Numerical Analysis
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Lecture 22
Numerical Solution of ODE 5
Example for RK-method of Order 2
Modified Euler's Method

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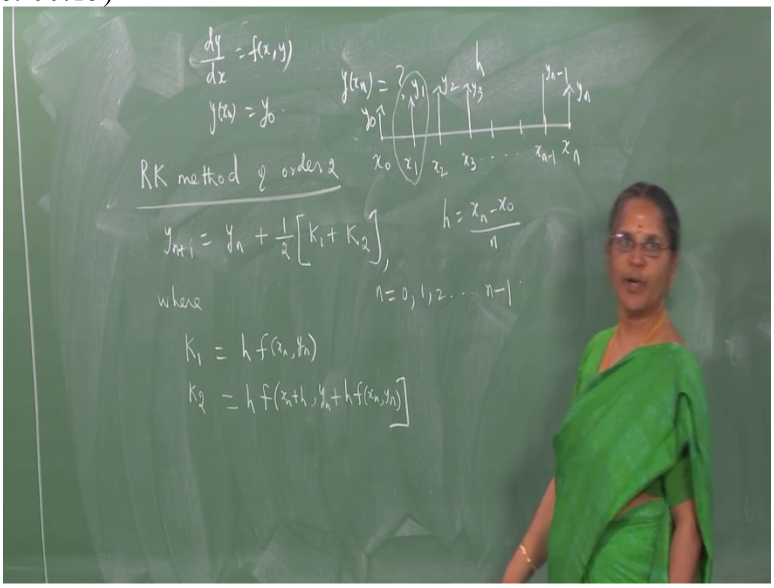

We derived Runge Kutta method of order 2 for solving an initial value problem dy by dx is equal to f(x,y) y(x 0) equal to y 0. And the method is given by y n plus 1 is y n plus half of k 1 plus k 2 where k 1 is h into f(x n y n) and k 2 is h into f(x n plus h) y n plus h into f(x n y n). So given the initial value problem we are required to compute the solution say at x n.

So we divide the interval x 0 to x n into n equal sub intervals of width h by means of points x 1 x 2 x 3 etc x n minus 1. So that h is x n minus x 0 divided by n. And obtain the values at x 1 which we call as y 1 at x 2 y 2 and so on at x n minus 1 it is y n minus 1 and at x n it is y n. And information is given at x 0 as y 0.

So we use this method for values of n equal to 0 1 2 3 etc upto n minus 1 and obtain start with information given at x 0 namely y 0 use this single step method which is an explicit method and compute what y 1 is namely it is y(x 1). Once you know solution at x 1 namely y 1 it is as though you are solving an initial value problem dy by dx is f(x,y) with y(x 1) is equal to y 1.

So you again use the single step method and compute y 2 which is y(x 2) and continue this till you reach this point x n at which the solution is y n and it can be obtained using the method that we have written down. This is what is Runge Kutta Method of order 2. This we have derived earlier and we would now like to make use of this method and solve a simple problem.

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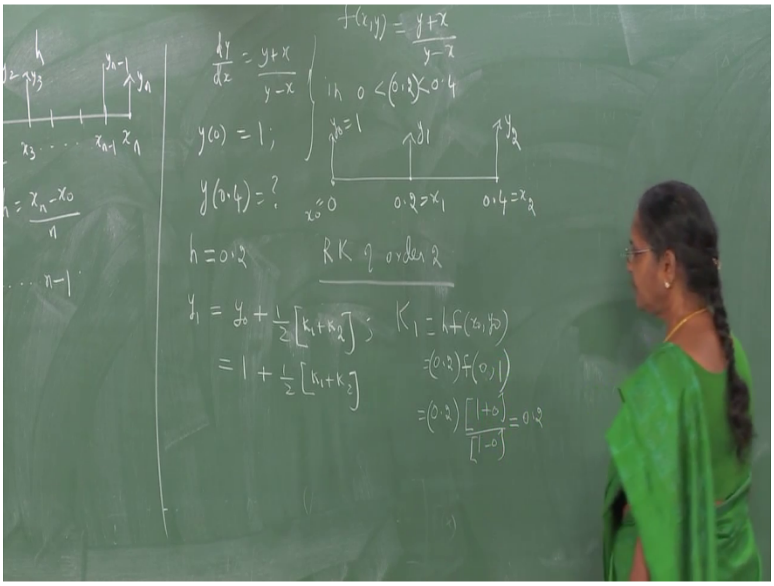

So let us solve the following problem, dy by dx is given to be say y plus x by y minus x. The initial condition is y(0) is equal to 1 your problem is compute y(0.4). By taking step size h as 0.2; sometimes this will be worded like solve this problem in 0 to 0.4 with step size as 0.2, so this indicates the step size that you need to take for solving this problem.

So you are asked to start at 0 and go upto 0.4 in steps of 0.2, so you observe that the mid point of this interval. So your x 0 is 0 your x 1 is 0.2 and your x 2 is 0.4. You have the information here and you are asked to get the information at x 1 and x 2 namely y 1 and y 2.

And we are asked to solve this problem by Runga Kutta method of order 2.So let us see how we can obtain the solution. We already have derived Runge Kutta Method of order 2 and we know that we have to use this to obtain the solution of that problem. So what is y 1 ? y 1 will be from here y 0 plus half of k 1 plus k 2.

So that will be 1 plus half of k 1 plus k 2. What is k 1? K 1 is h into f(x 0 y 0) so 0.2 times f (x 0) is 0 y 0 is 1. That is 0.2 times what is f(x 0 y 0) look at this the problem is dy by dx f(x,y).So your f(x,y) is y plus x by y minus x. So you have f(0, 1) so it should be 1 plus 0 by 1 minus 0. So the value is 0.2.

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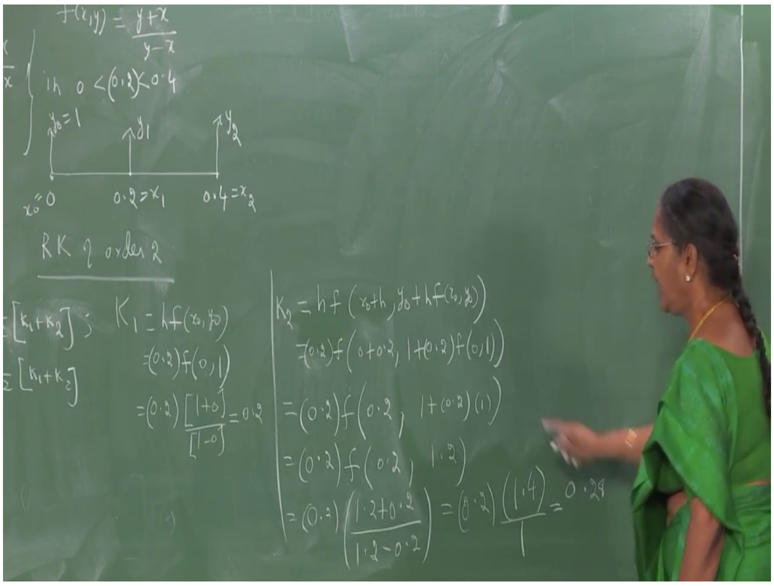

So we now compute what k 2 is? K 2 is some where h into f(x 0 plus h), y 0 plus h into f(x 0 y 0). So h is 0.2 into f(x 0) is 0 h 0.2 y 0 is 1 h is 0.2 f(x 0 y 0) is (0,1).

So this will be 0.2 into f(0.2, 1 plus 0.2 times what is f(0,1) we have just now seen it is (1 plus 0) by (1 minus 0) that is 1. This gives you 1.2. So you need to now evaluate f(0.2 ,1.2) which will be y plus x by y minus x. So you have 0.2 into f of this will be 1.2 plus 0.2 divided by 1.2 minus 0.2. So that will give you 0.2 into 1.4 divided by 1 and that is equal to 0.28, so k 2 is 0.28. So having computed k 1 k 2 we now can use them to find what y 1 is ?

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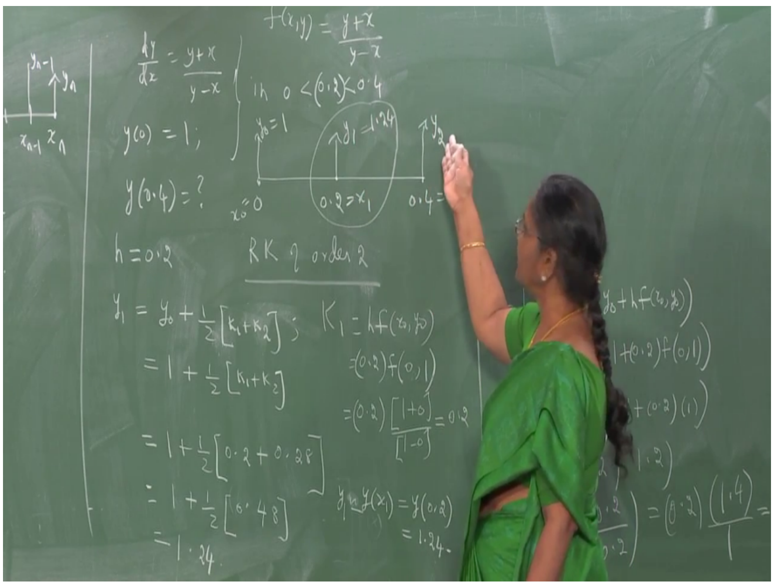

So y 1 will be 1 plus half of k 1 plus 1 to k 2 is 0.28, so that will give you 1 plus half of 0.48, so it will be 1.24. So it tells you that the value of y at x 1 is approximated by this y 1 and that is going to be value of y(0.2) and which is given by 1.24. So you have obtained the solution at this point x 1 and value here is 1.24.

So as I said earlier now we have information at this point and we have to solve the differential equation given that y(0.2) is 1.24 . So we again have to use RK Method of order 2 and move one step ahead to reach the point x 2 at which the solution is y 2 and it can be computed. So we work out the details of computations of y 2 so that you are sure of how the steps are going to be.

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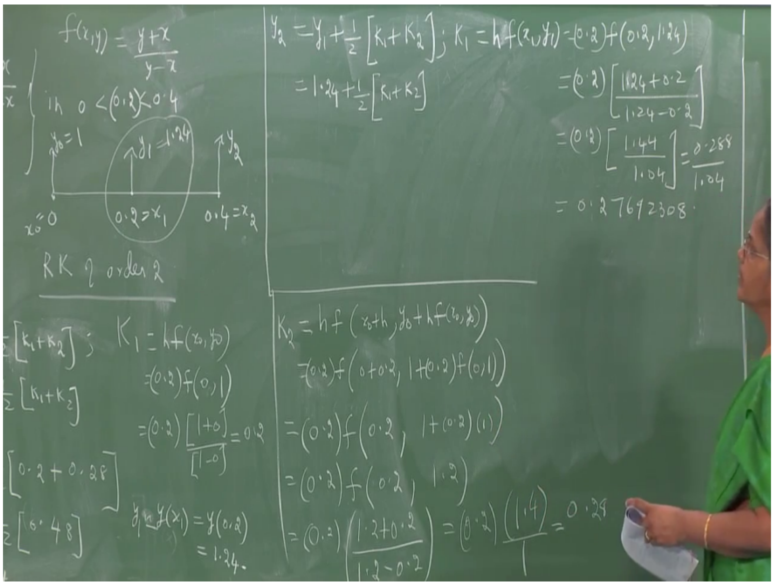

What is y 2 again from the method that we have written it is y 1 plus half of k 1 plus k 2, y 1 is computed just now as 1.24 plus half of k 1 plus k 2 so we compute k 1, k 1 will be h into f(x 1 y 1). So 0.2 times f(x 1) is 0.2 y 1 is (1.24) and f(x,y) is y plus x divided by y minus x. So we substitute for y and x appropriately and find out this result.

This turns out to be 0.2 into 1.44 by 1.04 and that gives 0.288 divided by 1.04 and it is 0.27692308, so k 1 is computed.

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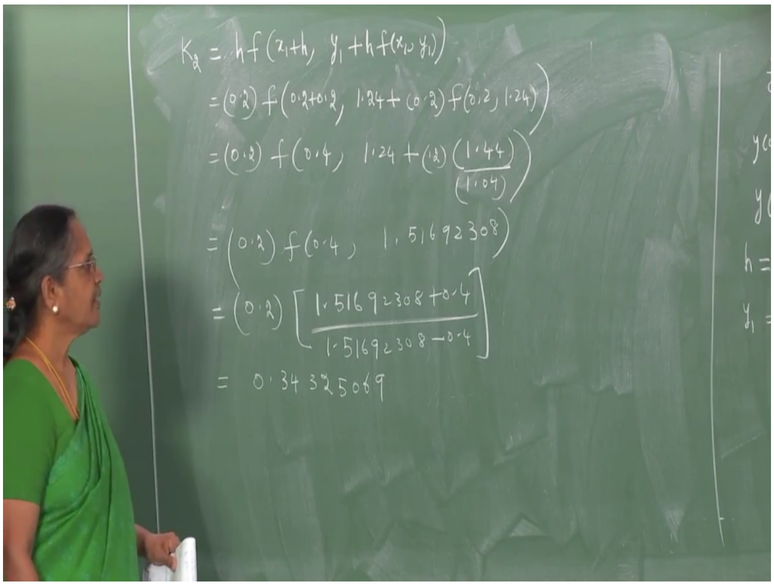

We now work out the details for computation of k 2. K 2 is computed as follows, k 2 is h into f(x 1 plus h), y 1 plus h into f(x 1 y 1). So it is 0.2 into f (x 1) is 0.2 h is 0.2; y 1 is 1.24 h is 0.2 into f(0.2), what is y 1? 1.24, this gives f(0.4), 1.24 plus 0.2 times 1.44 by 1.04.

So we have 0.2 times f(x,y) that turns out to be 1.51692308. So this function has to be now evaluated what is f(x,y) it is again y plus x by y minus x. If you simplify your computations you finally give this to be 0.34325069 and that is k 2. So having computed k 1 and k 2 we substitute in this expression and obtain what y 2 is.

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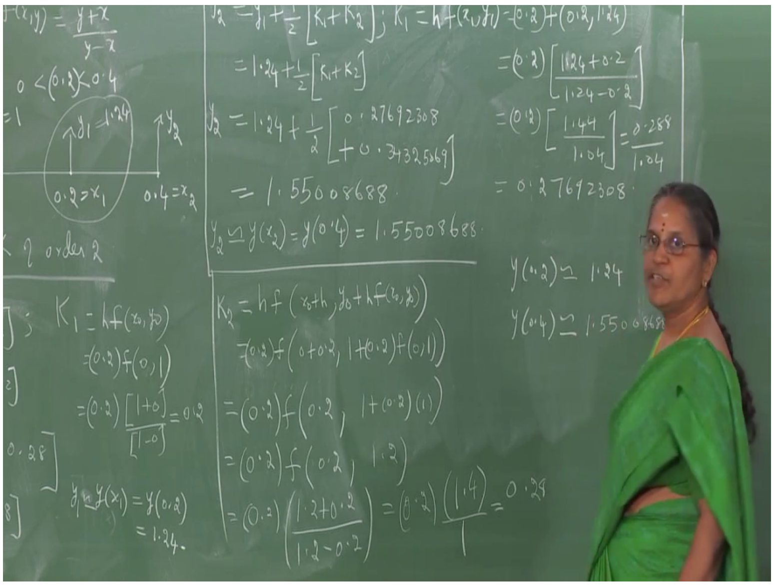

It is 1.24 into half of k 1 is 0.27692308 and k 2 is 0.34325069 and if you simplify you end up with value of y 2 as 1.55008688 so this y 2 approximates the value at x 2 namely y(0.4) and its value is 1.55008682 and you are asked to solve this initial value problem in this interval 0 to 0.4 with step size 0.2.

So you are required to get the solution at 0.2 and at 0.4 and you have computed them and so the final solution is 1.24 which is the value of y(0.2) and 1.55008688 which is the value of y(0.4). So you have to carefully work out the details by substituting the values of x i and y i and computing the function f(x i y i) while evaluating k 1 and k 2.

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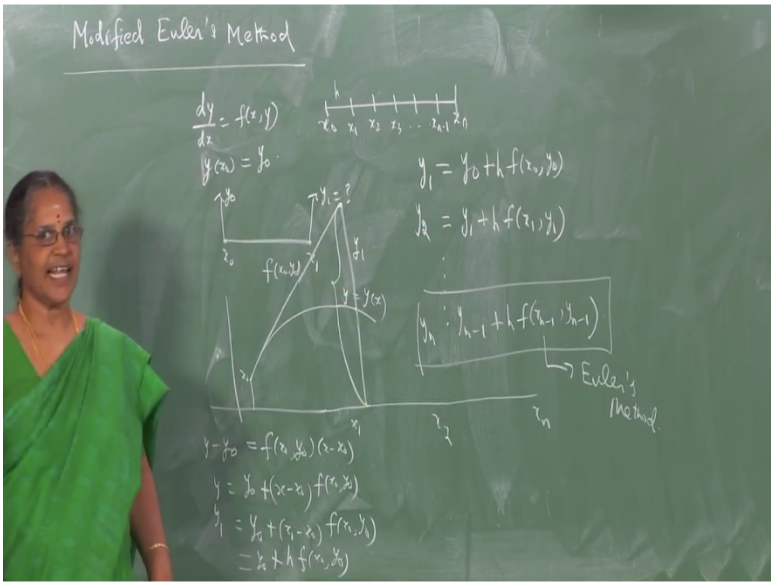

We now derive modified Euler's Method, we recall first what Euler's method is? Modified Euler's Method solves an initial value problem of the form dy by dx is equal to f(x,y) and initial condition is y(x 0) is equal to y 0. So we considered earlier Euler's method and geometrically said that in an interval say x 0 to x 1 given the information at x 0 plus y 0 when we want to find what is the value at x 1 namely y 1 if the solution curve in that interval x 0 to say x 1 is y is equal to y(x) then approximate this solution curve in this interval x0 to x 1 by means of a straight line which is a tangent to the curve at the point x 0 y 0 having slope f(x 0 y 0).

So we wrote down the equation of the line namely it passes through the point x 0 y 0 it has its slope as f(x 0, y 0) and so y will be y 0 plus (x minus x 0) into f(x 0 y 0). If I drop the ordinate at the point x 1 and the curve actually meets the ordinates say at this point but I do not know what the curve is I would like to approximate this ordinate, so I drop an ordinate at x 1 and if it meets this tangent line at some point I take that as the value of y 1 namely the ordinate at x 1.

So when y is y 1 x is x 1 and this gives me the value of y 1 namely y 0 if I take this to be h into f(x 0 y 0). So I have an approximation of the value at x 1 to be given by y 1 which is y 0 plus h into f(x 0 y 0). Now that I have the information x 1 y 1 I can move on to compute what is the solution at the next point x 2 which is such that x 2 is x 0 plus 2 h or x 2 is x 1 plus h.

Then in that case by the same argument y 2 will be y 1 plus h into f(x 1 y 1) and I can go on and reach the point x n at which I would like to find the solution namely it is y n which is y n minus 1 plus h into f(x n minus 1, y n minus 1).

So if I require the solution of this problem in the interval x 0 to x n I divide the interval into a number of equal e spaced points with step size h and determine a solution at these points using Euler's Method which is based on the function that the solution curve on any sub interval of the form x 0 to x 1 or any sub interval of the form x i to x i plus 1 is approximated by a straight line passing through the point x i, y i and having slope f(x i y i).

That is what is Euler's Method. It is a single step method it is an explicit method and we already have discussed the error that is involved at each step of the computation of the solution by Euler's method and we said that the step size h must be very very small and it is a very very slow process and so is there any possibility of modifying this method? Yes one can obtain the method which is known as modified Euler's method by making use of the following a function.

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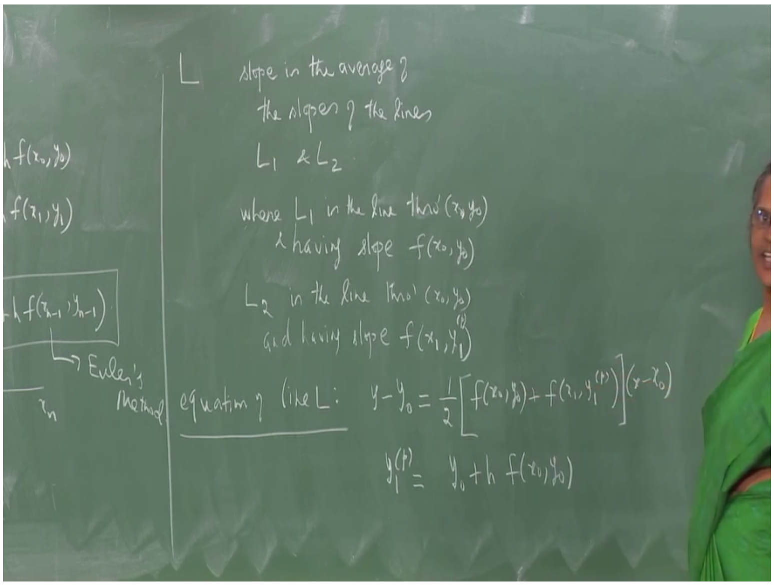

So what is the assumption in modified Euler's Method let us see, in modified Euler's Method the solution curve in an interval of the forms say x 0 to x 1 is approximated by a straight line segment say l whose slope is the average of the slopes of the lines L 1 and L 2 where L 1 is the line through x 0 y 0 and having slope f(x 0 y 0) and L 2 is the line through the same point x 0 y 0 and having slope given by f(x 1 y 1).

So you may ask me you do not know y 1 you are trying to find out what y 1 is at x 1 when you are given the information at x 0. So what is the modification Euler's Method? Take this y 1 as the value that is predicted by Euler's Method. You know how to get y 1 using Euler's Method.

So given an initial value problem first predict the value of y 1(x 1) by Euler's Method. Whatever you get as y 1 denote it by y 1 predicted value and evaluate f(x 1, y 1) p that gives you the slope of the straight line L 2 passing through the point say x 0 y 0. Now take the line L to have its slope to be the average of the slopes of the lines L 1 and L 2.

So let us now write down the equation of the line L what is equation of line L? This line passes through x 0 y 0 and has its slope to be the average of the slopes of these two lines. So the equation is y minus y 0 is equal to slope namely half of slope of L 1 is f(x 0 y 0) plus slope of L 2 is what f(x 1, y 1) predicted multiplied by (x minus x 0).

What is this y 1 predicted do you know that y 1 predicted is nothing but you go back to Euler's method. You predicted that y 1 using Euler's method. What is that it is y 0 plus h into f(x 0 y 0). So this is the equation of the line.

Then how do you determine now a new value of y 1 (x 1) because of this modification? So the new value is nothing but the ordinate at the point x is equal to x 1. So in this we substitute x as x 1 and y as y 1.

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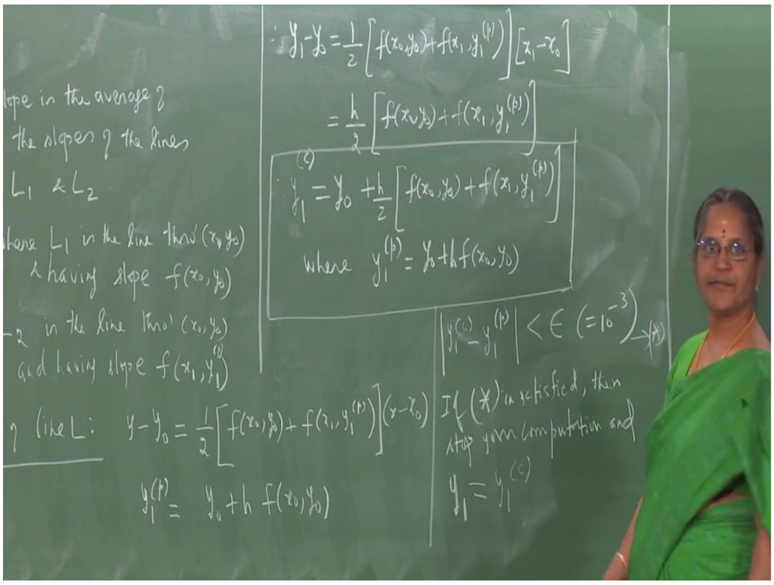

So we get y 1 minus y 0 will be equal to half of f(x 0 y 0) plus f(x 1 y 1) predicted multiplied by x 1 minus x 0 is the step size h. So it is h by 2 into f(x 0 y 0) plus f(x 1 y 1) predicted.

And therefore this gives you a new y 1 and that is equal to y 0 plus h by 2 into f (x 0 y 0) plus f(x 1 y 1) predicted. This is the y 1 which is obtained because of the modification that you incorporated in Euler's Method namely you modified the straight line approximating the solution curve in an interval x 0 to x 1 by taking that line to be 1 whose slope is the average of the slopes of the lines L 1, L 2.

So you obtain a new value a modified value at x 1 for y 1, so you say that you have corrected the earlier predicted value by Euler's method. So to end up with a corrected value of y 1 . And here as we have already seen the predicted value is given by Euler's Method. So this is the step that you should use to find solution at x 1 namely y 1.

So you have a predicted value of y 1 and a corrected value of y 1 which give you values at the point x 1. So you would like to check whether your corrected value and predicted value are such that the absolute value of the difference between the two is going to be less than the prescribed accuracy, say it is something like 10 to the minus 3.

If this happens, if call this as star if star is subside then you can stop your computations and you have the solutions correct to the desired degree of accuracy which is y 1 is equal to y 1 corrected. If suppose star is not satisfied let us see what has to be done.

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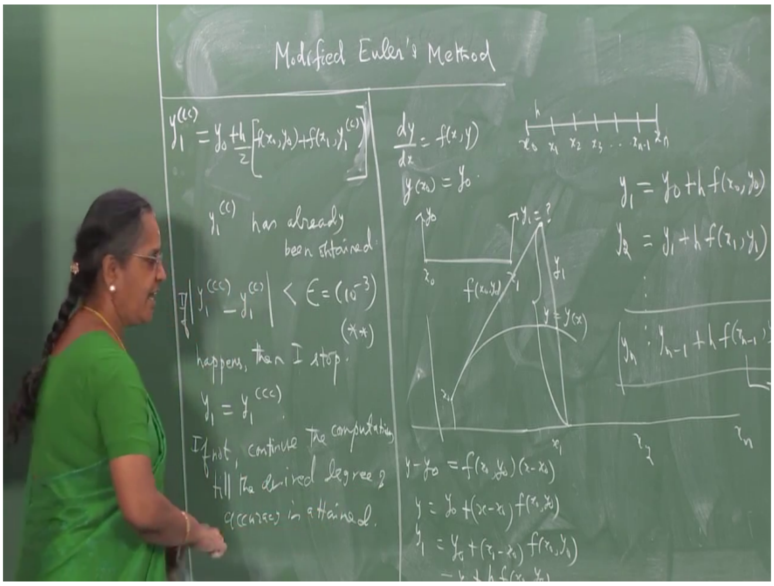

In that case we write down y 1second corrected value and compute this value. What is y 1 second corrected value? Its again by making use of this method instead of substituting y 1 (p) here for y 1 we can substitute y 1 corrected value that we have computed here to evaluate this function.

So that y 1 second corrected value will be y 0 plus h by 2 into f(x 0 , y 0) plus f(x 1, y 1) first corrected value. Earlier I used the predicted value by Euler's method and I corrected it. Now I am unable to satisfy my requirement of accuracy. And therefore I want to re correct it. And I compute the re corrected value and denote it by y 1 cc in which I make use of the first corrected value that I obtain from here.

So we substitute and compute what y 1 second corrected value here y 1 corrected value has already been obtained at this stage I check if y 1 second corrected value minus y 1 first corrected value is less than Epsilon, say 10 to the minus 3. If this happens call this as double star, if this happens then I stop the computation.

And write down the solution y 1 as y 1 second corrected value, if not continue the computations till the desired degree of accuracy is attained. At every step when you want to continue you must make use of the currently available corrected value of y 1. And whatever you get is the new corrected value.

And then you find the difference between the new corrected value and the previous corrected value and check whether your requirement is satisfied if so you stop if not you continue your computations. This is what is modified Euler's Method is. And we observe that we have a nice Predictor Corrector Method namely I am able to Predict the value of y 1 which is the ordinate at x 1 and I have scope for correcting it, not only correcting it once but I can make successive recorrections till the desired degree of accuracy is obtained.

So I can treat modified Euler's Method as a Predictor Corrector Method we can predict the solution at a particular point and correct it successively till the desired degree of accuracy is attained. We will develop some more Predictor Corrector Methods later on but we started with a modification in Euler's Method which is a single step method is an explicit method by simply modifying the assumption that in an interval the solution curve can be approximated by a line having its slope to be the average of the slopes of two lines passing through x 0 , y 0 we obtain a method.

This is again a single step method but we are able to get a Predictor corrector method which helps us to predict the solution and correct it, recorrect it successively till the desired degree of accuracy is obtained. So let us take an example and illustrate this method so that you know how to compute the solution using modified Euler's Method.

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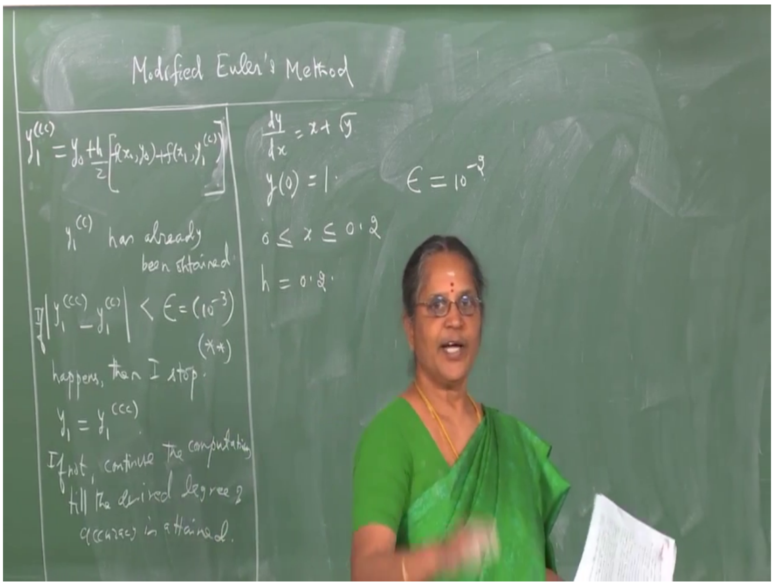

So let us take this addition value problem, dy by dx is x plus root y; initial condition given is y (0) equal to 1 and you are asked to solve this problem in this interval 0 to 0.2 by taking step size as 0.2 using modified Euler's Method. And determine the solution correct to the desired degree of accuracy say Epsilon which is 10 to the minus 2.One can ask for more accurate solutions since we are doing manually I take Epsilon to be 10 to the minus 2.

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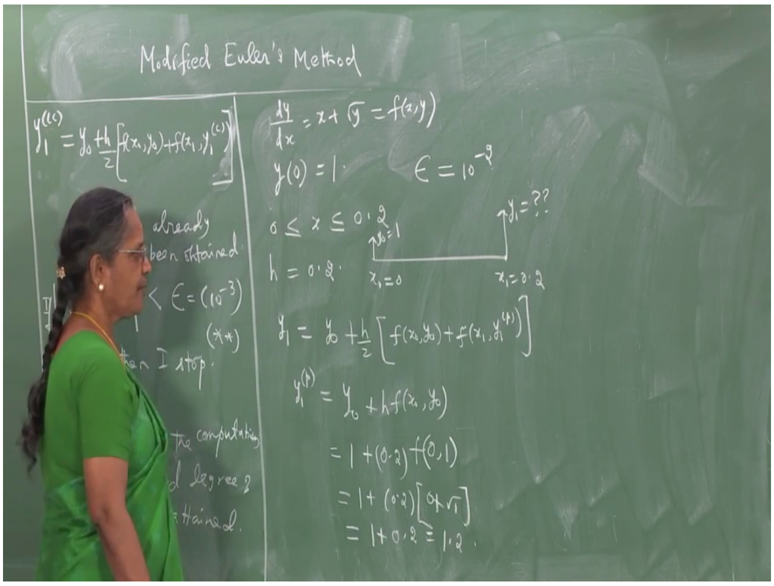

So let us work out the details. So always start with what is given write it out clearly so that we know what is it that we want to compute; x 0 is given to be 0 at which y 0 is 1. We are asked to get the solution at 0.2 and the step size is 0.2, so x 1 is 0.2 and the solution at x 1 is required correct to desired degree of accuracy and we are asked to use modified Euler's Method.

So the method will be such that if I want y 1 (x 1) it is y 0 plus h by 2 into f(x 0 y 0) plus f(x 1 y 1) predicted where y 1 predicted is going to be y 0 plus h into f(x 0 y 0). So let us first predict it. What is y 0? it is 1; h is 0.2 f(x 0) is 0 y 0 is 1. So this will give you 1 plus 0.2 into what is f(x,y)? This is f(x,y) which is x plus root y. So it is 0 plus root 1, so it is 1.2 so 1.2 is the predicted value and that is what I have to use here in getting what y 1 is.

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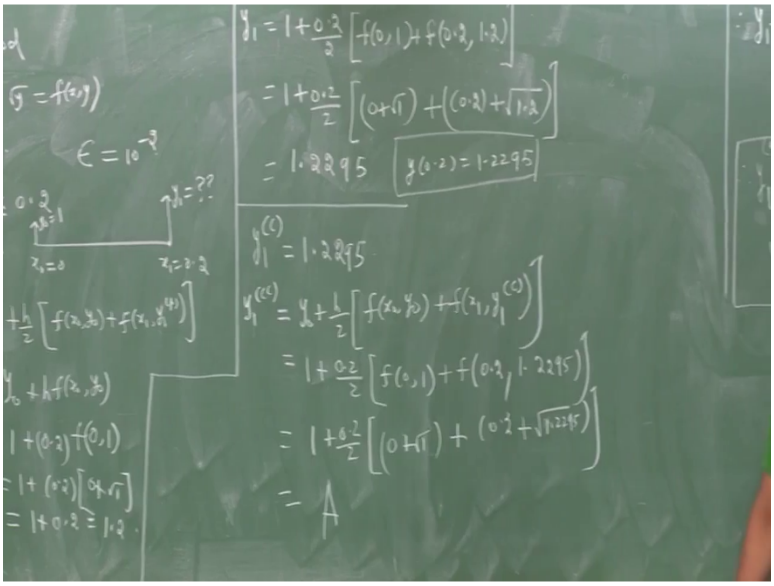

Let us compute y 1 now. So y 1 will be y 0 which is 1 plus h by 2 into f(x 0y 0) plus f(x 1) is 0.2l y 1 predicted is 1.2 which will be 1 plus 0.2 by 2 into f(x y) is x plus root y , f(x y) is x plus root y.

So if you compute this you end up with 1.2295 that is the value of y(x 1) namely y 1 obtained through modified Euler's Method if nothing is specified about the accuracy of your computation you can stop here and say that the solution y (0.2) is 1.2295 and stop your computations because we are not been asked to use modified Euler's Method as a Predictor Corrector method.

On the other hand some accuracy is specified and you are asked to compute the solution correct to the desired degree of accuracy and you are asked to use modified Euler's Method as a predictor corrector method you cannot stop your computations here. You will have to continue your computations and call the value of y 1 that you have obtained as the first corrected value and denote it as y 1 corrected which is 1.2295.

And now recorrect it by using modified Euler's Method. What is the recorrected value? Y 1 recorrected is going to be y 0 plus h by 2 into f(x 0 y 0) plus f(x 1 y 1) first corrected value so you need to use the first corrected value here. And you observe that these terms remain as they are y 0 plus h by 2 f(x 0 y 0) they appear also in your earlier computation.

They remain as they are, the only change that occurs here where instead of y 1 predicted value you should use y 1 corrected value. So this will be 1 plus h by 2 into f(0, 1 x 0 y 0) plus f (x 1) is 0.2, what is y 1 corrected value that is 1.2295 so it is going to be 1 plus 0.2 by 2 into x plus root y plus x plus root y.

So compute this and you obtain y 1 recorrected value suppose say you obtain its value to be some capital A. You can use your calculator and work out the details and get what this value of a is.

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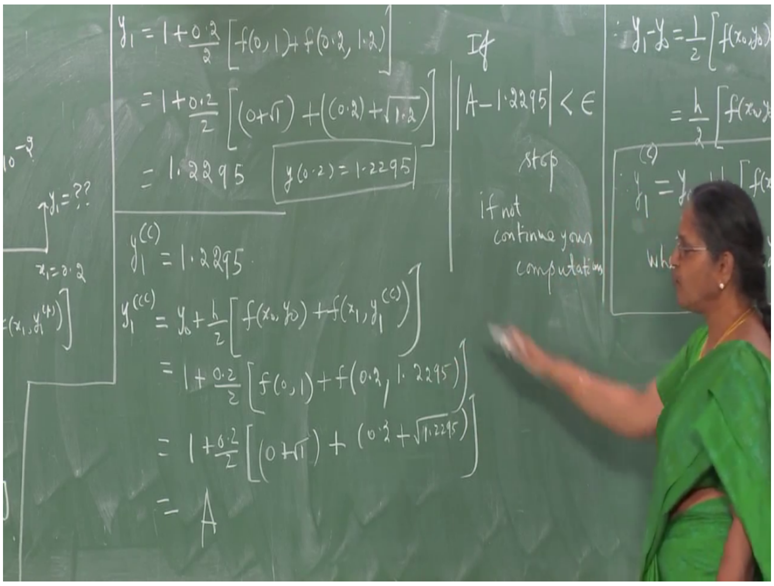

At this stage the difference between a second corrected value and the first corrected value and if it is less than Epsilon we prescribed accuracy then you stop if this happens you can stop.

If not continue your computations as explained earlier and get the solution at x1 correct to the desired degree of accuracy. You may ask me why at all you computed the second corrected value because we immediately observed that the first corrected value is 1.2295 and the predicted value is 1.2 so the difference turned out to be 0.0295 and we wanted an accuracy of 10 to the minus 2.

So at this stage namely the first stage we were not able to get our desired accuracy and that is why we went on to computation of y 1 again namely we recorrected this value and tried to obtain the value A and then we again have to check whether this condition is satisfied. If so we can stop otherwise we have to continue our computation still the desired degree of accuracy is attained.

I am sure you have understood the way in which the modified Euler's method has to be applied to an initial value problem and the solution can be obtained correct to the desired degree of accuracy. We will continue with the discussion of the Predicted Corrected Method in the next class one Predictor Corrector Method has already been obtained namely modified Euler's Method that was not our intention but we observed that it could be very nicely used as a Predictor Corrector Method.

So we shall develop other methods which are Predictor Corrector Methods for solution of initial value problem in the next class.