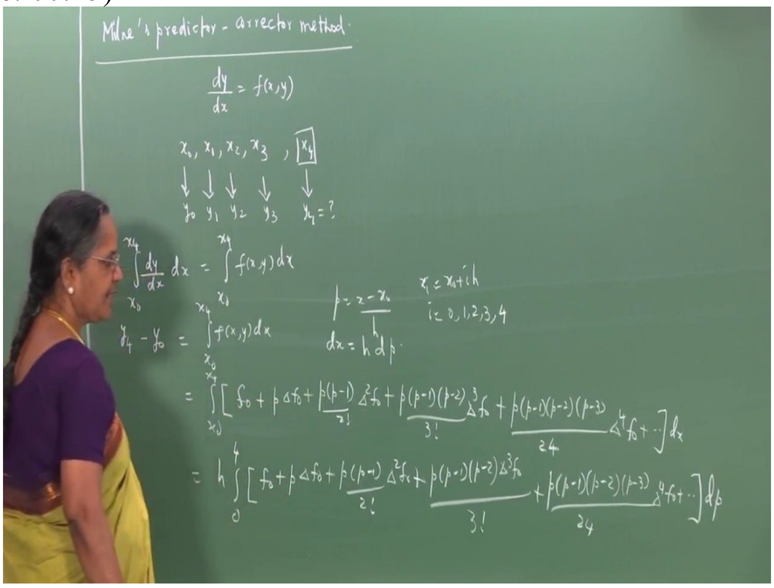
Numerical Analysis
Prof R Usha
Department of Mathematics
Indian Institute of Technology Madras
Lecture 24
Numerical Solution of Ordinary
Differential Equation - 7
Predictor - Corrector Methods
(Milne)

(Refer Slide Time: 00:15)



In today's lecture we shall consider another Predictor – Corrector pair which is a multi step method. This method is called Milne's Predictor – Corrector Method. This is again a Multi step method where we assume that the solution to the initial value problem dy by dx equal to f(x,y) is known at a set of previous equally spaced points namely the solution is known at x 0 x 1 x 2 x 3 and you are asked to get the solution at a point x 4.

Information about the values y 0 y 1 y 2 y 3 are known to you and you are asked to get the solution at the point x 4 namely what is y 4. So under this assumption we develop Milne's multi step method and also derive another multi step method and we will see how these two multi step methods can be considered as a Predictor – Corrector pair for computing solution at the point x 4 namely y 4.

So the integrate dy by dx with respect to x,y with respect to x between x 0 and x 4 integral x 0 to x 4 f(x,y) dx. So this gives you y between x 0 and x 4 so y 4 minus y 0 is integral x 0 to x 4 f(xy) dx.

We approximate f(x,y) by Newton's forward interpolation polynomial, it is f 0 plus p into delta f 0 plus p (p minus 1) by factorial 2 into del square f 0 plus p (p minus 1) (p minus 2) by factorial 3 into del cube f 0, p into p minus 1 into (p minus 2) into (p minus 3) by factorial 4 into del to the power of 4 f 0 and so and this must be integrated with respect to x.

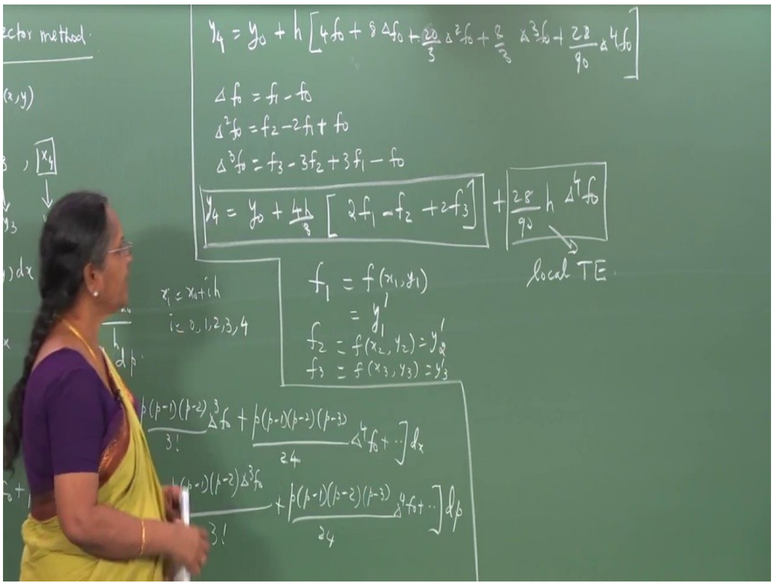
And what is p here p is x minus x 0 divided by h where h is the step size such that x i is x 0 plus i h so I take values 0 1 2 3 upto 4. So we now make use of this information and change the limits of integration from x 0 to x 4 accordingly.

Sp dx will be h into dp and when x is equal to x 0 p is 0 and when x is equal to x 4 p is x 4 minus x 0 by h. So it is 4h by h so it is 0 to 4 and since dx is hdp so this will be h and you have f 0 plus p delta f 0 plus p (p minus 1) by factorial 2 into del square f 0 plus the remaining terms as they are into del power f 0 plus etc.

Now integration as I said is with respect to p so it is dp. We have to evaluate this integral and then substitute for the forward differences delta f 0 del square f 0 del cube f 0 and del power 4 f 0 and simplify this result.

So when we perform integration we end up with the following step I have already indicated how you do it when we considered Adam Moulton Method So I just give the results of the integration of this step.

(Refer Slide Time: 05:26)



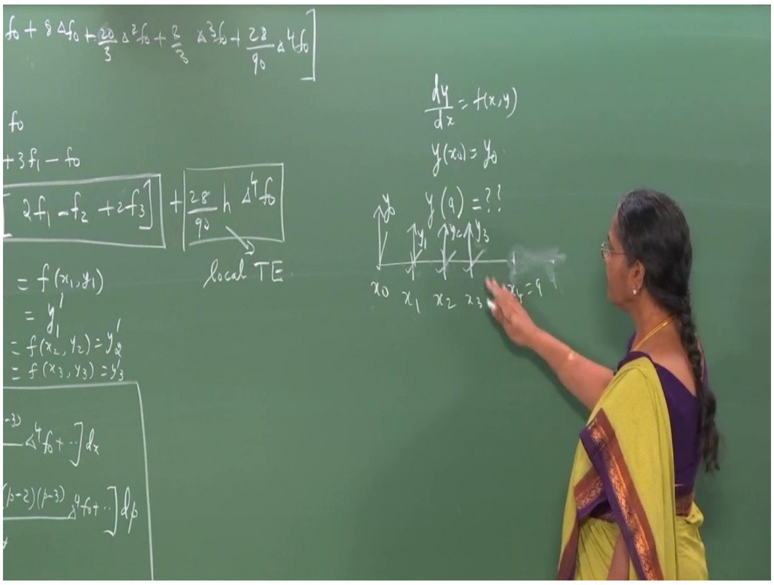
So it gives you y 4 equal to y 0 plus h into [4 f 0 plus 8 delta f 0 plus 20 by 3 into del square f 0 plus 8 by 3 into del cube f 0 plus 28 by 90 into del power 4 f 0] and we consider terms upto this.

And since delta f 0 is f 1 minus f 0 del square f 0 is f 2 minus 2 f 1 plus f 0 del cube f 0 is f 3 minus 3 f 2 plus 3 f 1 minus f 0 we substitute these forward differences in the terms which appear within the bracket and simplify. And we end up with y 4 as y 0 plus 4 h by 3 into 2 f 1 minus f 2 plus 2 f 3 plus 28 by 90 h into del power 4 f 0.

So as before this term gives you the local truncation error. So our method is that y 4 the value of y at x 4 depends on y 0 and then y 1 y 2 y 3 because what is f 1? F 1 is f(x 1y 1) and that is from the differential equation y dash, so y dash evaluated at x 1 y 1, so it is y dash 1.

Similarly f 2 will be f(x 2 y 2) and so it is y dash(x 2) and f 3 is f(x 3 y 3) and that will be y dash (3). So the solution at x 4 given by y 4 depends on the solution at the previous points namely x 0 x 1 x 2 and x 3. And that is why we assume that we already have some information about the solution at the previous 4 points.

(Refer Slide Time: 08:25)



Suppose say this information is not given and your problem is something like this , dy by d x is equal to f(x,y) and y(x 0) is y 0 and you are asked to say get the solution at some point y(a) is what is required.

And you are asked to solve this by Milne's Predictor Corrector Method. We will derive the method but I am just indicating here because we said that this method is based on the assumption that solution at the previous 4 points are known to us.

Here we are only given information at the point x 0. So what you do is you divide this interval a to x 0 into say intervals such that you have points x 1 x 2 x 3 and x 4 is going to be your a.

So divide the interval of width a minus x 0 into 4 equal parts by means of points x 0 x 1 x 2 x 3 and x 4 which is a. So information at this point is given by the initial condition. At these three points you require the information if you want to make use of Milne's Method namely the Predictor method or the Multi step method that we have derived here.

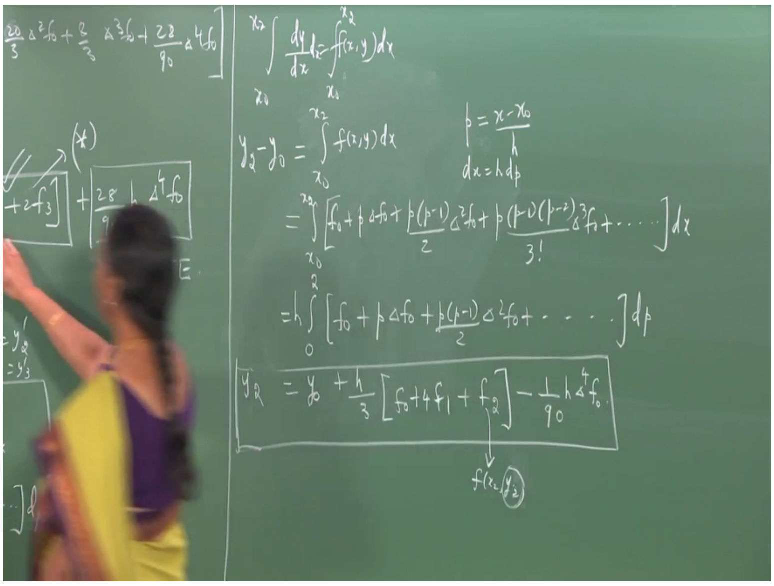
So what you do is you make use of either Runge Kutta Method of order 4 or Euler's Method or Taylor series method march from x 0 to x 1 obtain the solution by any one of those techniques and get the information here call it as y 1. And do that for each of these points namely you march from x 1 to the point x 2 use any of those methods that we have discussed namely Runge Kutta Method or Euler's Method or Taylor series method and compute the solution at x 2 which is y 2.

And by the same technique march to x 3 and then come and determine the solution at the point x 3 and call it as y 3. So now you have information at the previous 4 equally spaced points. So you must have take the points in such a way that you divide the width of this interval x 0 to a into 4 equal parts of step size equal to h and then march from 1 point to another point and determine the solution at each of these previous points.

Once you have the information at all the 4 points computation of the solution at a which you call as x 4 namely y 4 can be obtained from the multi step method you have derived. So either you are given information at the previous 4 equally spaced points.

If it is not available then you divide the given interval into 4 equal parts step size h and then use any of the single step methods to compute the solution at the previously located equally spaced points. And once you have this information you make use of the multi step method that you have derived now to get the solution at x 4 which is y 4.

(Refer Slide Time: 12:19)



So we now derive another multi step method to do that we integrate dy by dx is equal to f(x,y) with respect to x between say x 0 and x 2. And this will give you y between x 0 and x 2 so it is y 2 minus y 0 and that is equal to integral x 0 to x 2 f(x,y) dx.

So you approximate f(x,y) by again Newton's forward interpolation polynomial namely f 0 plus p into delta f 0 plus p into p minus 1 by factorial 2 into del square f 0 p (p minus 1) into (p minus 2) by factorial 3 into del cube f 0 plus etc and integrate it with respect to x. Here again x minus x 0 divided by h,

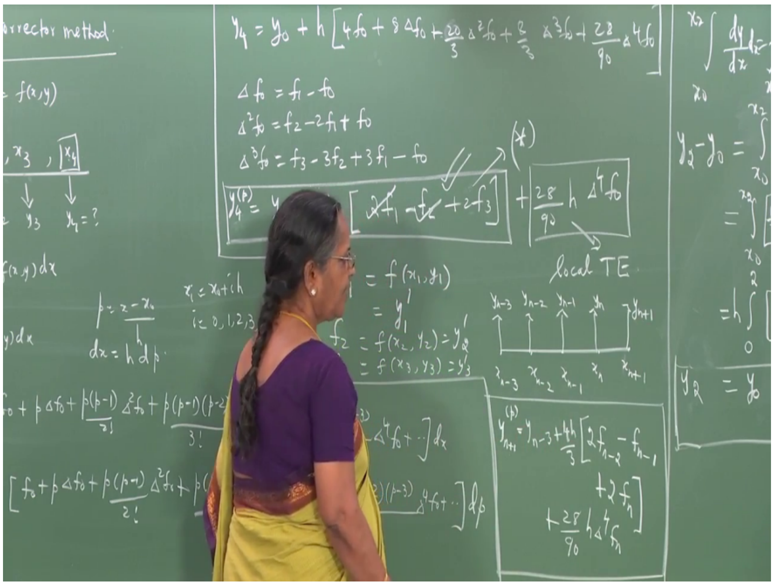
So making use of this you can change the variable of integration namely with respect to x. And we now change 2 integration with respect to p.So when x is x 0 p is 0 when x is equal to x 2 p is x 2 minus x 0 by h which is 2h by h and that gives you 2 right and that the terms within this bracket will have to be integrated with respect to p.

So dx is hdp so you integrate with respect to p between the limit p equal to 0 to 2. So the result of integration and then replacing the forward differences by means of the values of the function at x 0 y 0 x 1 y 1 and so on will result in the following method, namely y 2 is equal to y 0 plus h by 3 into f 0 plus 4 f 1 plus f 2 minus 1 by 90 times h into del to the power of 4 f 0.

So you have another method we observe at this stage that we have y 2 to be given by y 0 plus h by 3 f 0 plus 4 f 1 plus f 2 where f 2 is f(x 2 y 2). This requires knowledge of y 2 so that f can be computed and that can be made use of here but what is the value of y 2 is the question. This suggests that the value of y 2 that we have here can be obtained from the multi step method that we have derived earlier.

So we can call that as a Predictor and use that value over here so that f can be evaluated and using that information the new value of y 2 can be obtained. So that this can be called as a corrected value. So let us now properly denote the nodes and write down in general what the two methods are ?

(Refer Slide Time: 16:30)



Say in the first method I am required to compute the solution say at x n plus 1 which is y n plus 1 I require knowledge of the solution at the previous 4 points. So I shall call those 4 points as x n , xn minus 1 , x n minus 2 and x n minus 3 where the values are y n minus 2 y n minus 1 and y n. So the predictor that we have obtained can now be written as y n plus 1 which we want to predict clear?

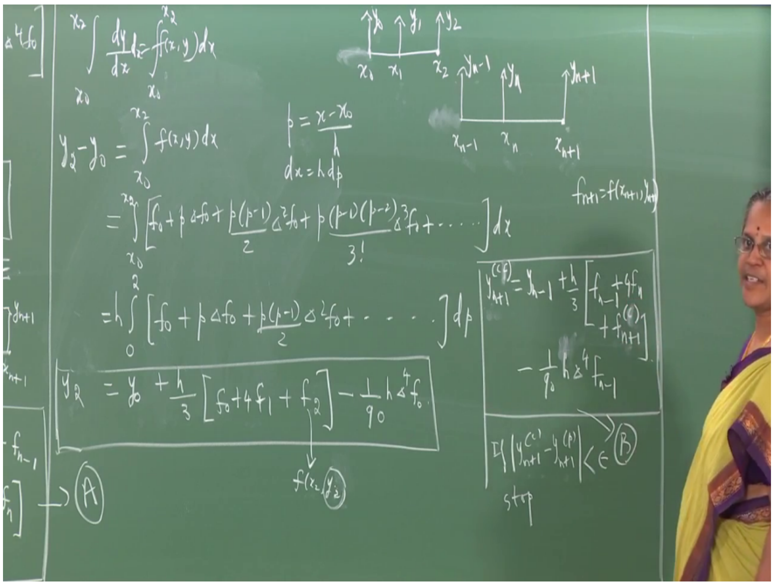
Is y 0 Just look at this picture right? Y 4 which has to be predicted is given by y 0 which is the solution at x 0. So if I want to put this in general and want the value of y n plus 1 at x n plus 1 then y n plus 1 predicted is y n minus 3 plus 4h by 3 multiplied by twice what do I have here f 1 that is f(x 1 y 1).

So generalizing this is going to be f (x n minus 2) (y n minus 2) which I call as f n minus 2 then I have minus f 2 namely it is value of f 1 x 2 by 2.So correspondingly this is going to be minus f(n minus 1) and then I have twice f 3 namely the value of the function at x 3 y 3.

So in this case it is going to be plus twice f n and local truncation error is 28 by 90 into h into del to the 4 f n. So we have a predictor which predicts the value at x n plus 1 which is given by y n plus 1 and it makes use of the information at a previous 4 points.

It is this predicted value which I have to use in the other method which we have derived and then correct it so that we can call the other method as a Corrector method.

(Refer Slide Time: 19:21)



So let us see here what does this formula tell us if you want the solution at x 2 namely y 2 the previous equally spaced points are x 1 and then x 0 at which the solutions are y 1 y 0.

So the method gives you the solution at this point. Now if I want to make use of this as a corrector and get the solution at this point I call this point as x n plus 1 at which I want to get y n plus 1. So the previous point will be x n and the 1 before that is going to be x n minus 1.

If I want to generalize this to a result involving these three points then the method tells me y n plus 1 is equal to y 0 which corresponds to the solution y n minus 1 then plus h by 3 into f 0 which is the solution here. So it is going to be f(x n minus 1) (y n minus 1) so I call it f n minus 1. Then plus 4 times f 1 corresponding to that here it is going to be f(x n y n) which I denote by f n.

And then I have f 2 namely a solution at this point and corresponding to that I should write down the solution at the point f n plus 1 minus 1 by 90 h into delta to the 4 f n minus 1 which is the local truncation error.

So I have the method written in general in terms of x n minus 1 , x n and x n plus 1 and the function values there. So as before I observe that this f n plus 1 is f (x n plus 1) y n plus 1. So I require information about y n plus 1 where do I get it from, I get that information from this predicted value.

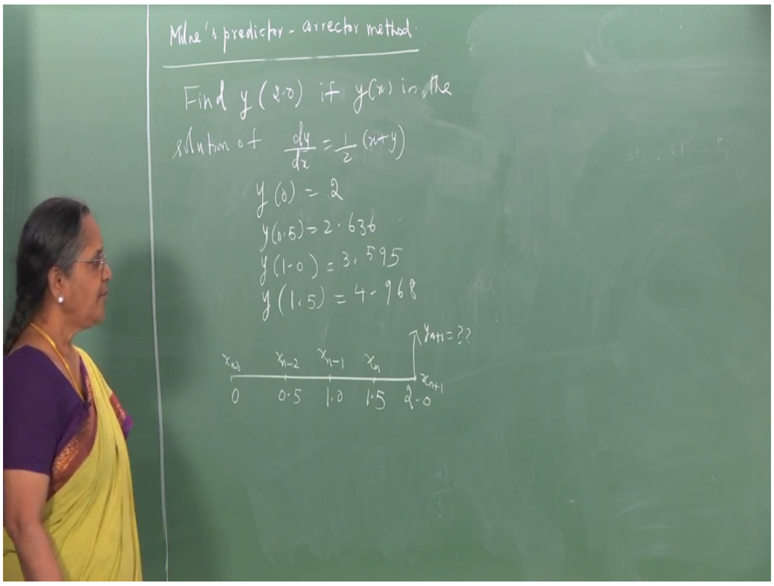
So i call this f n plus 1 as being obtained from the predicted value or as being obtained by using the predicted value of y n plus 1 from that method. So I compute f n plus 1 call it f n plus 1 predicted. So i know every term on the right hand side so I get the value of y n plus 1. Since I have obtained a new value of y n plus 1 I call this y n plus 1 corrected value.

So I predicted it by using formula A and I corrected it using formula at this stage I check whether the absolute value of difference between the corrected and predicted values of y n plus 1 is less than Epsilon. If that happens then we stop our computations, if this does not happen then what do you do? I make use of this method once again and get the second corrected value where I make use of the y n plus 1 first corrected value in computing f n plus 1.

So I shall denote this by f n plus 1 corrected value and hence I get the second corrected value. And once again I check if the second corrected value minus the first corrected value are such that the absolute value of the difference between them is less than the desired accuracy. If so we stop, if not we recorrect this again using this multi step method.

So this method which we have derived is corrector method that is the one which we derive earlier is a Predictor Method, so we make use of A and B as a Predictor Corrector pair in obtaining solution to dy by dx is equal to f(x,y) when information about the solution is known at the previous four points which are equally spaced. And this method is called Milne's Predictor Corrector method.

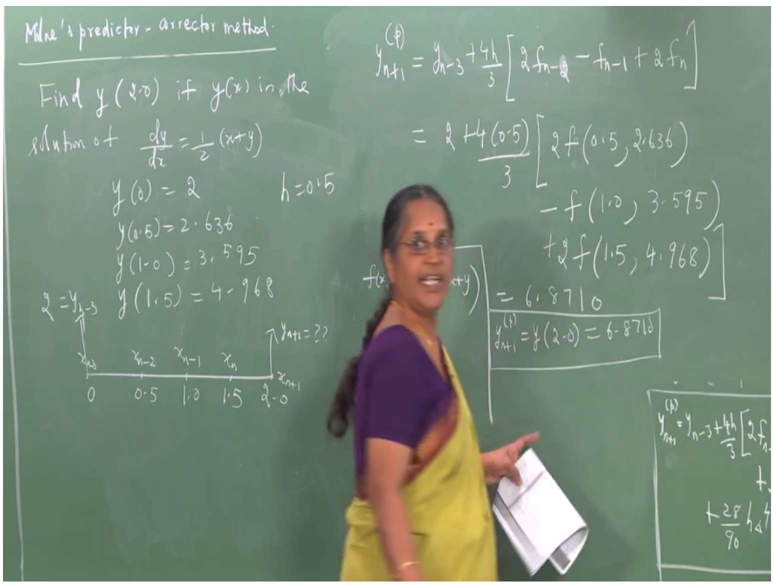
(Refer Slide Time: 24:57)



Let us now consider the following example and solve it using Milne's Predictor Corrector Method. Find y(2.0) if y(x) is the solution of dy by dx equal to half of (x plus y). Given that y(0.5) is 2.636 y(1) is 3.595 and y(1.5) is 4.968.

So we are asked to solve this problem using Milne's Predictor corrector Method and we are required to get the solution at 2 and we are given information at 1.5, 1, 0.5 and 0. So we have information at previously located equally spaced points which we call as x n x n minus 1 , x n minus 2, x n minus 3. And the solution is required at x n plus 1 namely y n plus 1 is required.

(Refer Slide Time: 26:42)



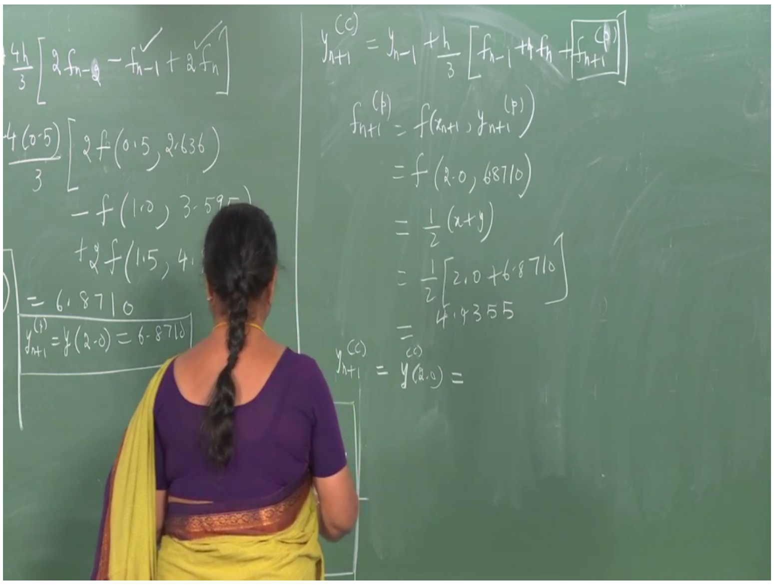
So we use the Predictor given by y n plus 1 Predictor is y n minus 3 plus 4h by 3 into twice f n minus 2 minus f n minus 1 plus twice f n. What is y n minus 3 ? y n minus 3 is the solution at 0. So it is y 0 which is required and y 0 is given to be 2. So it is 2 plus 4 into what is h the points are equally spaced and the step size is 0.5. So 4 into 0.5 by 3 into twice f(n minus 2),

So let us look at x n minus 2 so it is 0.5 and information at 0.5 is that it is 2.696 then you have minus f(n minus 1) x n minus 1 is 1 so 1 and at x n minus 1 the solution is 3.595.And then we have plus twice f n that is f(x n y n) and x n is 1.5 and the solution y n is given to be 4.968.

So it is always better to write down each step systematically so that you know you evaluate your function at these points x i y i and then use the differential equation. What does it say? It says f(x,y) is half of x plus y. So f(0.5,2.636) will be half of 0.5 plus 2.636. So similarly you can evaluate each of these functions which appear here.

And if you simplify that you end up with the predicted value as 6.8710. So this is a predicted value namely this is the value of y at x n plus 1 which is 2. So y n plus 1 predicted is y evaluated at 2 and that is 6.8710. But we need to solve the problem using Milne's Predictor Corrector Method. So we have a way to correct this solution. How do we do it?

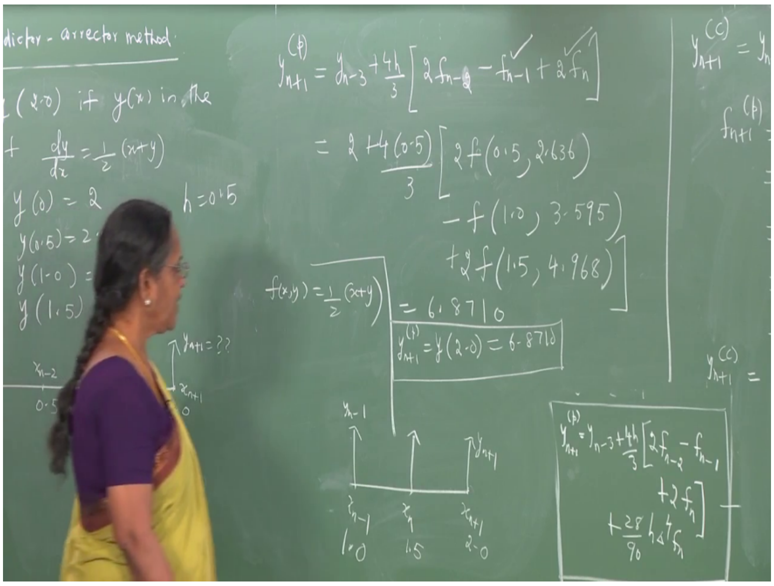
(Refer Slide Time: 30:05)



We use the Corrector Method which we have derived. What does the corrector say? It says if you want to correct the solution at x n plus 1 namely y n plus 1 then the corrected value is y n minus 1 plus h by 3 into f n minus 1 plus 4 f n plus f n plus 1 Predicted value. So make use of this corrector and then get the solution. What is this f n plus 1 predicted value? That is f (x n plus 1, y n plus 1) predicted. So that is f at what is x n plus 1 to what is y n plus 1 predicted we just now computed it which is 6.8710.

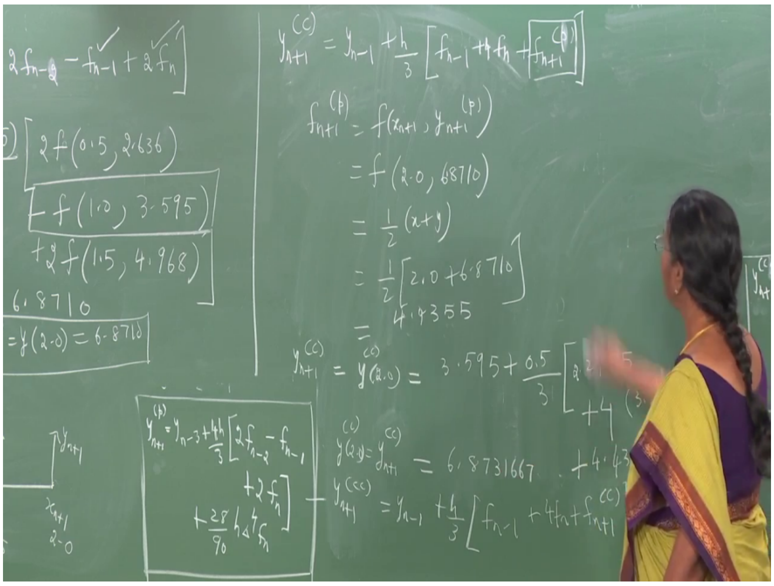
So evaluate this what is f(x,y) it is half (x plus y) that is half of 2 plus 6.8710 and that is equal to 4.4355. That is the only value that you need to compute the other values f n f n minus 1 are already available to you because you have used them here. so those values are available to you and you can take from there and substitute these values that you already have . And so the corrected value of y n plus 1is y(2) corrected and that is equal to y(n minus 1).

(Refer Slide Time: 32:10)



So when you use this method you have to again mark those points x n plus 1, x n and x n minus 1. The method says the value of y to be corrected here is based on the value of y here which is y n minus 1 and the function value f(x n minus 1, y n minus 1) plus the function value here namely x n y n. So if this point is true then x n will be 1.5 x n minus 1 will be 1. So mark clearly what the points are and what the function values are that is very important and when you substitute those values.

(Refer Slide Time: 33:07)

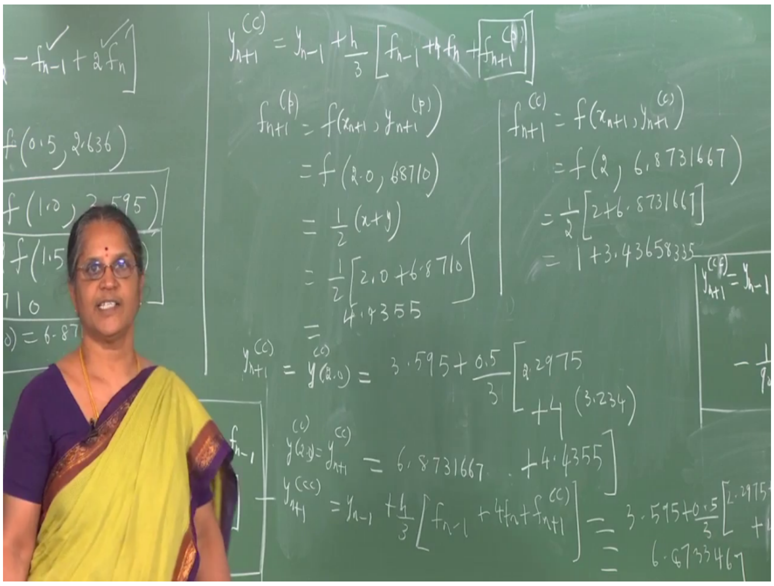


So can you tell me what is y n minus 1 it is the value of y at x n minus 1 namely it is the value of y (x n minus 1) which is 1. So y (1) which is 3.595 plus h that is 0.5 y 3 into what is f n minus 1? It is f (1,y n minus 1) that is 3.595 namely the value that you have obtained here. So that turns out to be 2.2975 and then 4 times f n namely f(1.5, 4.968) which turns out to be 3.234 and finally f n plus 1 predicted that you already have computed so 4.435.

So simplify this and that gives you 6.8731667 so this is the corrected value of y (2). So let us try to apply this corrective again and recorrect the solution and see how it can be done. Before doing that what is the formula that we have to use we write down the formula.

So y n plus 1 second corrected value is y n minus 1 plus h by 3 into f n minus 1 plus 4 f n plus f n plus 1 now here this f n plus 1 is corrected using the information y n plus 1 corrected. So f n plus 1 is f (x n plus 1, y n plus 1) corrected, y n plus 1corrected has been obtained and x n plus 1 is 2.

(Refer Slide Time: 32:25)

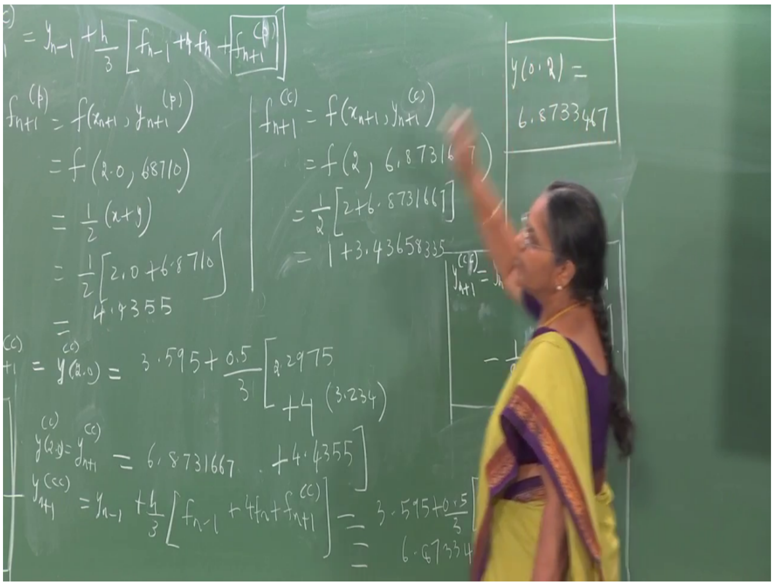


So we can now compute what is f n plus 1 corrected value, so it is f(x n plus 1, y n plus 1) first corrected value namely it is f (2) and the first corrected value is 6.8731667 and that is half (2 plus 6.8731667). So that will give you 3.595 plus 0.5 by 3 into 2.2975 plus 4 into 3.234 plus the value which you have got here. What is it? It is 1 plus 3.43658335 and so that will be 1 plus 4.43658335 so when you simplify this you get the solution as 6.8733467.

Suppose say I have asked you to perform 2 iterations or correct the Predicted value twice then it is enough that you stop your computations at this step because you have performed the correction twice.

On the other hand if suppose you are specifying some accuracy then you will have to check whether the absolute value of the difference between successive iterations turn out to be less than the prescribed tolerance and if so you stop otherwise you recorrect your solution till the desired degree of accuracy is attained. So since I have said in this case stop your computations.

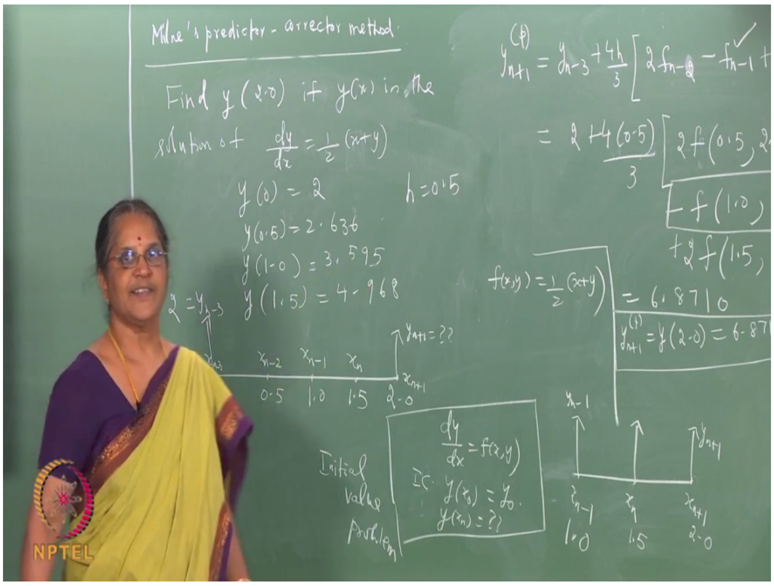
(Refer Slide Time: 37:47)



After two corrections you can write down the solution to the problem as y at 0.2 is 6.8733467. So you have used Milne's Predictor Corrector Method to obtain solution to this problem when you are given information at the previously equi spaced points. So far we have considered three Predictor Corrector Methods namely modified Euler's Method turned out to be a Predictor Corrector Method then we derived Adam Moulton or Adam Bash force Method which is again a Predictor Corrector Method.

This method is a multistep method and in this class we have derived Milne's Predictor Corrector Method, so we have derived two multi step methods and we make use of these two multi step methods as Predictor Corrector pair then try to obtain solution to the first order differential equation correct to the desired degree of accuracy.

(Refer Slide Time: 39:14)



So in the next class we move on to problems which are boundary value problems all along we have been discussing problems which are called initial value problems where we have tried to obtain solution to dy by dx is equa l to f(x,y) and some initial condition namely at the point x 0 the solution is specified for it is known say y 0.

This class problem is called initial value problems given this information at x 0 we are asked to get the solution at some point x n. And we have developed some single step methods we have also developed multistep methods and we have used them as Predictor corrector pairs and we have been able to obtain solutions to initial value problems.

We now move on to what I referred to as Boundary value problems, So in the next class we understand what we mean by boundary value problems and develop numerical methods by means of which such boundary value problems can be solved. We focus our attention only on linear boundary value problems in this course.

The solution of non linear value boundary problems is beyond the scope of this course and you will be doing it at a higher level. So we consider boundary value problems namely linear boundary value problems and develop numerical techniques in the next class.