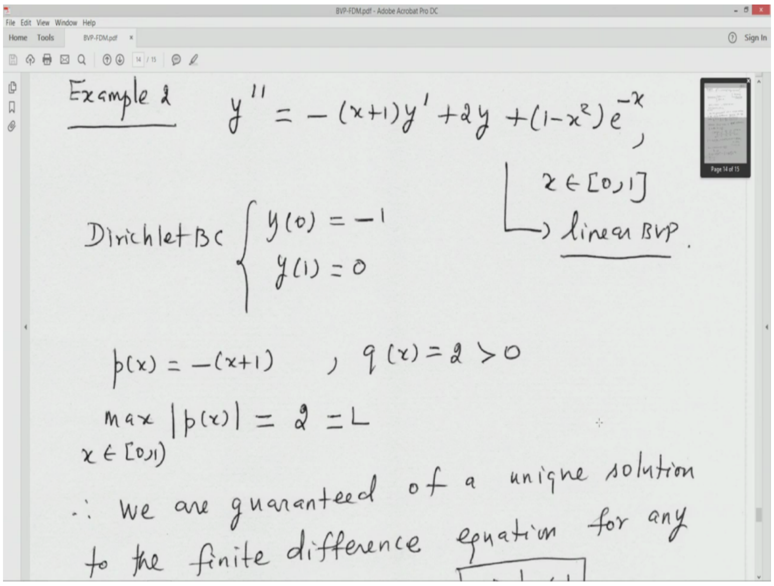

So this is another example, which is again a boundary value problem governed by second order differential equations. Let us again solve this problem subject to Dirichlet boundary condition, so the differential equation is given here and interval of interest is 0 to 1 and it is a linear boundary value problem and the conditions are y of 0 is – 1 and y of 1 is 0, so the unknown function is specified at the endpoints of the interval, so they correspond to Dirichlet type of boundary conditions.

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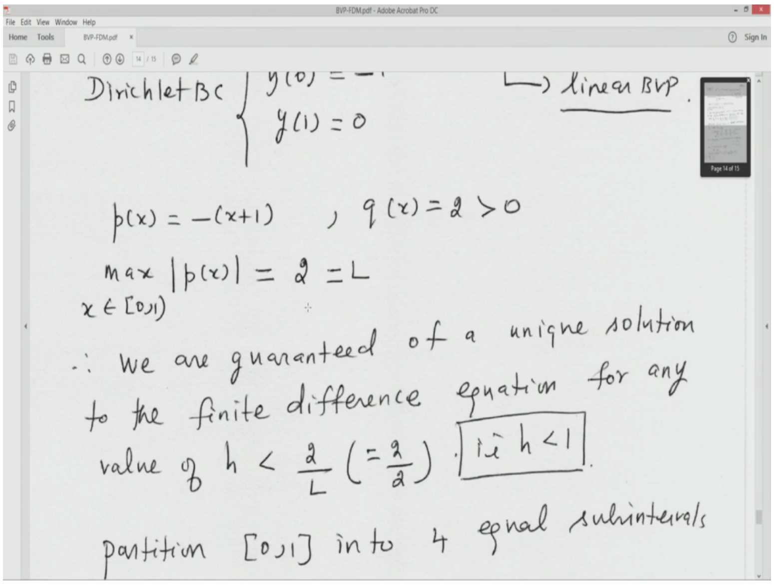

Now let us find out what are p of X, q of x in this differential equation. p p of x is – of x + 1, q of x is 2 which is positive, and we observe that maximum of modulus of p of x for x in this interval is 2 so this gives us L, so what should we choose our step size h as? It should be less than 2 by L, L here is 2 so take our step size h to be less than 2 by 2 that is less than 1. So we are guaranteed of a unique solution of the finite difference equation for any value of h, which is less than one. What is it that we should do to solve this system?

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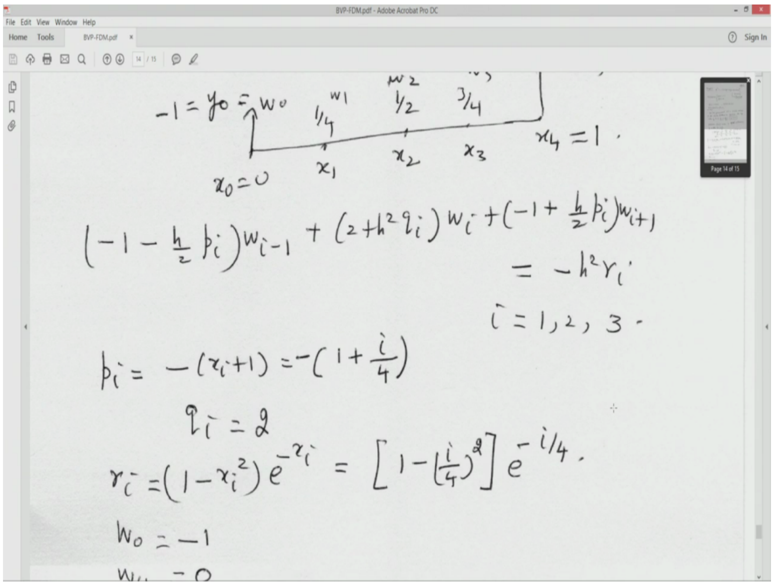

Step 1 is to partition the interval 0 to 1 into 4 equal subintervals, so step size would be 1 by 4. So what are these points, the points are x 0, x 2, x 3, x 4 of which x 1, x 2, x 3 are interior points, x 0 and x 4 r boundary grid points. So we replace the derivative which appears in the differential equation by appropriate second order accurate central difference formula multiply fraud by – h square and then collect the like terms and then obtain the finite difference equation, which is given by this, so this is a lead for i = 1, 2, 3, which are the interior points.

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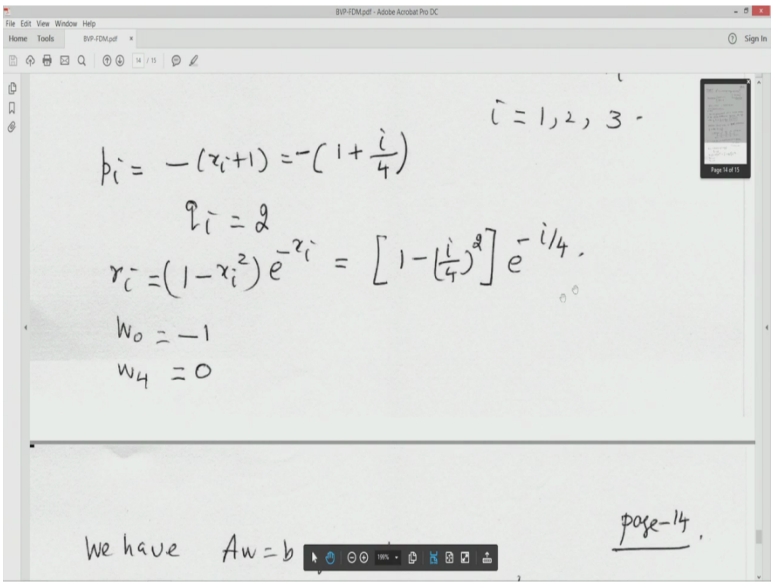

What are p i? p i are given by – of x i + 1, what is x i, x i is x 0 + i h, x 0 is 0, h is 1 by 4, so x i is i by 4, so – of x i + 1 is – of 1 plus i by 4. What about q? q of x is 2, so q of x i that is q i is 2. What about r of x? It is 1 – x square into e power – x, so that will r i is 1 – x i square into e power – x i and we know x i is i by 4 so we have substituted for x i as i by 4 and we have r i to be given by this.

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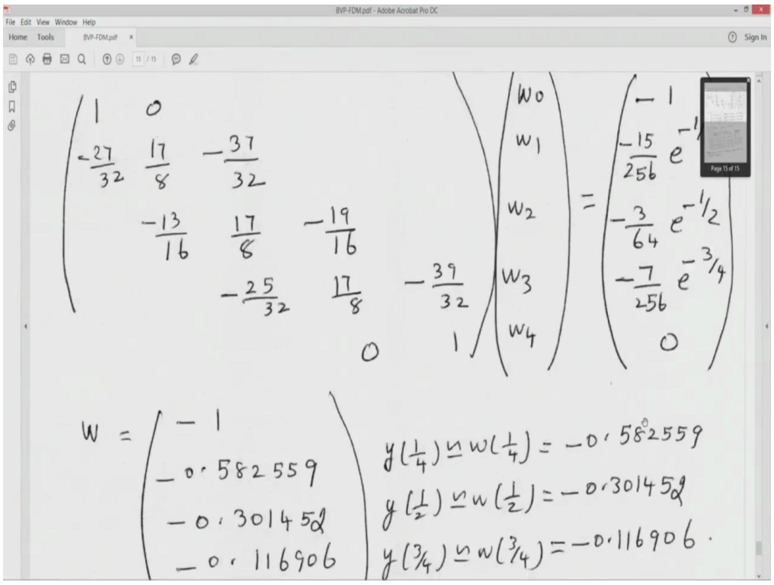

What other information do we have? We have w 0 to be – 1 and w 4 to be 0, so we now can write down the algebraic system equation in matrix form namely A w = b, so the first equation the w 0 = – 1 is given by the 1st equation here; 1 into w 0 + 0 into w 1 + 0 into w 2, etc is – 1. Then interior points we have the finite difference equation, from there we write down these 3 equations. The last equation tells us that w N is 0, what is w N? That is w 4, so w 4 is 0 and that is what is expressed here in the last row of this system. So we solve this system and obtain the solution vector w as this, which gives us the approximate solution at the interior points; 1 by 4, half and 3 by 4, which approximates the exact solution y at these points and the solution is given by this. We also know that the exact solution of the system is y of x = x –1 into e power – X, so we would like to find out what the absolute error is.

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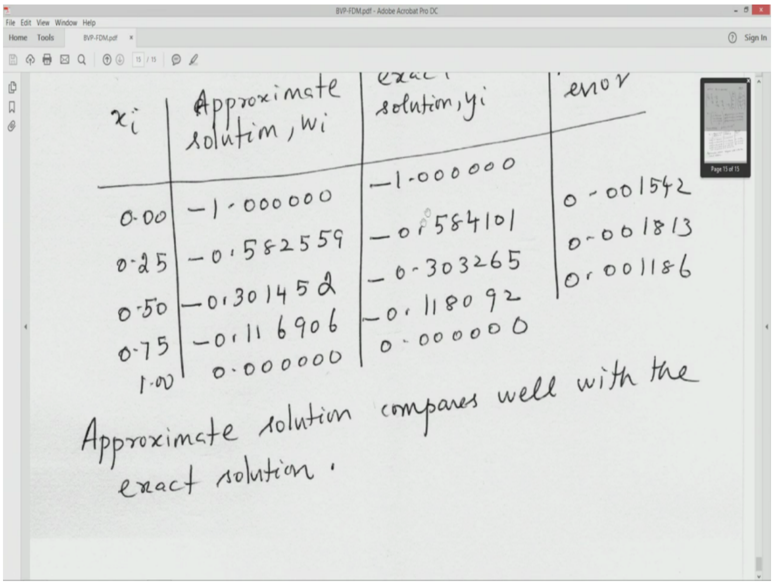

So in the table we present the x i values, the corresponding exact solution at these x i and the approximate solution at these x i and compute the absolute error. And we observe that the approximate solution compares very well with the exact solution since the absolute error is very small. So again we see that the approximate solution matches well with the exact solution in this case and the conditions that we have imposed on p of x and q of x are satisfied and they are sufficient to ensure that the system has a unique solution and this approximate solution compares well with the exact solution. So we have discussed how to solve linear boundary value problem subject to Dirichlet boundary condition by using a finite difference method.

The steps involved are such that we replace the derivatives by finite definite approximation and arrive at a system of algebraic equation and the solution of this system gives us the approximate solution of the interior points, which are obtained by dividing the interval into N equal subintervals. And we also have determined the conditions under which his beliefs labour boundary value problem, which is converted into a system of algebraic equation posses a unique solution, so we take care to choose the step size h in such a way that the conditions specified are satisfied, then we are guaranteed that with a unique solution for the system A w = b and we will be able to obtain the approximate solution to this boundary value problem by finite difference method of the interior points at which we seek solution.

Now the question comes, what do we have to do if we have a boundary value problem, which is governed by boundary conditions which are not of the Dirichlet type? Namely boundary conditions are either of the Robin type of the Neumann type, so we shall discuss the details now. The procedure is analogous to what we have done for the Dirichlet problem with slight modifications namely with boundary conditions are of the Robin type or the Neumann type, so we have to make some modifications to write down these boundary conditions and incorporate them into algebraic system, so let us see how this can be done.

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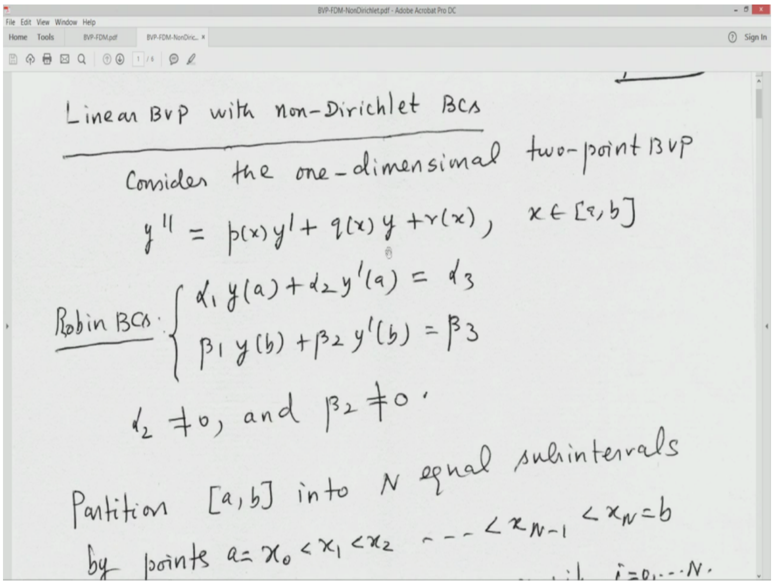

We now consider linear boundary value problem with non–Dirichlet boundary conditions. Consider the one–dimensional 2 point boundary value problem, it is a linear boundary value problem, so the differential equation is y double prime = p of x y prime + q of x into y + r of x for x in the interval a, b and the boundary conditions are of the Robin type. We impose the condition that Alpha 2 is different from 0 and Beta 2 is different from 0 because we are now interested in boundary value problem with non–Dirichlet boundary conditions. So as before you partition the interval a, b into N equal subintervals by points x 0, x 1, et cetera, x N – 1, x N, where the step size h is b – a by N, so the points x i are given by x 0 plus i h for i = 0 to N.

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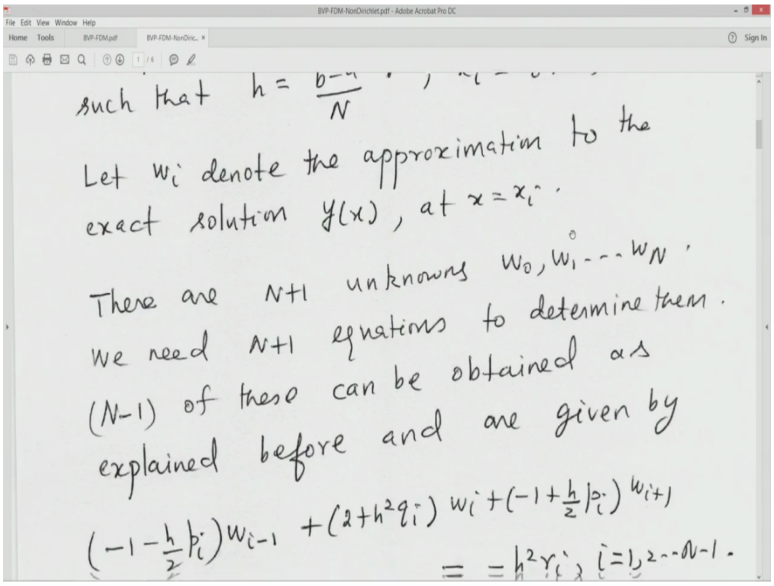

So as before, let w i denotes the approximation to the exact solution y of x at letter x = x i, so there are N + 1 unknowns w 0, w 1, et cetera, w N so we require N + 1 equations to solve for these N + 1 unknowns out of which N – 1 equations can be obtained as before by replacing the derivatives which appear in the differential equation by means of appropriate finite difference approximation. And using the resulting finite difference equation at the interior points i = 1, 2, 3 up to N – 1. So if we do that then we end up with this equation, which is valid or i = 1 to N – 1, this is exactly obtained in the same way as we have done in the previous case where we have explained the method of solution of boundary value problem with Dirichlet boundary conditions. Now, we have to apply the boundary conditions and see what we get from the boundary conditions.

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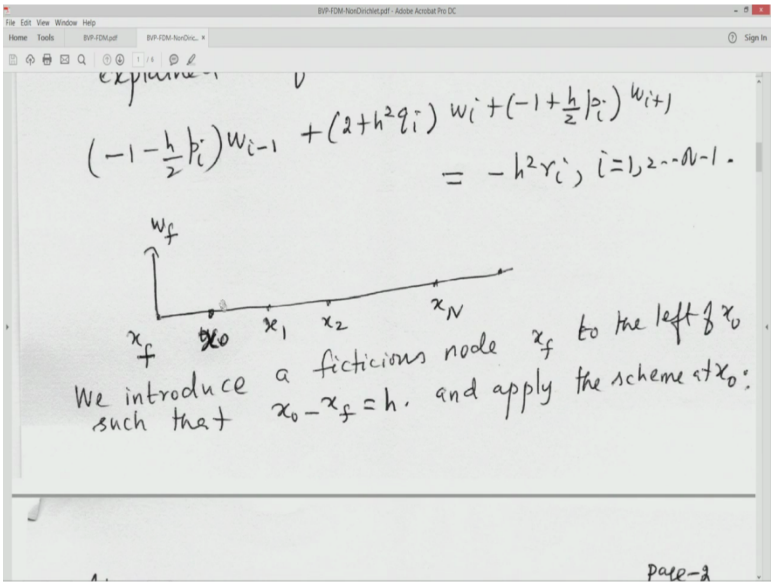

Let us look at boundary conditions, they are either of the Robin type or they are of the Neumann type must so we deal with the boundary conditions as follows; namely when we are given the boundary condition at the left endpoint namely x 0, which is A we introduce a fictitious node x f to the left of x 0 such that x 0 – x f is h, the step size that we have already taken. I repeat, when we want to apply the boundary condition at the point x 0 and get the corresponding equation, which should be incorporated in the algebraic system we take a fictitious point letter x f to the left of x 0 at a distance of x 0 and call the approximate solution at that point as w f.

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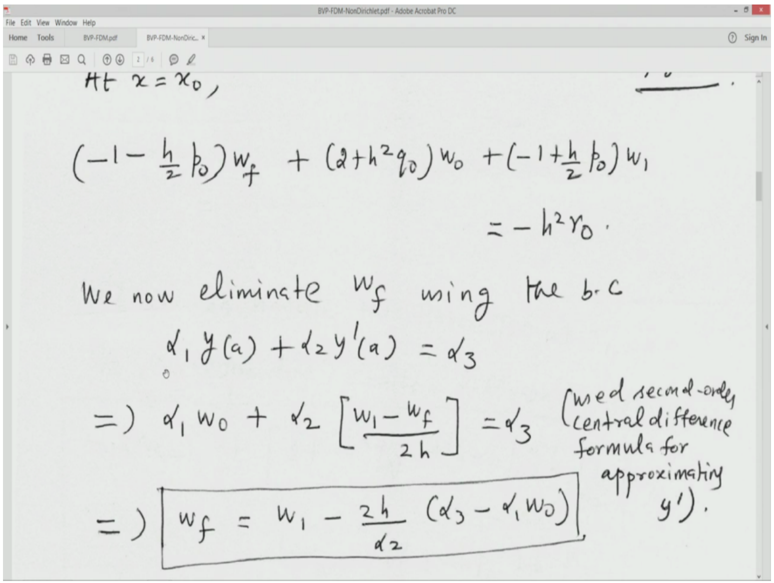

And I apply at x = x 0 what happens to the finite difference equation? We see that the finite difference equation turns out to be this, when I take i = 0 in the equation that I have shown. This involves w f, do I know w f? Yes the 1st boundary conditions which is prescribed at the left–hand point A will help me to find out what w f is, let us see how it is obtained. What is the boundary condition? Alpha into y of a + Alpha 2 y prime a is Alpha 3, so this is Alpha 1 into y of a is y at x 0 and its approximation is w 0, so Alpha 1 into w 0 plus Alpha 2 into we need replace the derivative y prime at x 0, what is y prime at x 0? We replace it by a second order accurate finite difference method what will that be, that is going to be w 1 – w f by 2 h, let us see how we have obtained.

So we want to apply the second–order finite difference approximation to the 1st derivative at this point x 0, so y prime x 0 is equal to the value of y at this point – the value of y at this point x f divided by the distance between the 2 points namely 2 h and that is what we have written here. So y prime at a is replaced by w 1 – w f by 2 h and the boundary conditions tells us that this is equal to Alpha 3. And from here we can obtain what w f is, so w f will be w 1 – 2 h by Alpha 2 into Alpha 3 – Alpha 1 w 0. Of course w 0, w 1 are unknowns, so w f is also unknown that appears in the differential equation and since x f is a fictitious point, I am not interested in obtaining the solution there so I express the solution at x f in terms of the solution at the other points at which I seek the solution.

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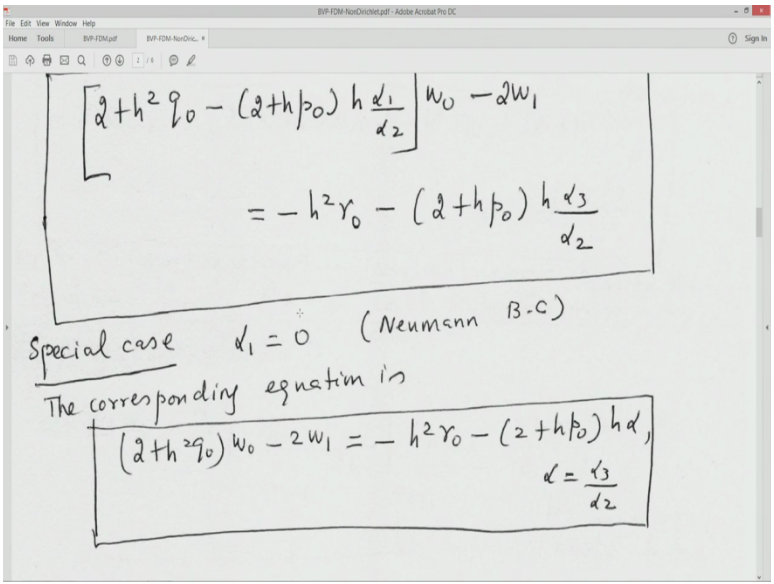

So x = x 0 = a, the equation that I should use is this which is obtained by substituting for w f in this equation, so this is going to be the equation that I should use at x f = x 0, which is a. Now the situation has been obtained when we have a Robin boundary condition. Let us consider the special case, when we have Alpha 1 to be equal to 0 then our boundary condition turns out to be Alpha 2 y prime a is Alpha 3, so y prime at a is Alpha 3 by Alpha 2 which I denote by Alpha. So y prime at a is Alpha and therefore that corresponds to a Neumann boundary condition because the derivative alone is specified at one endpoint a so in that case I can reduce the equation that I should use from here by substituting Alpha 1 = 0 in this, so that gives you the equation as this.

So if a Neumann boundary condition is specified as the first boundary condition, then the finite difference equation that should be used and should be incorporated in the algebraic system is given by this. So I repeat, if it is a Robin boundary condition this will be used in the algebraic system as the first equation. If Neumann boundary condition is specified as the first boundary condition at the endpoint a, this equation will be used as the first equation in the algebraic system. And from second to N – 2 equations are obtained from the differential equation by replacing the derivatives by appropriate finite difference approximations and so on. And what happens when the boundary condition which is specified at the other endpoint namely B is of the Robin type?

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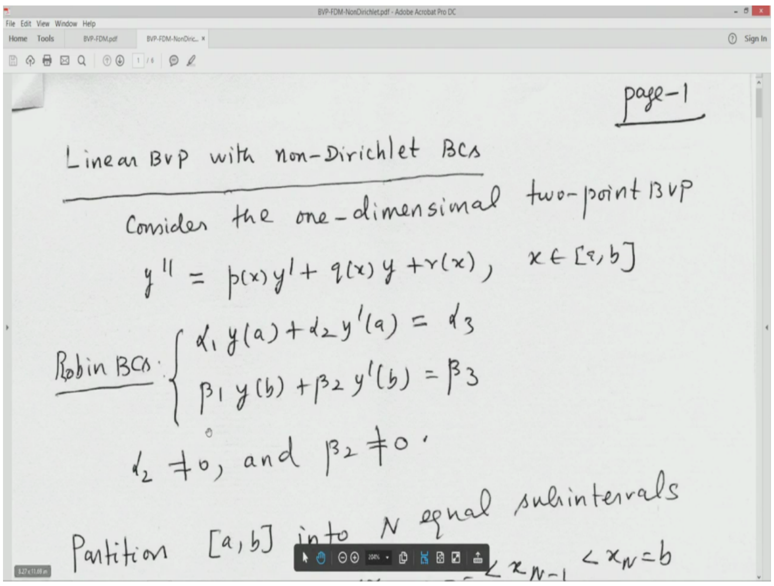

Namely, Beta 1 into y of b + Beta 2 y prime at b = Beta 3, in this case again this y prime b will have to be replaced by a second order accurate central difference formula that would require an additional point to the right of b so we introduce a fictitious point, b is our x N so the fictitious point, which appears on the right–hand side is going to be x N + 1 such that x N + 1 – x N is h, so the derivative at little x N is going to be value at x N + 1 – value at x N – 1 divided 2 h, so it is going to be y prime at b will be replaced by w N + 1 – w N – 1 divided by

2 h. So replacing 1 prime b in that way the equation that should be incorporated as the last equation in the algebraic system will be obtained by substituting for w N + 1 obtained from this boundary condition. So we have the following system when we complete doing all this which when solved will give us an approximate solution

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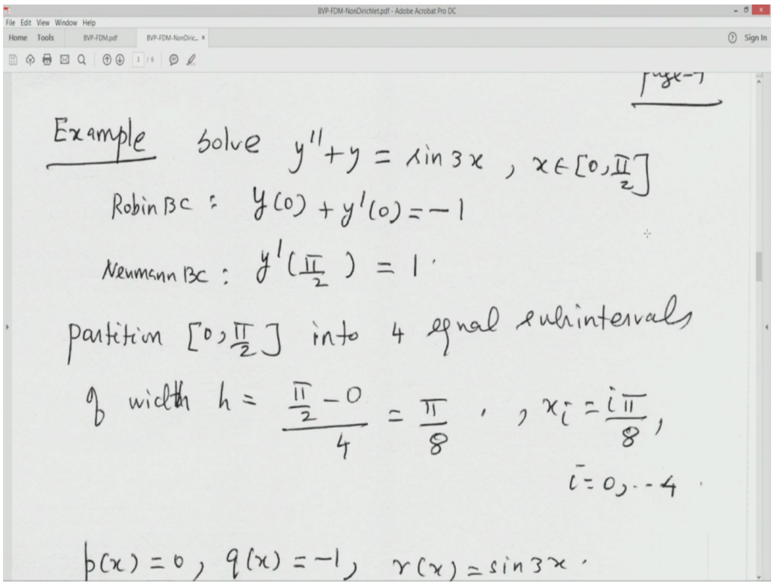

So let us consider an example to illustrate how we can solve the boundary value problem for non–Dirichlet boundary conditions. We now consider an example to illustrate the method of solving boundary value problem governed by differential equations of second order with non–Dirichlet boundary conditions. So the given differential equation y double prime y = Sine 3X in the interval 0 to Pie by 2. The first boundary condition is of the Robin type, y of 0 + y prime of 0 = – 1 and second boundary condition is a Neumann boundary condition, y prime at Pie by 2 is 1, so we partition the interval 0 to Pie by 2 into 4 equal sub intervals of width h, which is Pie by 2 – 0 by 4 which is Pie by 8 so our x i will be i Pie by 8 because x 0 is 0 and i will vary from 0 to 4.

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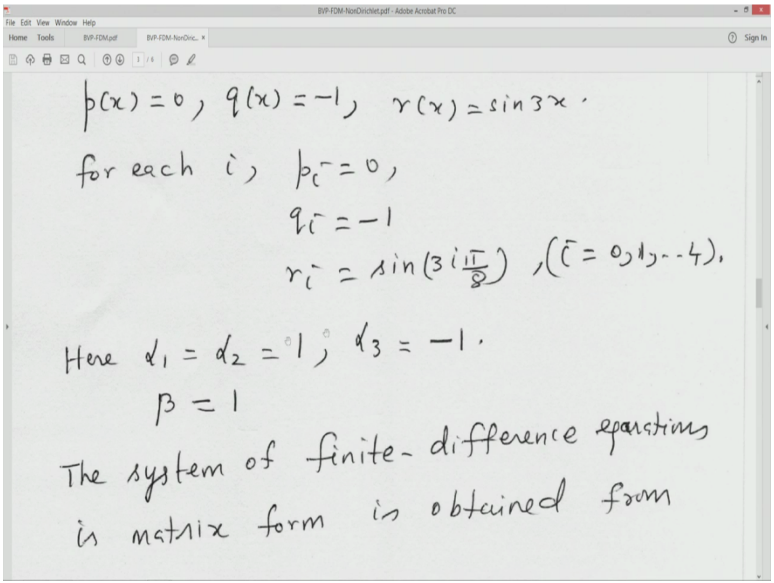

Let us find out what are p of x and q of x and r of x. p of x is zero, q of x is –1 and of x is Sine 3 X. So for each i p i is 0, q i is – 1 and letter r i is sine 3 i Pie by 8 for i = 0 to 4. And we also know that Alpha 1 and Alpha 2 take the value 1, Alpha 3 is – 1 and beta is 1 because the second boundary condition is a Neumann type of boundary condition. So we obtain now the system of finite difference equations in matrix form. What do we do, we replace the derivatives in the system by appropriate finite difference methods so we would like to replace them by second order accurate finite difference formulas in this problem and then evaluate the differential equation at the interior points and obtain the resulting equation for the unknowns which are approximations to the exact solutions of the differential equation, namely the unknowns are w i.

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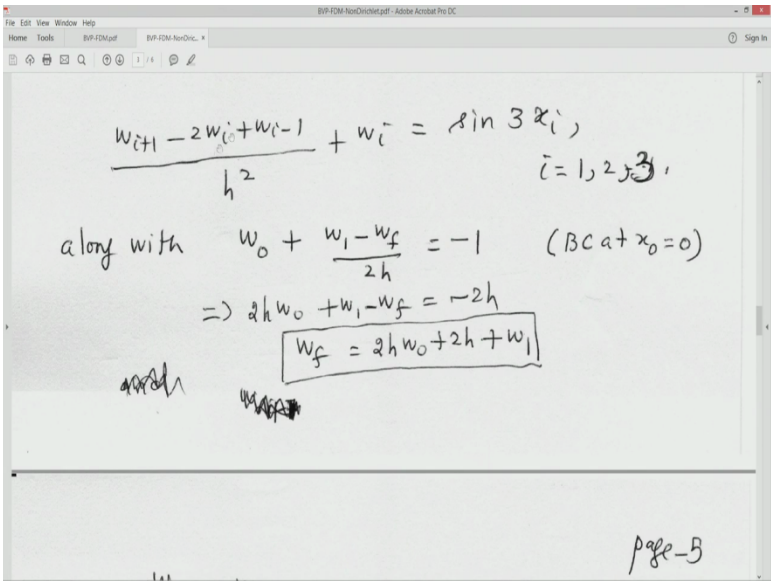

When we do all this we end up with this equation namely; w i + 1 – 2 w i + w i – 1 by h square + w i is Sine 3 x i for i = 1, 2, 3 at the interior points. Now the 1st boundary condition is a Robin boundary condition and therefore we have to replace the derivative there by means of a second order accurate central difference scheme, when we do that we get w f which is the approximate value of y at x f, where x f is a fictitious point introduced to the left of x 0. So w f turns out to be this and it is expressed in terms of w 0 and w 1, so this is obtained from the 1st boundary condition which is of the Robin type.

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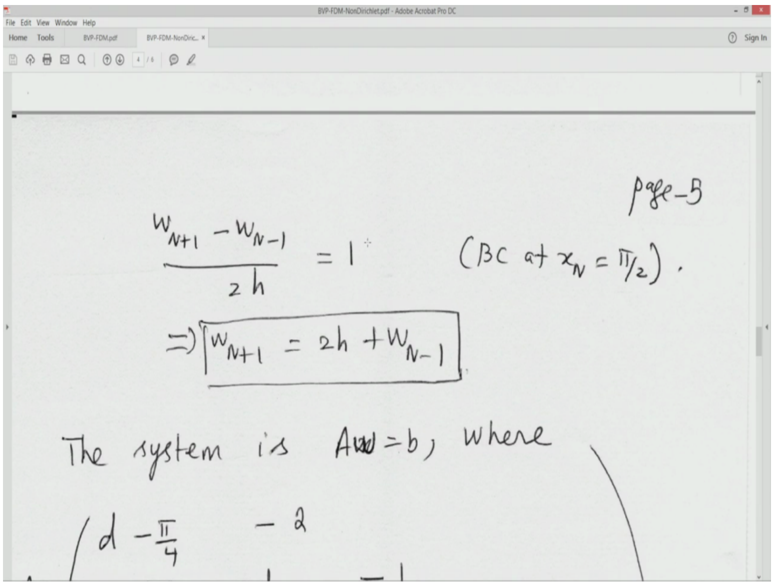

The second boundary condition is a Neumann type of boundary condition and therefore we replace the derivative there by means of the second order accurate finite difference approximation. So y prime and the endpoint Pie by 2 is specified to be 1, so we introduce a fictitious point x N + 1 to the right of x N and replace the derivative at x N by the value at this point – the value at the left point, which is x N – 1 divided by the distance between the 2 points which is 2 h and that is what we have done here; w N + 1 – w N – 1 by 2 h is 1, this is the approximation that we get when we replace the first derivative at the endpoint namely x N = Pie by 2. From here we get what w N + 1 is which is the approximate solution value at the fictitious point which we have introduced namely x N + 1.

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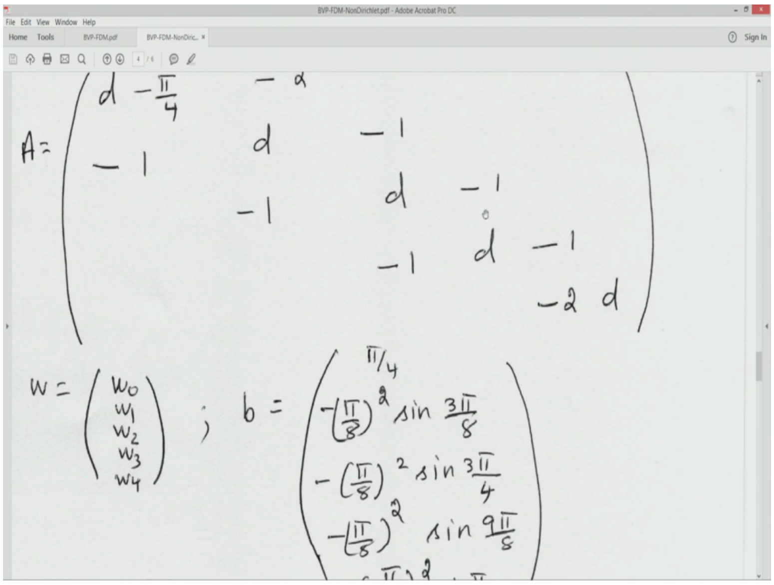

So all the equations that have to be incorporated into the system are available with us so we write down the system A w = b as this and the coefficient matrix and w has components w 0 to w 4 and vector B is given by this. In this coefficient matrix we have d which is given by 2 – Pie by 8 the whole square so we solve this system and obtain the solution vector as this. So we have been able to use finite difference method to obtain approximate solutions to the boundary value problem where non–Dirichlet type of boundary conditions are specified namely; Robin type of boundary conditions and Neumann type of boundary conditions are specified. So we would like to now see whether we can obtain the absolute and check whether our approximate solution to this problem is reasonably good, so let us find what the exact solution is.

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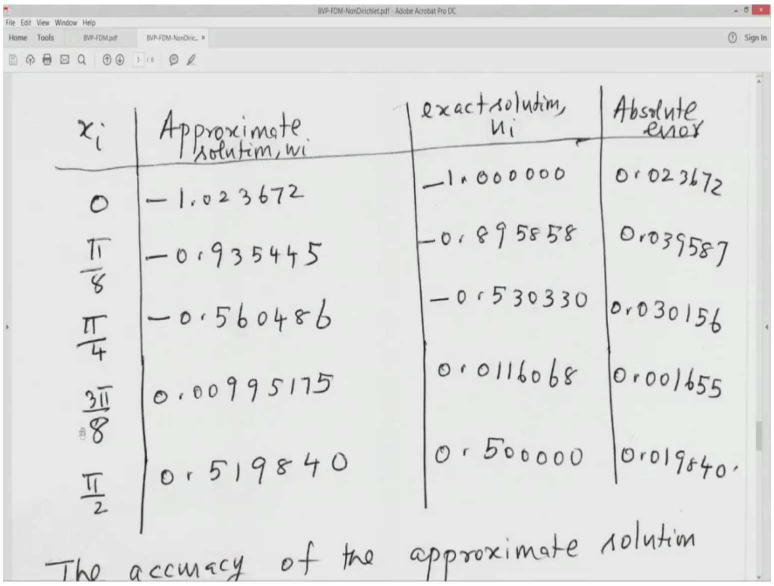

We observe that the exact solution is given by y of x = – Cos x + 3 by 8 Sine x – 1 by 8 into Sine 3 x. So we write down in this table the x i values the corresponding approximate solutions that we have obtained using finite difference method, the exact solution and then compute what the absolute error is at each of these x i, and we observe looking at the column in which the absolute error which is the difference between the exact solution and the approximate solution at each of the x i, we observe that the approximate solution agrees quite well with the exact solutions since the absolute error is small. So the accuracy of the approximate solution is quite reasonable in spite of the fact that the step size h chosen by us namely Pie by 8 is not very very small.

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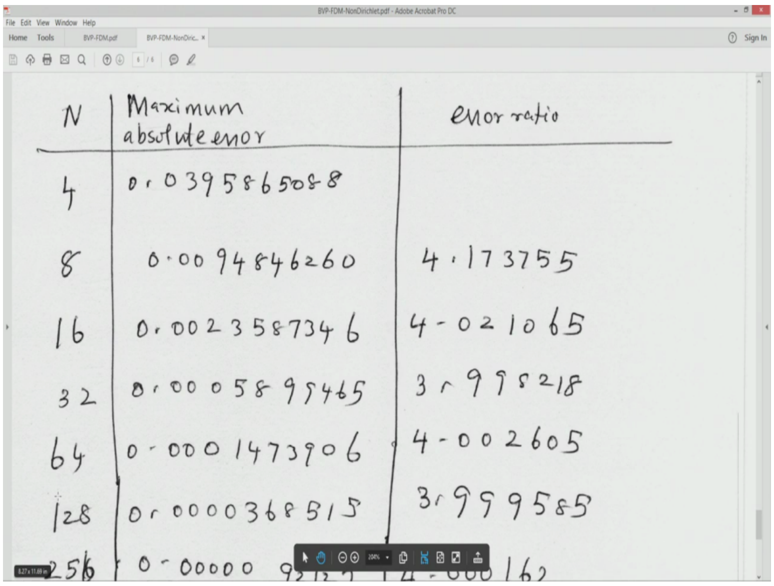

In addition we would like to check the numerical accuracy of the second order scheme that we have used to solve this boundary value problem. So what do we do in order to access this? We increase the number of sub intervals into which we have partitioned the interval 0 to Pie by 2. We used 4 sub intervals in our computations earlier and obtained the approximate solution, so we double this number of sub intervals namely we divide the interval 0 to Pie by 2 into 8 equal sub intervals of step size Pie by 2 by 8 so that Pie by 16 is h, so h is halved as a result of doubling the number of sub intervals and we continue to do this and in each of these cases we list down what the maximum absolute error is and see what the error ratio is, Namely the ratio with N = 8 by N = 4 of the maximum absolute error that we have obtained.

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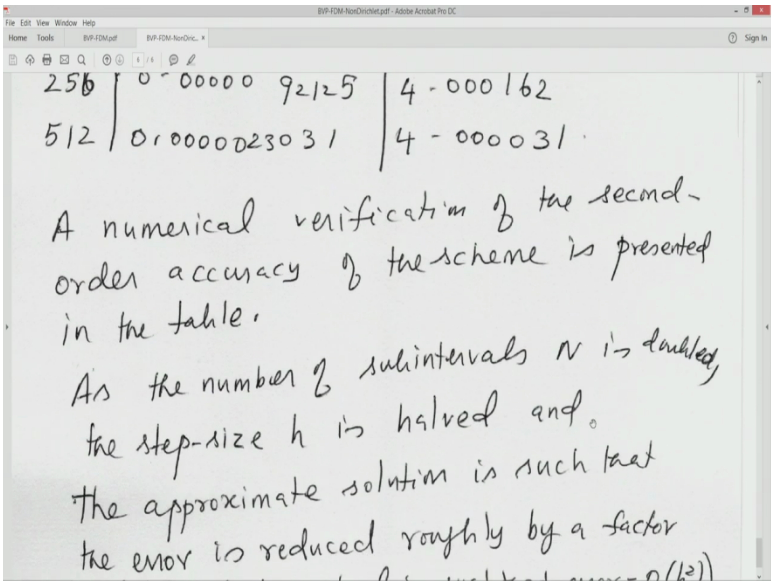

We observe that the error is roughly reduced by a factor of 4 when we double the number of sub intervals say from 4 to 8, which clearly indicates the numerical accuracy of the second order finite difference scheme that we have used in solving this boundary value problem by finite difference method. So this happens each time we double the number of sub intervals in which case the step size h is halved, so h become smaller and smaller and we observe that the error ratio in this case is close to 4 showing that our approximate solution is very close to the exact solution and the second order scheme that we have used has helped us to obtain solution which are very close to the exact solution. So in this lecture we have discussed finite difference methods of solving linear boundary value problems, which are governed by second order differential equation subject to either Dirichlet type of boundary conditions or Neumann type of boundary conditions or Robin type of boundary conditions.

So we close our discussions on the finite difference techniques of solving boundary value problems and in the next class we shall consider another technique which is known as a Shooting technique for solution of boundary value problems, which are linear boundary value problems