Numerical Analysis
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Lecture No 25
Numerical Solution of Ordinary Differential Equations-8-Linear Boundary Value Problems (Finite difference method)

Good morning everyone, in our discussion on numerical solutions on ordinary differential equations, all along we were concerned with the numerical solution of initial value problems governed by 1st order ordinary differential equations of the form d y by d x = f of x, y subject to some initial condition y of x not equal to y 0. We now move on to the boundary value problems, which are governed by 2nd order ordinary differential equations along with boundary conditions which are specified at the endpoints of a closed interval. So our goal is to develop numerical techniques for approximating the solution of 2 point boundary value problems.

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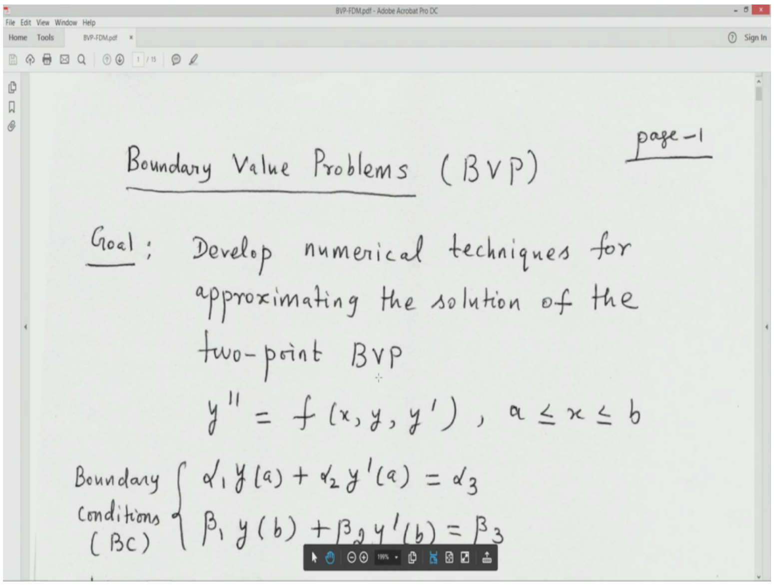

They are in general written in the form y double prime = f of x y prime. Here y is the unknown function, it is a function of x, prime denotes derivative with respect to x and here x is an arbitrary function of 3 of its arguments x, y and y prime. This differential equation defined in the interval a–b, which is a closed interval and y also satisfies the boundary conditions namely the boundary conditions are of the form Alpha 1 y at a + Alpha 2 into y prime at a is Alpha 3. Beta 1 y at b + Beta 2 y prime at b = Beta 3, here Alpha 1, Alpha 2, Alpha 3, Beta 1, Beta 2 , Beta 3 are constant which are known to us because the boundary conditions are prescribed.

And if the right–hand side function f is such that it is of the form p of x into y prime + q of x into y + r of x, for some function p, q and r then we say that the boundary value problem is a linear boundary value problem and it is governed by the 2nd order differential equation y double prime = p of x into y prime plus q of x into y plus r of x along with the boundary conditions which are given here. If it is not of this form, then we say that the boundary value problem if a non–linear boundary value problem. Now we would like to understand these boundary conditions, we observe that we have a linear combination of the unknown functions at one endpoint and its derivative at that endpoint is prescribed as Alpha 3 in the 1st boundary condition.

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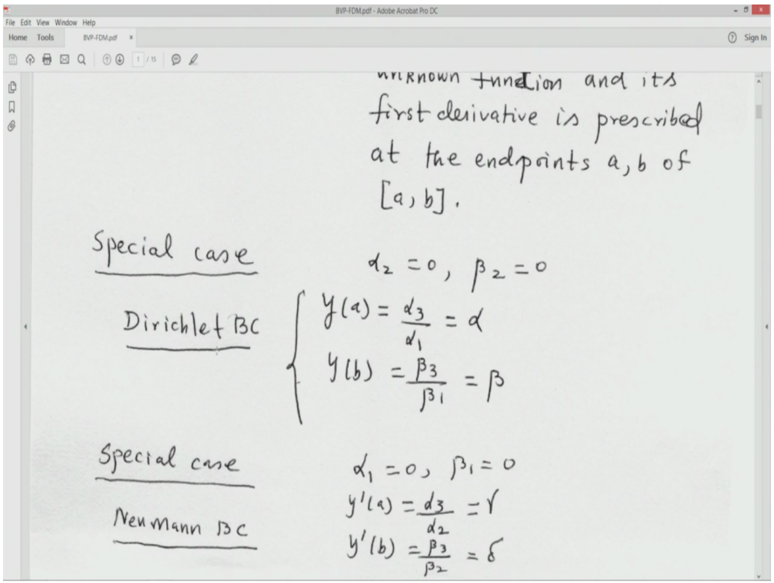

And in the 2nd boundary condition similarly we have a linear combination of the function value and its derivative at the other end point b is prescribed and is given as a Beta 3, so such type of boundary conditions are known as Robin or Mixed boundary conditions. There are special cases, if suppose Alpha 2 is 0 and Beta 2 is 0, let us see what it means. If Alpha 2 is 0 and Beta 2 is 0 then we have Alpha into y of a is Alpha 3 or y of a is Alpha 3 by Alpha 1. And Beta 1 y of b = Beta 3, so y of b is Beta 3 by Beta 1, so in the special case when Alpha 2 and Beta 2 are 0 then the boundary conditions given above reduced to conditions of the form y of a = Alpha and y of b = Beta, where Alpha and Beta are given to us, such type of boundary conditions are called Dirichlet boundary condition.

So by a Dirichlet boundary condition we mean the unknown function is specified at the endpoints of the interval a, b in which the differential equation is satisfied. Then there is another special case where Alpha 1 is 0 and Beta 1 is 0 let us see this case. If Alpha 1 is 0 and

Beta 1 is 0 then we have boundary conditions which are given only in terms of the derivative of the unknowns at the 2 endpoints a and b, so in this case the boundary conditions are of the form y prime at a is some Gamma and y prime at b is some Delta so the conditions are given in terms of the derivative of the function at the boundary points namely a and b, then such type of boundary conditions are called Neumann boundary conditions.

So boundary value problems governed by the 2nd order differential equation subjects to boundary conditions, which are of the Robin type or of the Dirichlet or of the Neumann type will be given to us and we now try to develop numerical methods for solving any of these boundary value problems using finite difference method. So let us try to understand this finite difference method and see how we can apply the finite difference scheme for solving boundary value problems which are linear boundary value problems.

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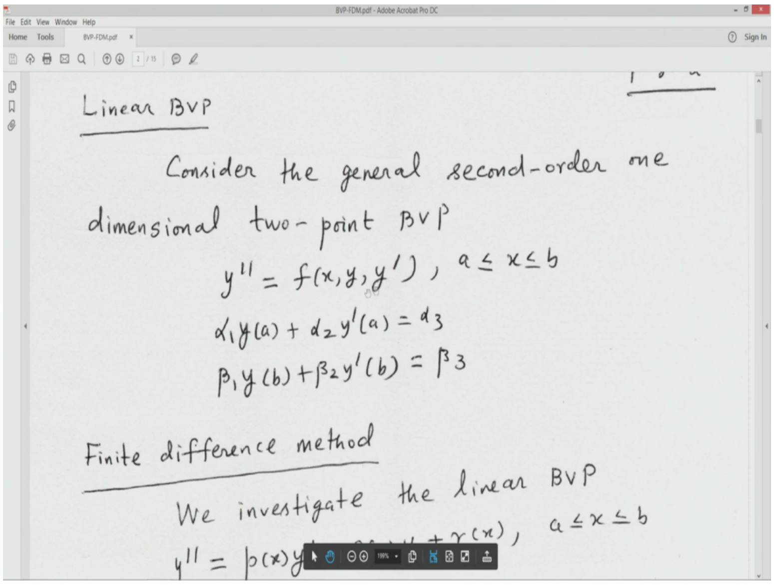

So we consider general second–order one–dimensional 2 point boundary value problem; namely y double prime is f of x, y, y prime, which is defined for x in the interval a, b along with this Robin type of boundary condition. So as we said we would like to understand finite difference method of solving linear boundary value problems subjects to the boundary conditions which are specified in this form and we focus our attention on the linear boundary value problem and so right–hand side f is of the form p of x into y prime + q of x into y + r of x, where x lies between a and b.

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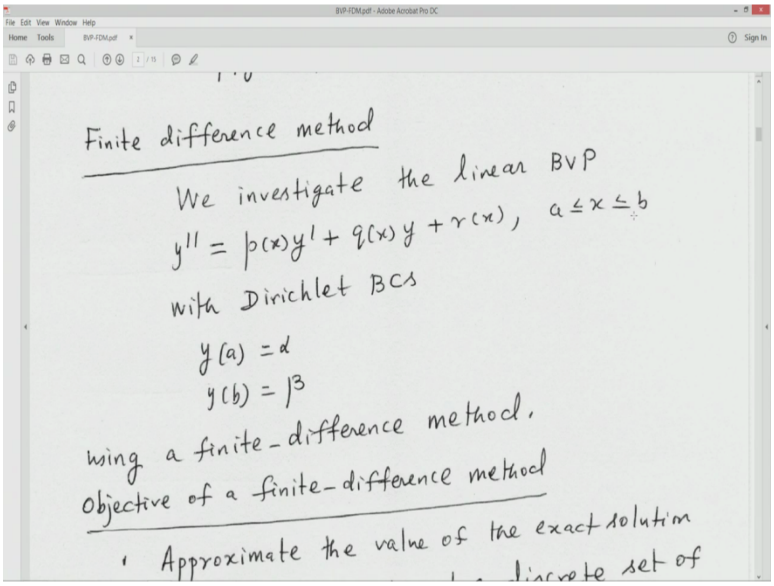

So this is our 2nd order differential equation and let us 1st focus on the case then we are given, Dirichlet type of boundary conditions namely the unknowns are prescribed at the 2 endpoints a and b, so the conditions are y at a is Alpha and y at b is Beta, so the question is solve these boundary value problem subject to Dirichlet around the conditions using a finite difference method, so what is the objective of the finite difference method?

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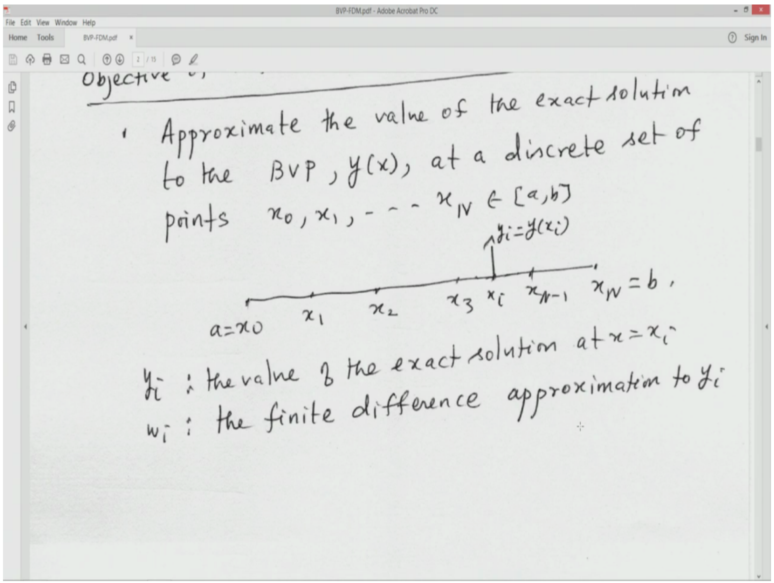

The objective is that here we approximate the value of the exact solution to the boundary value problem y of x at a set of points namely x 0, x 1, etc x n, which belong to the interval where different equation is defined. So this interval a, b is one in which the differential equation is defined and we select points x 0, x 1, etc , x n in this interval and try to get the approximate solution at these discreet points that is what will be obtained using finite difference method. So to do that let us be not by y i the value of the exact solution at the point x = x i. And let us denote by w i the finite difference approximation to this exact value which is y i at any point x i, so what do we do in finite difference method?

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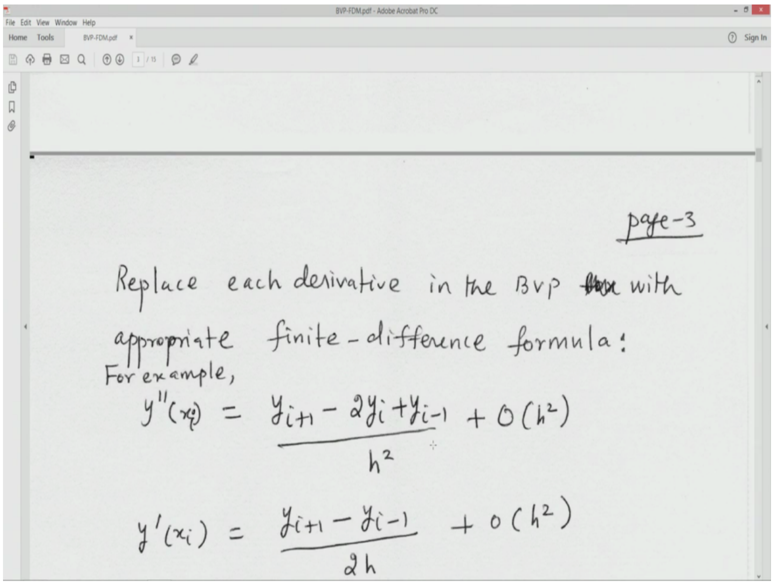

We are given a 2nd order differential equation satisfied in an interval so we try to replace each of the derivatives, which appears in the differential equation by appropriate finite difference formula. And I recall the finite difference formulas which we have developed a number discussion on numerical differentiation, so we already know how to approximate the derivatives by means of finite difference approximation. So for example, if the 2nd order derivative appears in the differential equation, you can replace the 2nd derivative y double dash of x i by 2nd order accurate finite difference formula, which is a central difference formula given by this and the error is of order of h square. And if there appears 1st order derivative in the differential equation, then replace that again by a 2nd order accurate finite difference formula and the order of this method is 2 and the error is of order of h square.

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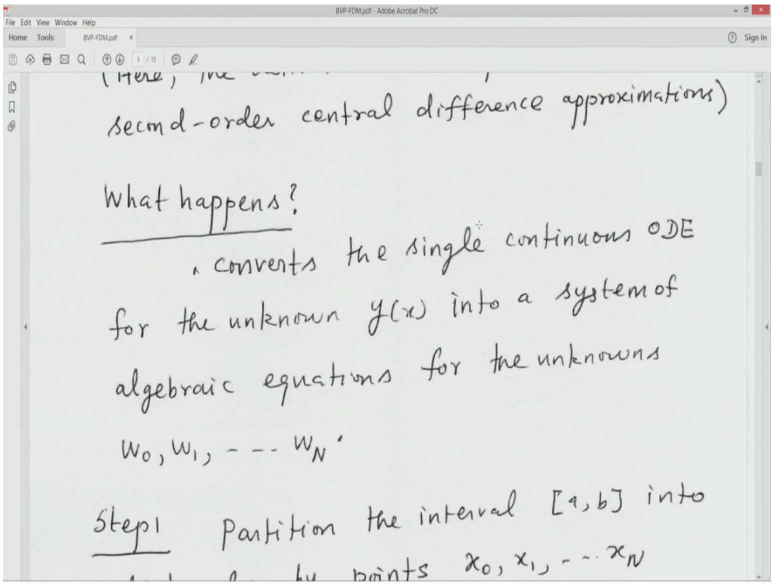

So you replace the derivative, which appear in the linear boundary value problems governed by the 2nd order situation which has the 2nd order derivative and the 1st order derivative, by means of these finite difference approximations. Then what happens as a result of this you will see that when you replace the derivatives by appropriate finite difference approximations then that single continuous differential equation for the unknown y of x is reduced to a system of algebraic equations for the unknowns w 0, w 1, w 2, et cetera, w N, which are values at the point x 0, x 1, x 2, etc, x n and these are approximate solutions of this differential equation or the boundary value problem, so let us understand how we can do it.

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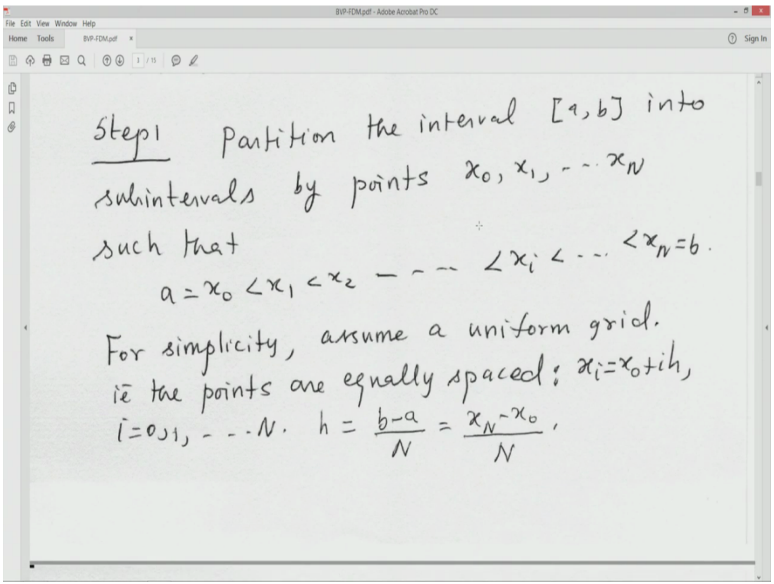

What is step 1? Step 1 will be to get these points x 0, x 1, etc, x n at which we seek approximate solution of the differential equation, so how do we get these points? We take the interval a, b, we partition this interval into number of sub intervals by means of points x 0, x 1, x 2, etc, x n such that they satisfy the condition. So for simplicity we assume a uniform grid, what does it means? We choose these points x 0, x 1, x 2, etc, x n in such a way that they are equally spaced. So how do we get it, we divide the interval a, b into sub intervals of equal width h, what is the step size h?

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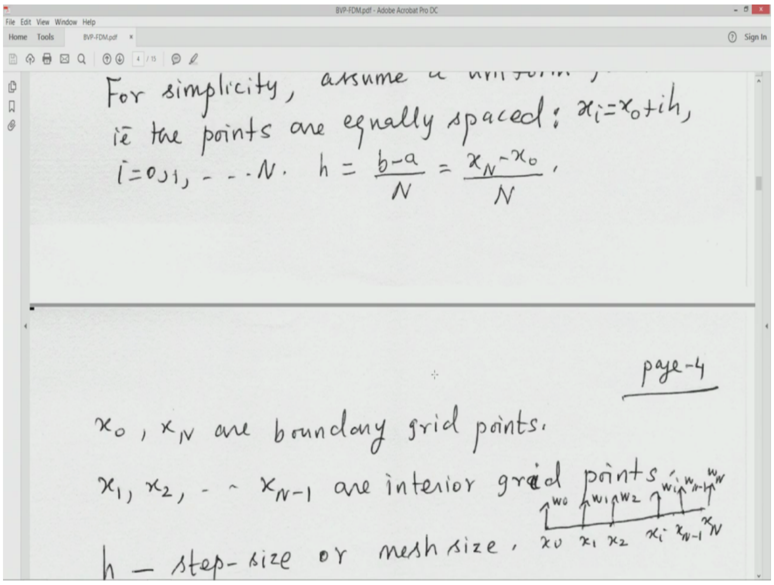

The step size h will be x n – x 0 divided by capital N, if we want divide the interval a, b into N equal sub intervals. So our step size h will be x n – x 0 by N, so our points x i which appear on in the interval r of the form x 0 + i h, so the distance between any 2 successive points is going to be h, which is b – a divided by N. So having chosen these points, these are the points which are referred to as grid points. The endpoints a and b, where a is x 0 and b is x n, they are called the boundary grid points.

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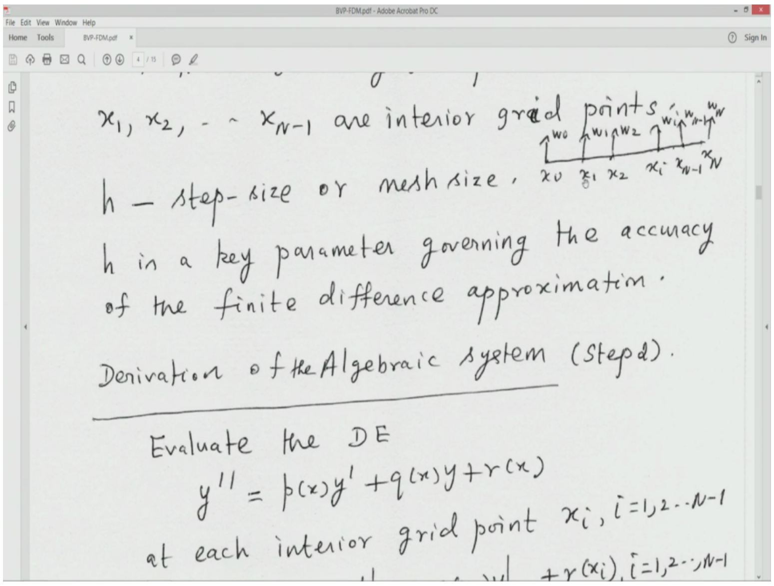

On the other hand, the other points which are x 0, x 1, x 2, etc, x n – 1, which appear in the interior they are referred to as interior grid points. At each of these points x i the solution, the exact solution is y of x i which we denote by y of x i. And approximation to this exact solution y i are denoted by w i, w i is an approximation to the exact solution at the point x i namely it is an approximation to the exact solution y of x i, which we denote by y i. And what is this h, h is the step size or the mesh size and h plays a vital role namely it is a key parameter, which governs the accuracy of the finite difference method, so what is the next step? Having partitioned the intervals and obtain the points of which we seek approximate solution, we move on to getting what is known as the algebraic system from the finite different equations so let us understand how we do this.

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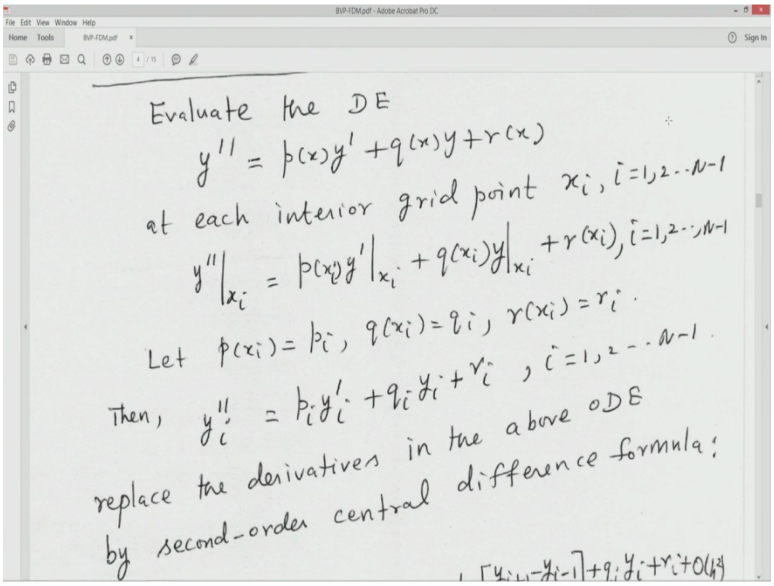

So we take the given differential equation, which describes this boundary value problem and our focus is going to be on linear boundary value problems so our differential equation is of the form y double prime = p of x into y prime + q of x into y + r of x, so we evaluate we know that the differential equation is satisfied at each point x in the interval a, b so we evaluate the differential equation at each of the interior points that we have chosen in the interval a, b. What are those interior points, x 1, x 2, etc, x n – 1, so we have y double prime evaluated at x i will be p at x i into y prime at x i + q of x i into y at x i + r of x i and this is for i = 1 to N – 1, the differential equation is satisfied in the interior of the interval is reflected here, now what is p of x i, it is denoted by p i just on notation, q of x i is q i and r of x i is r i.

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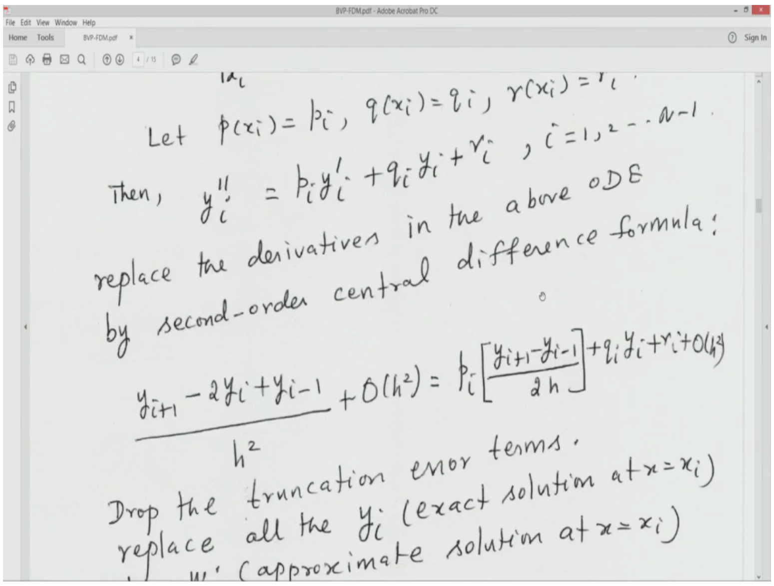

Since the differential equation is given to us, we know what are p of x, q of x r of x and therefore we know what are p i, q i and r i at each of the i, which is 1, 2, 3 up to n – 1. So we use the notation and write down the step as y double prime at i is p i into y prime at i + q i into y at i + r i for i = 1 to N – 1. What is the next step, what did we say? Our goal in finite difference method is to replace each derivative that appears in the differential equation by an appropriate finite difference formula so we replace now the 2nd order derivative, the 1st order derivative by appropriate finite difference formulas. In what follows, we shall consider the 2nd order accurate Central difference formula for replacing the 2nd order derivative and the 1st order derivative which appear in the differential equation governing the boundary value problem.

So what is y double prime i, finite difference formula for y double prime i is that it is y i + 1 – 2 y i + y i – 1 by h square and the method is of order 2, so the error is of order of h square and that is equal to p i into y prime i, so again we replace the 1st derivative by 2nd order accurate finite difference scheme, which is the central difference formula so it is y i + 1 – y i – 1 by 2 h + q i into y i + r i right + here this is a 2nd order method, so plus order of h square. So now I dropped the truncation error term, so when i drop the truncation error term I have to replace all these exact values at x i by the corresponding approximate values.

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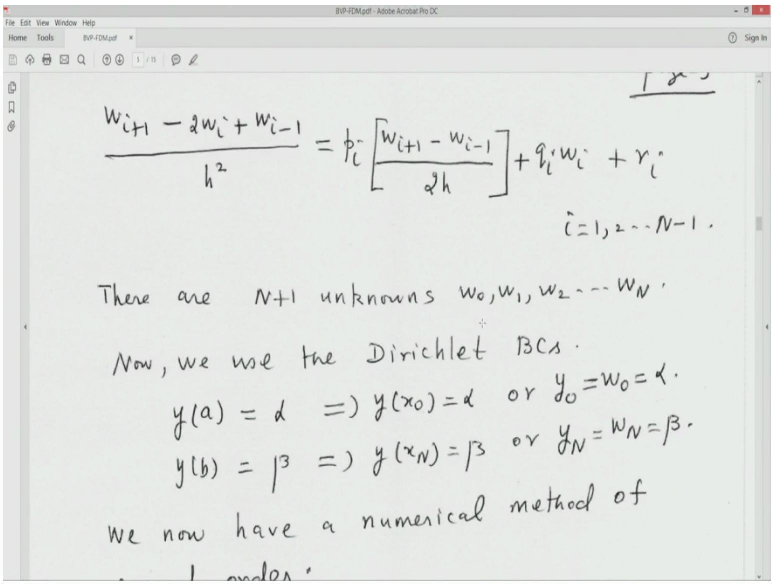

As a result i will get the scheme to be w i + 1 – 2 w i + w i – 1 by h square and that is equal to p i into w i + 1 – w i – 1 by 2 h + q i into w i + r i and this is satisfied for all i, which is 1, 2, 3, etc up to N – 1. So this will give us N – 1 equations in the unknowns w i. In addition there will also be w 0 and w N appearing here, when will that come when i = 1 see you have i

– 1 so that will give you w 0. And when i = N – 1 when you substitute it here, for that you get w N – 1 + 1 and that is w N, so you will also have w 0 and w N appearing when i is 1 and i is n – 1 respectively, so totally there will be n + 1 unknowns namely w 0, w 1, etc, w N. But we have not made use of the boundary conditions which are given to us so far, so this is the time that we make use of these boundary conditions, let us see what it gives us.

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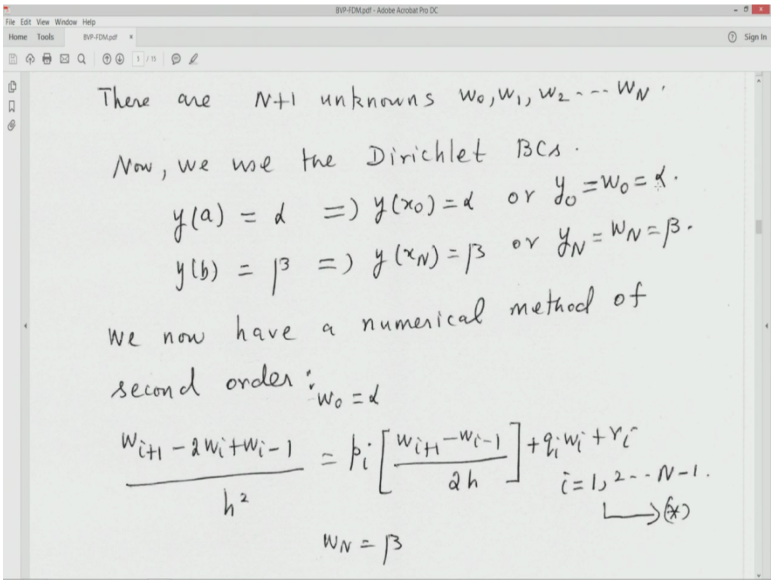

So we use the Dirichlet boundary condition, the 1st condition is the unknown y is specified at one endpoint namely the left endpoint a. What is it, y of a is Alpha and a is our x 0, so y of x 0 is Alpha, y of x 0 is denoted by y 0 so y 0 is Alpha and how do we denote the approximation to this in our notation, it is w 0, so w 0 = Alpha is prescribed so we know w 0 is. Similarly y at b the other endpoint is prescribed to be Beta so y at x n is Beta, in our notation it is y suffix n and that is Beta and how do we denote this approximate value at x n or our notation says that it is the w N, so w N is Beta and Beta is prescribed so we also know w N. So from the boundary conditions we are able to w 0 = Alpha and w N = Beta, so we now have a numerical method of 2nd order, why 2nd order because we have replaced the derivative which appear in the differential equation by a 2nd order accurate Central difference formula.

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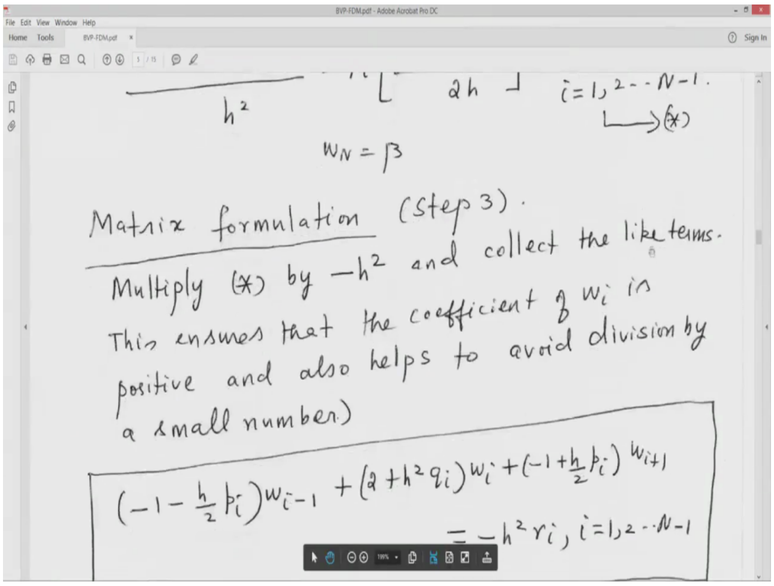

So we have a numerical scheme which is of 2^{nd} order and the scheme is w i + 1 – 2 w plus w i – 1 by h square is p i into w i + 1 – w i – 1 by 2 h + q i w i + r i and this is true for i = 1 to N – 1 and in addition we have w 0 = Alpha and w N = Beta, with the help of which we can solve for the N – 1 unknowns w 1, w 2, etc, w N – 1, so we would like to rewrite these equations in such a way that we can put them in a matrix form say a w = b, where a is an n – 1 cross n – 1 matrix, w is an n – 1 cross 1 vector and b is an N – 1 cross 1 vector, let us see whether this is possible. So to do that we multiply this equation star by – h square and collect the like terms namely what are the coefficients of w i – 1, what are the coefficients of w i and the coefficients of w i + 1.

So we collect these terms and then write them in a systematic way so that when we take values i as 1, 2, 3, et cetera up to letter N – 1, this will give letter N – 1 algebraic equation in letter N – 1 unknowns. Why do we multiply by – h square and collect the terms, the modification of this by – letter h square so will result in the coefficient of w i to be positive. In addition it will help us to avoid division by a small number that is why we multiply the entire equation by – h square and then we collect the like terms. And we observe that if we collect w i – 1 coefficient, so i have multiplied by – h square so this will have a negative sign so i have a – 1. There also appears to w i – 1 here, so when i multiply by – h square this will become positive but I take this term to the other side and so this will have coefficient – h by 2 why, this is multiplied by – h square so h square by 2 h that will give you h by 2, it goes to this side and therefore I have a – h by 2.

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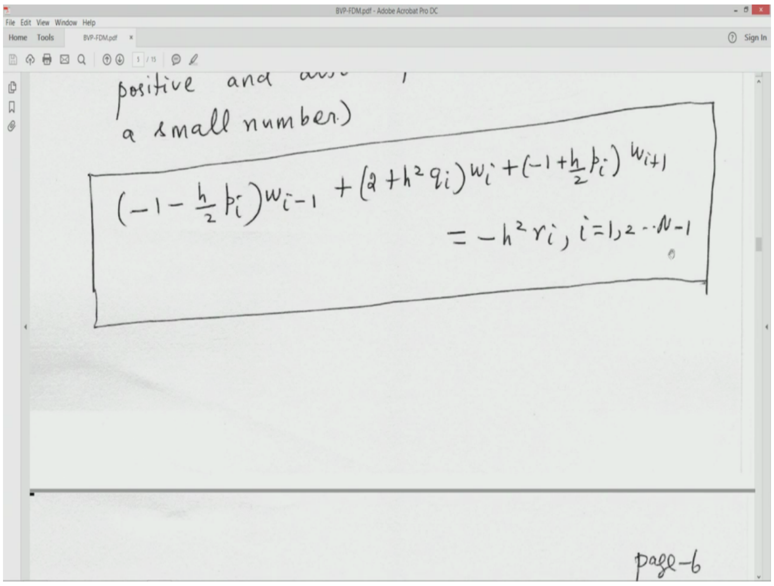

So the coefficient of w i – 1 is – 1 – h by 2 into p i namely the coefficient, which appears here. These are all the terms with w i –1 in this equation, now i move to coefficients of w i. i have multiplied by – h square so this coefficient will be plus 2 that appears here and i have a w i, I multiply by – h square, so the term is – h square q i but I take it to this side so that becomes + h square into q i, so coefficient of w i is 2 + h square into q i. Then i collect coefficient of w i + 1, so here I have – 1 as the coefficient and from here it is going to be + h by 2 into p i and that is what I have written here. So we have collected the like terms and what will appear on the right–hand side, I have multiplied by – h square, so – h square R i will appear on the right–hand side, so this gives us the equations in the algebraic equation for the unknowns w 1, w 2, et cetera, w N – 1.

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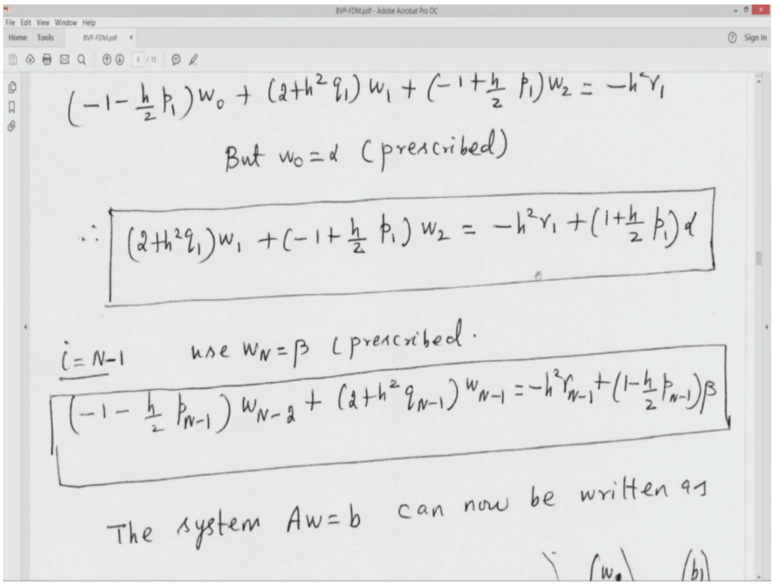

So if I substitute i as 1, we already have seen this. When I substitute i as 1, distance out to be w 0 and i know what is w 0 is, it is equal to Alpha so I make use of that and rewrite this equation when i = 1 as follows. So this is the equation when I take i = 1 and I use the fact that w 0 is Alpha, so that leads to the equation in the unknowns w 1 and w 2 as this. Similarly, when i = N – 1 we have seen that this will give you w N and w N is prescribed to be Beta and so I can make use of that information here and take this term to the right–hand side. So when you do that, you end up with an equation in the unknowns w N – 2 and w N – 1, the term with w N goes to the right–hand side and we have used the fact that w N is Beta. So for i = 1 this will be the equation, for i = N – 1 this will be the equation and so we write down the system of equations in the form A into w = b.

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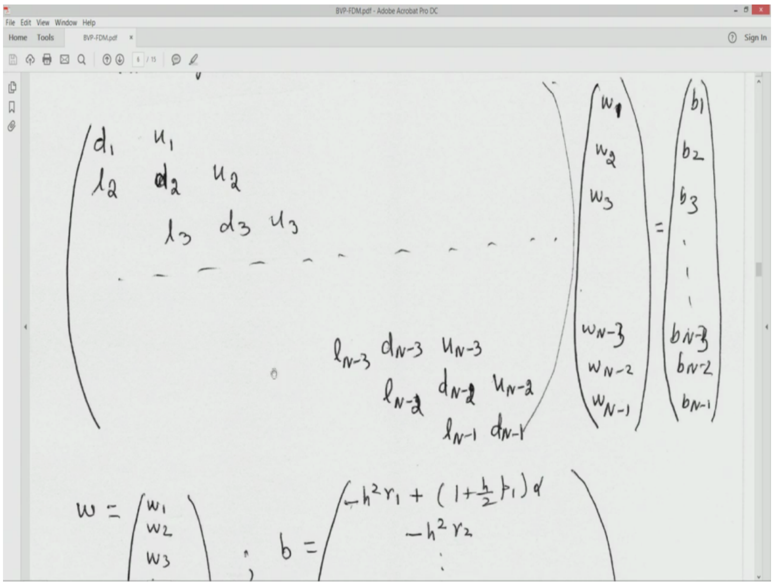

Let us denote the elements in A as follows; the elements which appear on the title are denoted by d i, the elements which lie above the diagonal are denoted by u i and the entries which lie below the diagonal are denoted by l i.

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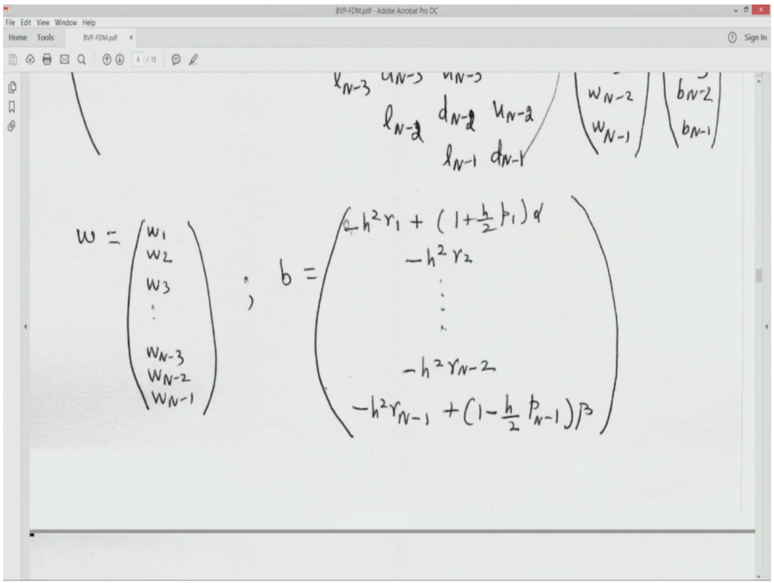

What is the unknown vector? W and it has components w 1 to w N – 1, what is the right–hand side vector? It is b it has components b 1, b 2, et cetera, b n – 1 and they are written here. So we have the system of the form A into w = b, let us now give the expression for d i, u i and l i, which are obtained from the equations which we have derived.

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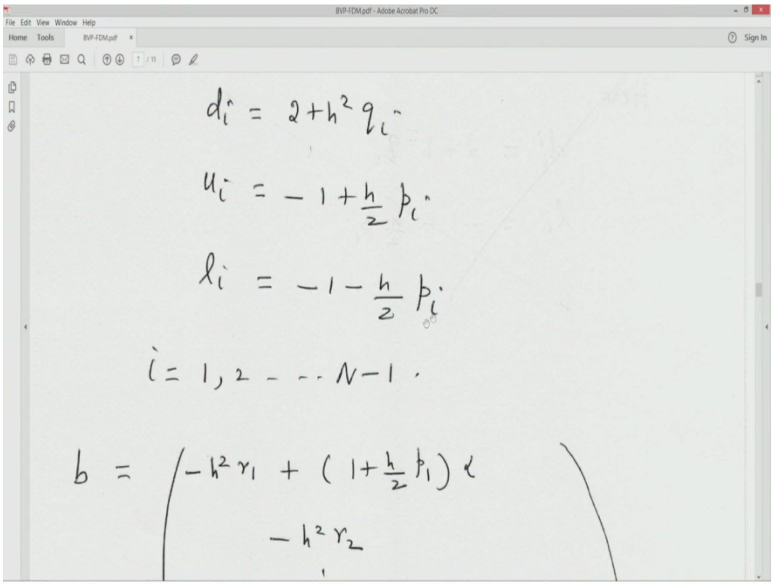

So d i is the coefficient of w i, so it is 2 + h square q i, u i is the coefficient of w i + 1, so it is – 1, so it is – 1 + h by 2 p i and l i are the coefficients of w i – 1, so l i is – 1 – h by 2 into p i, so for i = 1, 2, 3 up to letter N –1, the entries d i, u i, l i can be obtained from here

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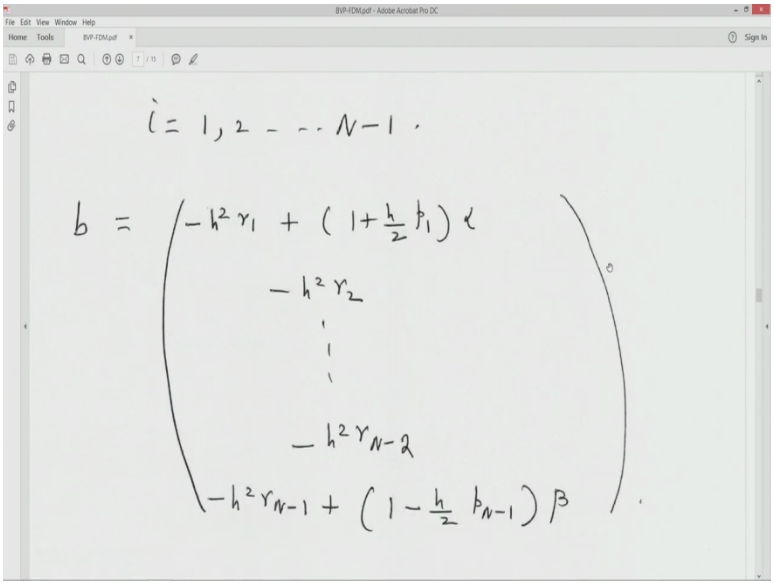

So I have again written down the right–hand side vector b, so we have now a system of equations of the form A w = b, so we now see that when we replace each of the derivatives that appear in the 2nd order differential equation, which governs the boundary value problem along with the boundary conditions, the single continuous equation for the unknown y of x is reduced to system of algebraic equations, which can be put in matrix form letter A w = b, so what is it that we need to do, we have to solve this system. When we solve this system, we will end up with those unknown values w 1 to w N – 1, which give us the approximation to the exact solutions y of x i, where letter x i are the interior points in the interval which we have selected by dividing the interval into equal number of sub intervals.

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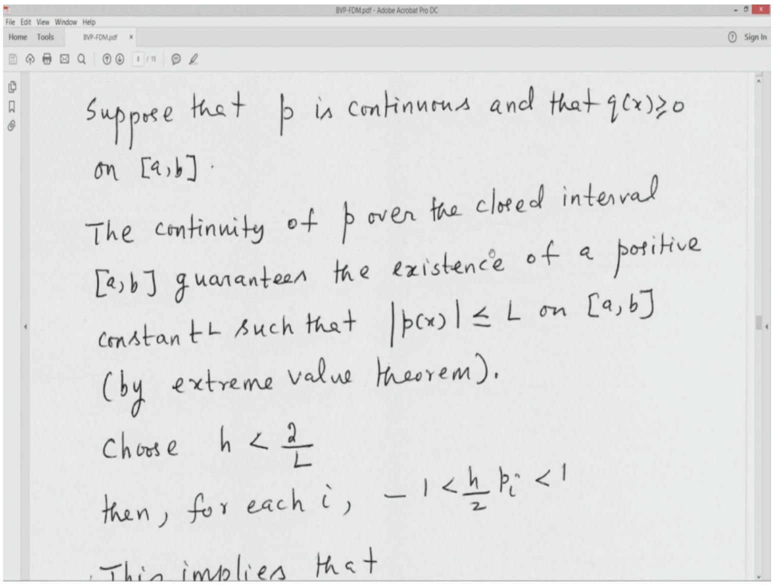

Before that we need to answer a question, what is that what can you say about the solution of the system? Is the solution a unique solution? Yes, under certain conditions we can show that the system possesses a unique solution, so let us look into all those conditions. Suppose that this function p is a continuous function and this q is such that q of x is greater than or equal to 0 for every letter x in this interval a, b, so let us start with this condition. Then the continuity of p on this closed interval a, b guarantees the existence of a positive constant l such that mod p of x will be less than or equal to l on the interval a, b and this is by extreme value Theorem.

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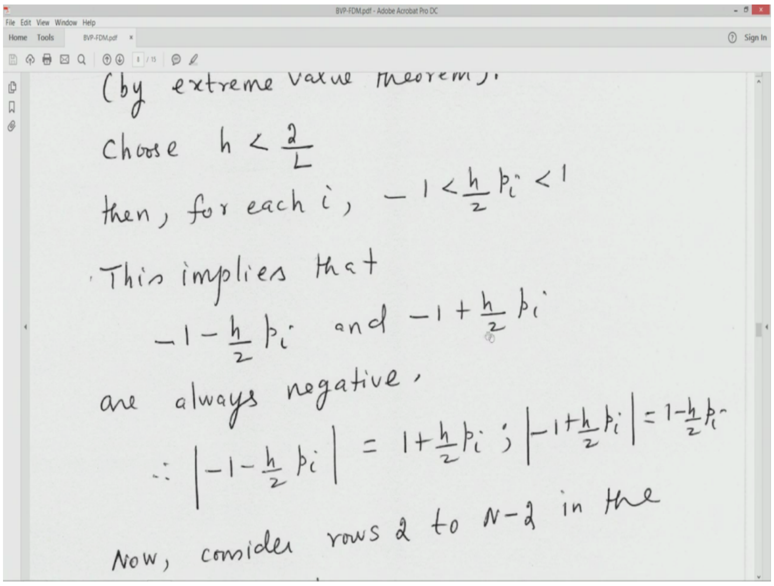

So I choose my step size h to be less than 2 by l, what is the step size h? We divide the interval a, b into m equal intervals by means of points x i such that h will be b – a capital N. So we choose our step size h in such a way that h is less than 2 by l, where l is such that modulus of p of x is less than or equal to l on the interval a, b in which the differential equation is defined. Then what happens, for each value of i, h by 2 into p i will be less than 1, why, h by 2 into capital l is less than 1, so h by 2 into p i will be less than and it will be greater than – 1, what does that imply? Let us look at – 1 – h by 2 into p i and – 1 + h by 2 into p i, which appear in system A w = b as l i u i, so let us find out what is the sign of these terms.

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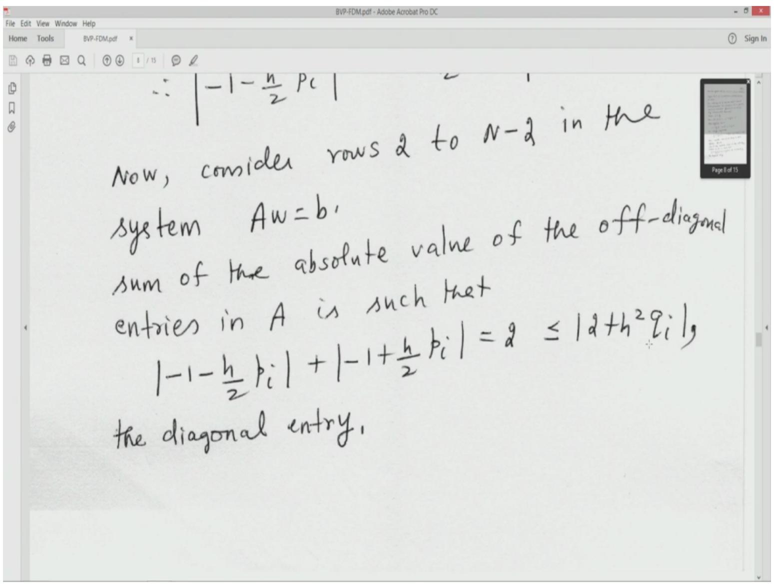

Since h by 2 p i lies between – 1 and 1, these will be always negative and therefore the absolute value of – 1 – h by 2 p i is 1+ it is by 2 p i and absolute value of – 1 + h by 2 p i is 1 – h by 2 into p i. Now let us consider rows to 2 and – 2 in the system A w = b, these are the absolute values of the off diagonal entries, so let us consider the sum of the absolute value of the off diagonal entries namely modulus of – 1 – h by 2 p i plus modulus of – 1 + h by 2 p i, so that turns out to be 2 and that is less than or equal to 2 plus h square q i in absolute value, what is this? This is the diagonal entry d i in that row, so what have we shown for all the rows i from row 2 to row N – 2, modulus of l i plus modulus of u i is less than or equal to modulus of d i. That is some of the absolute values of the off diagonal entries is less than or equal to the absolute value of the diagonal entry, let us see the rows 1 and N.

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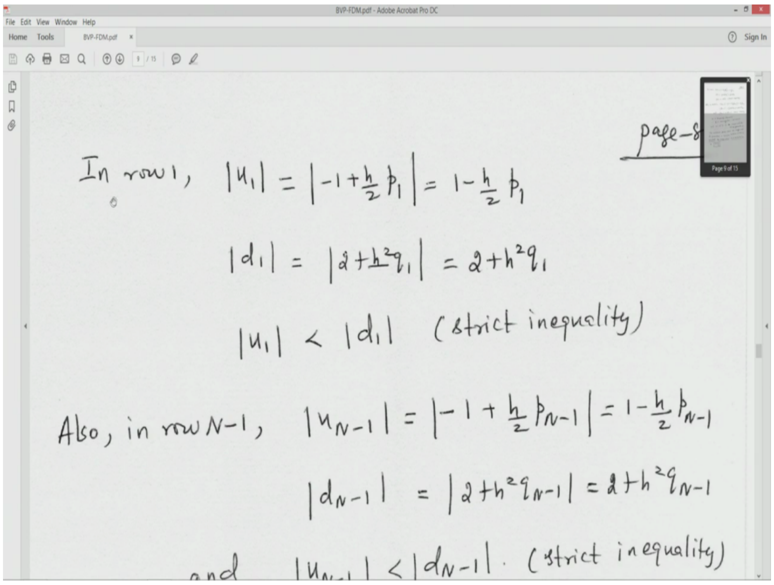

Now in row 1, we have modulus of u 1 to be modulus of – 1 + h by 2 p 1, which is 1 – h by 2 p 1. And mod d 1 is modulus of 2 + h square q 1, and so it is this and we observe that mod u 1 is strictly less than mod d 1. Similarly, in the N – 1 row, mod u N – 1 is this and mod d N – 1 is 2 + h square q N – 1 and we observe that mod u N – 1 is strictly less than mod d N – 1. So what does it says? It says that diagonal entries in the matrix A are such that the absolute value of the diagonal entry from row 2 to row N – 2 is greater than or equal to some of the absolute values of the off diagonal entries in that particular row, this happens in each row from row 2 to row N – 2. And in row 1 and row N – 1 there is strict diagonal dominance, A is a diagonally dominant matrix and therefore, A is non–singular so A inverse exists and therefore, the system A w = b has a unique solution given by w = A inverse b.

And this is what we wanted to find out whether the system has a unique solution. Yes it has any solution under the conditions that p of x is a continuous function on the closed interval and q of x is greater than or equal to 0 on the interval a, b and the step size h is less than 2 by l, where l is such that modulus of p of x is less than or equal to l for x belonging to the interval a, b. So the conditions given above are sufficient to guarantee a unique solution of the system provided the step size h satisfies the condition that h is less than 2 by L. Now that we know the system has a unique solution, the only step that remains to be done is, solve this system and obtain an approximate solution, we will continue this in the next class.