Numerical Analysis

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Lecture -2, Part - 2

Mathematics Preliminaries, Polynomial Interpolation-1

At each step of scientific computation it is important and necessary that we have knowledge of how much of error has been incurred in that particular computation. So we try to measure this error at each step of computation in terms of the following measures.

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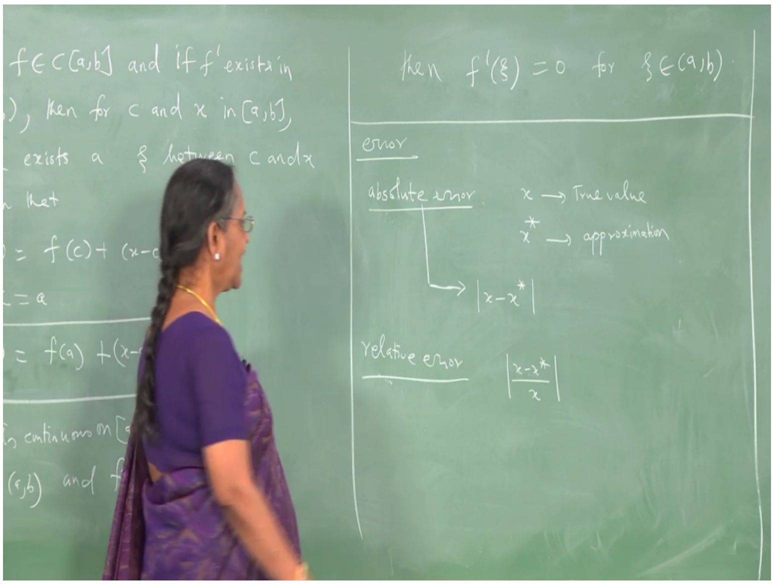

One such measure is what is called the absolute error namely if x is the true value and we approximate this by x star say an approximation that comes out as a result of implementing some numerical algorithm then at that step we have incurred an error whose absolute value is x minus x star.

And by a relative error we mean the absolute value of x minus x star relative to the true value x and that is what is the relative error that has been incurred in approximating x by x star.

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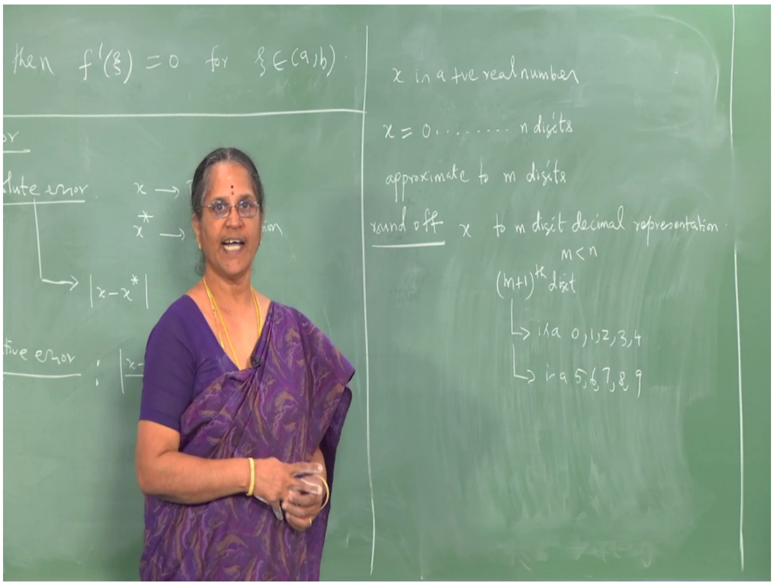

Suppose a x is a positive real number and I represent x by n digit decimal representation. By that I mean that x is 0. there are n digits after the decimal so this is a number which I want to approximate to n digit decimal number namely I would like to round off x to m digit decimal representation then what do I do? With m less than n. I want to approximate this n digit decimal representation to m digit representation with m less than n.

So given this x I would look at x and see what is the m plus 1 th digit in that representation which is given in the form of n digit representation. So my rounding off of that number depends on what the value of m plus 1 th digit is. If the m plus 1 th digit is 0,1,2,3 or 4 then I round off the number which is given as n digit representation such that the first m digits are the same as that of these x and I discard all the digits beyond the m plus 1. Namely m plus 1 th digit and the digits after that are all discarded.

So you discard all the digits beyond the n th digit. On the other hand if the number is 5,6,7,8 or 9 then I add 1 unit to the m digit and discard beyond the m digits. The resulting number that I get I denote it as x star.

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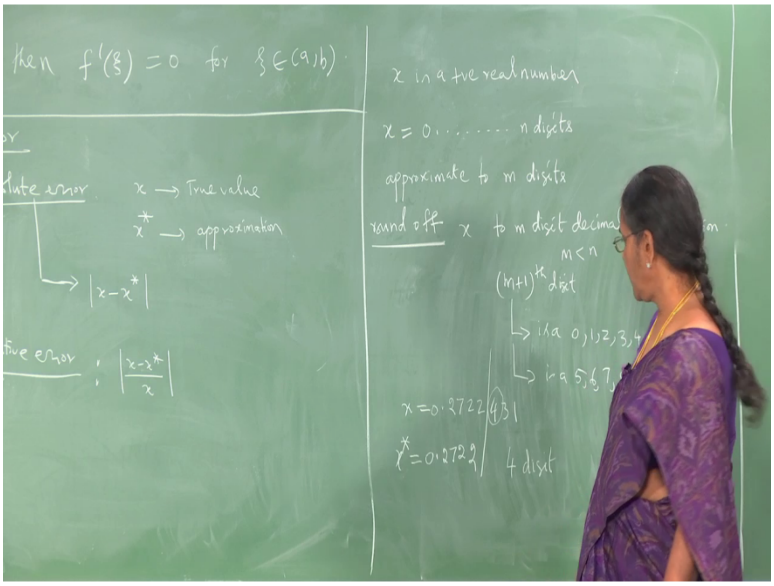

So let us take an example and illustrate. If suppose the number is 2,7 2,2 4,3,1 so I am given a 7 digit decimal representation of the number and I want a 4 digit representation of the number namely I would like to round off this number x to 4 digits. Then what do I do I look up the digits upto 4 digits and look at what the fifth digit is. It is a 4 and therefore I roundoff this number as x star by writing down the first 4 digits as they are and discard the remaining digits in the representation of x. So this x star is the rounding of x to 4 digit decimal representation.

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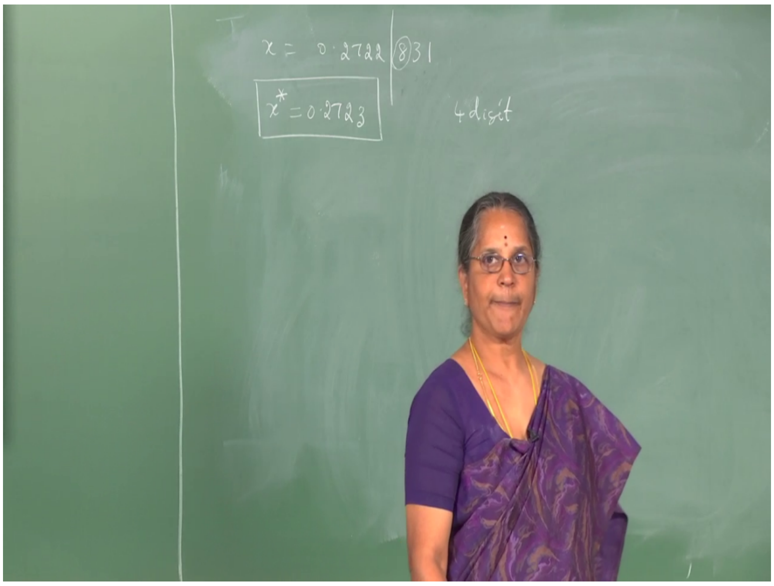

On the other hand if x is given by say point 27 22 831 if the number is this again I have a 7 digit decimal representation of a number x. Then I require a 4 digit rounding off this number so what do I do I again look at first four digits and see what the fifth digit is and the fifth digit is a 8 and therefore the rule is I add 1 unit to the fourth digit so that I get 2723 and then discard rest of the digits from x. So this x star is obtained after rounding x to 4 digit decimal representation.

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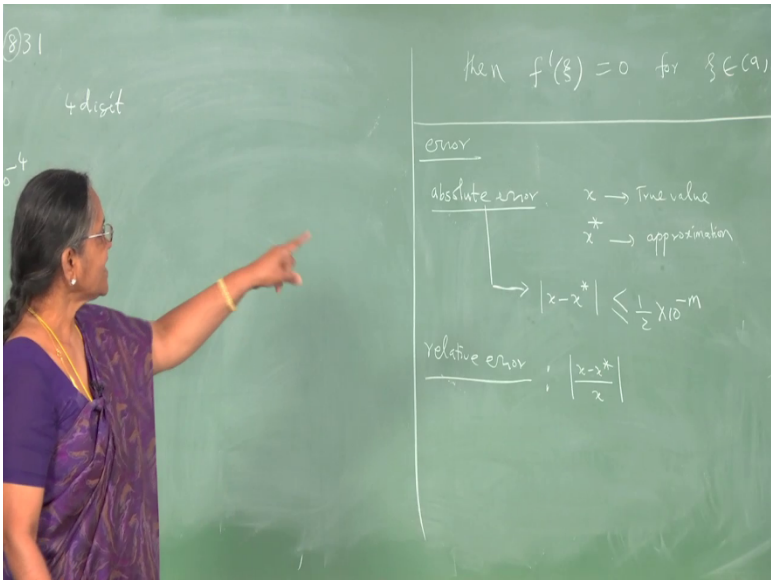

If I do this then this absolute error explained as x star is clearly less than or equal to half of 10 to the power of minus 4. So in general if you write down the if you obtain your approximation correct to n decimal places by rounding the true value x then your absolute error will be less than or equal to half 10 to the power of minus m.

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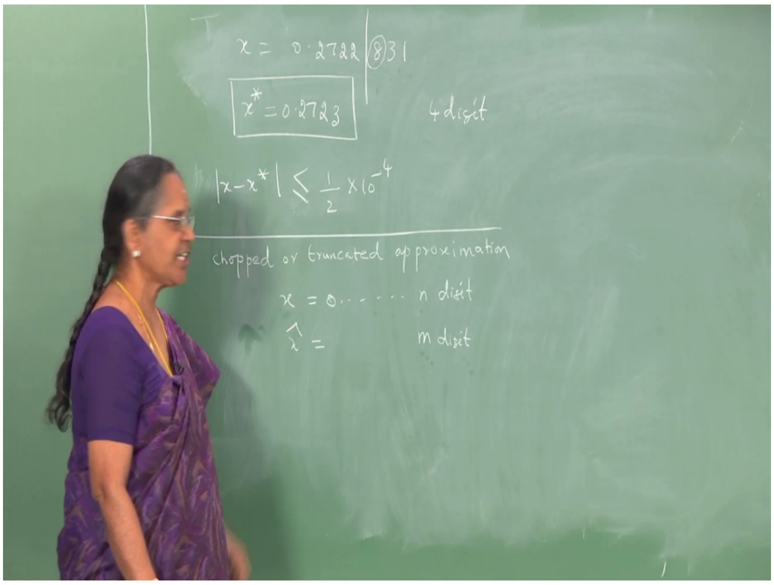

Let us now see another way of approximation of a positive real number namely its called chopped or truncated approximation. So in this given a number x right and has n decimal representation and I want to obtain m digit chopped or truncated approximation of this number then suppose I denote it by x hat it is obtained by simply discarding all the digits in x beyond the n the digit. You do not worry about what the value at the m plus 1 th location you truncate or chop the digits in x and discard all the digits beyond the m digits in the n digit decimal representation of x.

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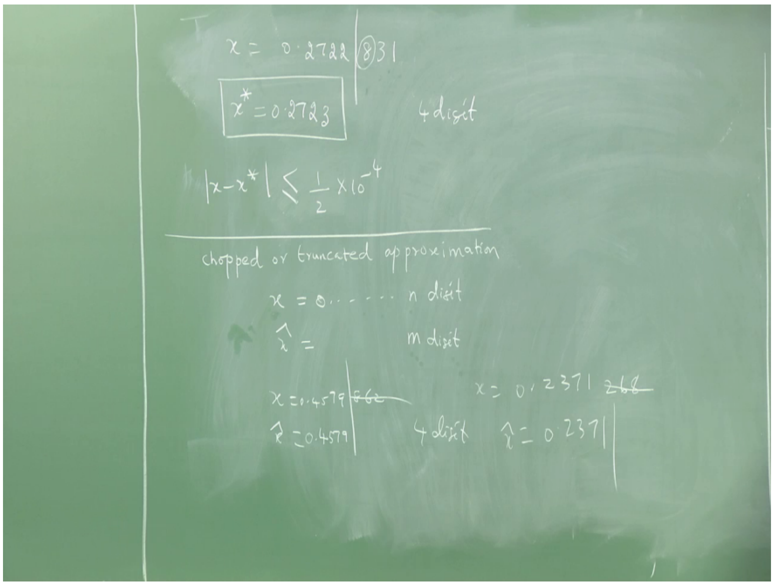

So if I consider x is equal to point 4579862 this is x, and what is x hat if I require a 4 digit decimal representation I do not worry about what digits appear beyond the fourth digit I simply chop them off discard them and then write down the approximation as x hat is equal to point 4579. If x were 0.2371 26 8 and i require a 4 digit approximation it is 0.2371 I do not worry about what happens beyond 4 digits.

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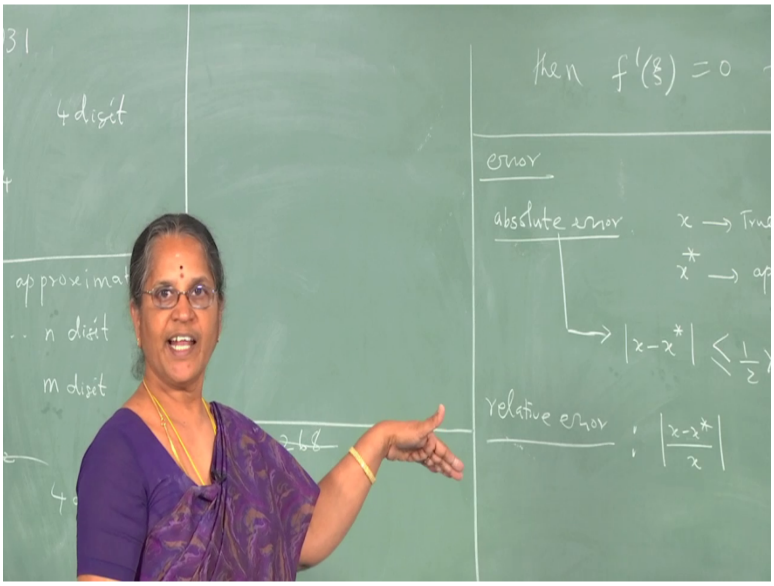

Now you may ask me why do we have two different measures of error that is a natural question. Isn't it not enough if we have single measure of error and we make use of this definition of error and then see how we can control this error after computing error at each step of our computation. I mean when we look at the absolute error this is the natural measure of accuracy of approximation.

Namely you have a true value you have an approximate value the absolute value of the difference gives you the error that has been incurred. That is a natural measure of accuracy of approximation but we will show by means of an example that the absolute error is highly misleading you cannot judge about the accuracy of your computation based on the absolute error. So it is always better to use the relative error rather than the absolute error.

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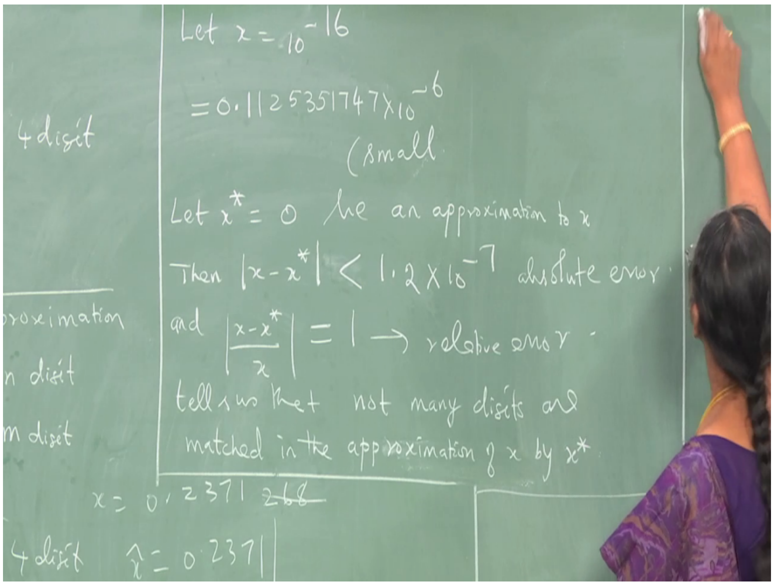

So let us take x to be 10 to the power of minus 16 whose value is 0.1125351747 into 10 to the power minus 6 it is a very very small number. So let me approximate this by 0 be an approximation to x. So the absolute error in this case x minus x star is going to be this minus 0 which is less than1.2 into 10 to the minus 7. So that is the absolute error. On the other hand if I compute x minus x star by x then it is nothing but 1.

So this is the relative error what does it tell so this tells us that not many digits are matched in the approximation of x by x star. Because the relative error is 1 you have made 100 percent error. So your x star and x they are not matched namely many of the digits in x and x star are not matched with each other. That is what the relative error tells. Whereas when you look at the absolute error this tells that the difference is very very small so it looks as though many of the digits are matched.

Whereas that is not true what is given by the information of relative error. So you may say that this is because you have taken a very very small number. Let us see what happens if I take a large number and then find out the approximation.

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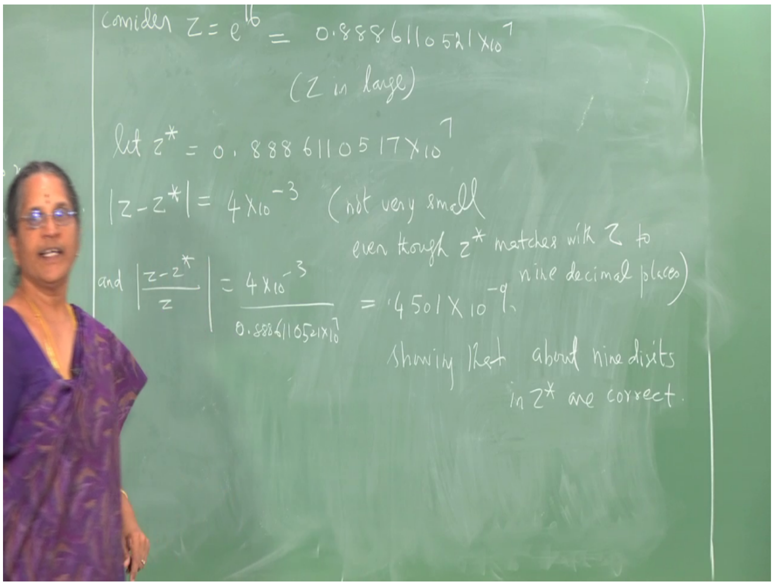

So now I shall consider a very large number say z which e to the power of 16. So let us write down e power 16 0.8886110521 into 10 to the power of 7. It is clear that z is very large. So let us take z star to be an approximation. 0.8886110517 in to 10 to the power of 7. So when I look at this and compute z minus z star it is 4 into 10 to the minus 3. But this is not very small even though z star matches with z to 9 decimal spaces. Although this happens the difference is 4 into 10 to the power of minus 3.

On the other hand if I compute the relative error which is z minus z star by z it is 4 into 10 to the minus 3by z which is e power 16 0.8886110521 into 10 to the power of 7. This turns out to be 0.4501 into 10 to the power of minus 9. What does this show this shows that about 9 digits in z and z star are correct.

So as I said earlier you are able to show that in this case z is very large and looking at z star you will immediately think that your absolute error is going to be a very very small quantity that does not happen, its not small even though z and z star match up to 9 digits. On the other hand if you compute the relative error it clearly shows that the 9 digits are correct in approximation of z star by taking the true value as z.

So the absolute error although is a natural measure is misleading and therefore it is preferable to use the relative error in measuring the amount of error we incur at 8 step. Of course it is clear practically it is not possible because we have no knowledge of what the true value in any problem that is what we are trying to solve and we would like to get approximations.

So in practice it is difficult however we can say how we can make use of the relative error as well as absolute error in giving some information about what the error is at each step of our computation. And give also methods controlling this error.

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We will now move on to polynomial interpolation. So we have studied calculus in earlier semesters. And we have understood that calculus provides us tool with the help of which we are able to understand behaviour of functions. But it is necessary that these functions are either continuous or differentiable. But there are several real life situations and applications where the relationship between two quantities are given in the form of discrete data and therefore we will not be able to use directly the tools provided by calculus and this requires we seek a continuous function which can be obtained from the given data.

So a polynomial is very easy to work with. So when the information about a function is given at a set of discrete points we try to seek a polynomial of suitable degree depending upon the available information. Such that these polynomial interpolates the function at a set of discrete points. By this I mean that this polynomial passes though the points say x 0 x 1 etc x n at which the information about the function is given in terms of function values.

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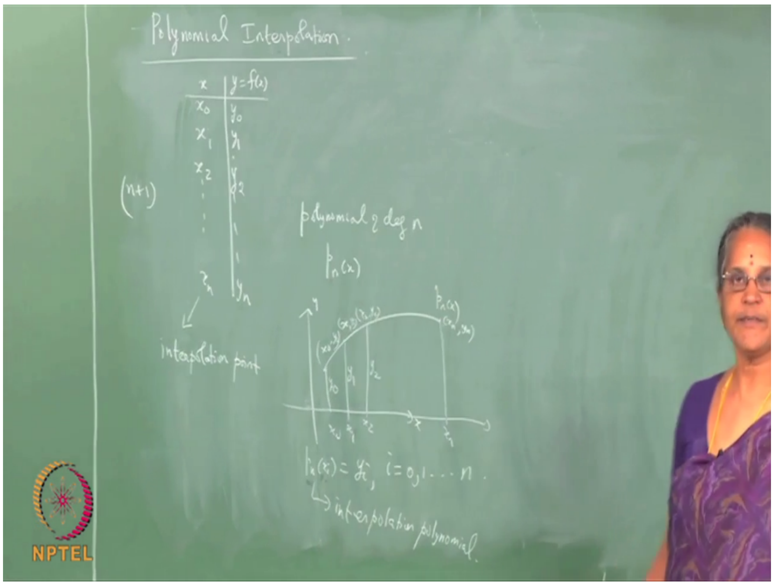

Or in other words if you want to mathematically formulate this problem we are giving information about the independent variable at effect of discrete points say x0 to x n. And we are also given the value of some functions which may have been obtained by some experimental observations or some measurements. Say in the form y 0 y 1 etc y n. So I would like to get a reconstruct this function f(x) as a continuous function.

So as I said earlier I want to seek a polynomial of degree n say I denote it by P n(x) such that if I draw a smooth curve in the xy plane of this polynomial P n(x) it should pass through the points x 0 x 1 x 2 etc say x n such that it should take the values y 0 y 1y 2 at the points x 0 x 1 etc. So I want the graph of p n(x) in xy plane to pass through these points.

Or in other words I require P n(x i) to by y i for i is equal to 0 1 2 3 upto n. In this case we say that the polynomial interpolates the function f(x) at the points x i for i is equal to 0 to n. We call these points x 0 to x n as interpolation points. And the polynomial is called the interpolation polynomial. So mathematically we want to obtain a polynomial of degree n that fix a given set of n plus 1 data points.