Numerical Analysis
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Lecture 20
Numerical Solution of ODE 3
Examples for Taylor Series Method
Euler's Method

Good Morning in the previous class we discussed about numerical solution of initial value problems governed by first order ordinary differential equations. And we also discussed about Taylor series method for solving such the first order differential equation in which an initial condition is specified. So let us recall Taylor series method of order n and use this method to solve some initial value problems and illustrate the method.

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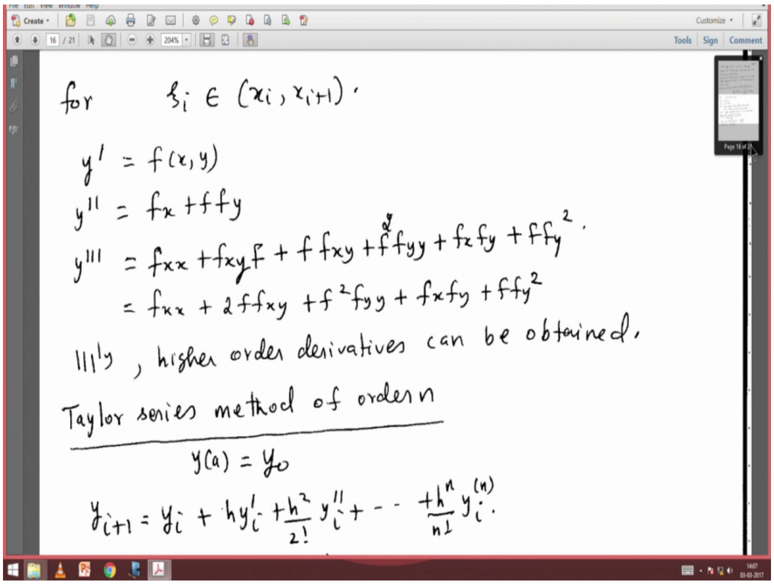

So we said that if we had a initial value problem governed by dy by dx is equal to f(x,y) in the interval a to b subject to an initial condition y (a) equal to y 0. And suppose that the solution y(x) to this initial value problem has (n plus 1) continuous derivatives then we can expand the solution y(x) in terms of its n th Taylor polynomial about the point x i and evaluate y at the point x i plus 1 which is x i plus h.

So y(x i plus 1) will be y (x i plus h) y dash(x i plus h square) by factorial 2 y double dash(x i) plus etc plus h power n by factorial into n th derivative of y (x i) plus the error term which is of power of h power n plus 1. Since we want to use Taylor series method for a n. And Psi i belongs to (x i , x i plus 1).

So here the various order derivatives of y which appear in this expansion are obtained by taking derivative of the right hand side function f (x , y).

So the successive derivatives are presented in terms of the partial derivatives of the different orders of f with respect to x and y. So when we compute the derivatives of y with respect to x of several orders namely upto order n is what we require to write down the solution at x i plus 1, Then we have Taylor series method of order n given by y(a) is y 0 that is the initial condition.

Then we know we can compute y 1 with i is equal to 0 by substituting the derivatives evaluated at the point x 0, y 0. Then we have y 1 when we use this expansion to obtain solution at x 2 namely y 2 and continue till x n at which we require the solution which is denoted by y n which approximates y (x n). The local truncation error is of order of h power n plus 1. So we have derived this method in the previous class and we have just recalled what the method is.

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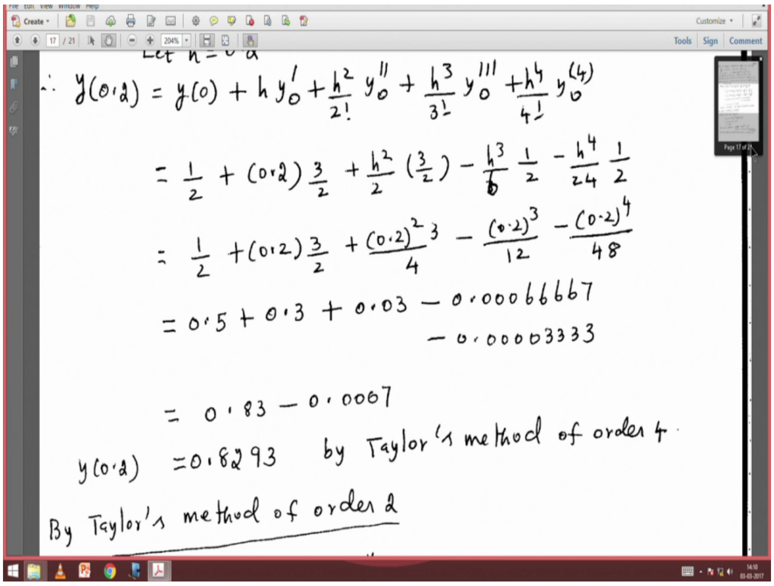

So let us consider some examples and then see how we can apply these methods to the solution of initial value problems governed by first order differential equations. So let us take dy by dx to be y minus x square plus 1 in the interval 0 to 2 initial condition is y (0) equal to half. And say we are asked to get the solution by Taylor's method of orders two and four. So we require derivatives of y upto order 4, so let us take y prime which is y minus x square plus 1 and compute this successive derivatives.

And then we can evaluate these derivatives, so similarly we evaluate the third order derivative at the initial point, first order derivative at the initial point and keep these values ready. And then take the step size to be h is equal to 0.2. So let us take h to be 0.2 and then obtain the solution.

So y (0.2) will be y(0) plus y dash 0 , since we want to use Taylor series method of order 4 let us compute the solution with order 4 first. So we have terms upto h power 4 by factorial 4 into fourth derivative and we already have the values computed with us, so substitute then you end up with a solution y (0.2) is 0.8293 by Taylor series of order 4.

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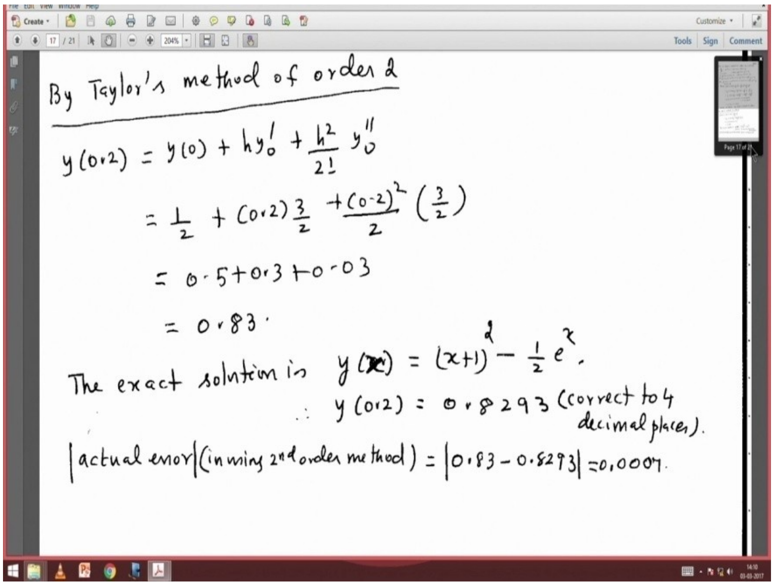

So now let us compute the solution using Taylor series method of order 2, so we just require terms upto order of h square in the expansion and we already have these values we substitute and we get the value to be equal to 0.83. So we have solutions y(0.2) using Taylor series method of order 2 turns out to be 0.83. So let us see what the exact solution is? The exact solution is (x plus 1) the whole square minus e power x by 2 and this time we immediately verify by computing the derivative y prime.

So since I have compute the solution 0.2 I evaluate y(x) which is 0.2. So substitute x as 0.2 and that gives you y(0.2) to be 0.8293 and this is correct to 4 decimal places. So what is the actual error when we use second order method. The solution computed numerically is 0.83, the exact solution gives 0.8293 so the error that is occurring in this computation is 0.0007 whereas at this point when we require solution correct upto 4 decimal places and use Taylor's Method of order 4 we see that the method of order 4 and the exact solution match upto 4 decimal places.

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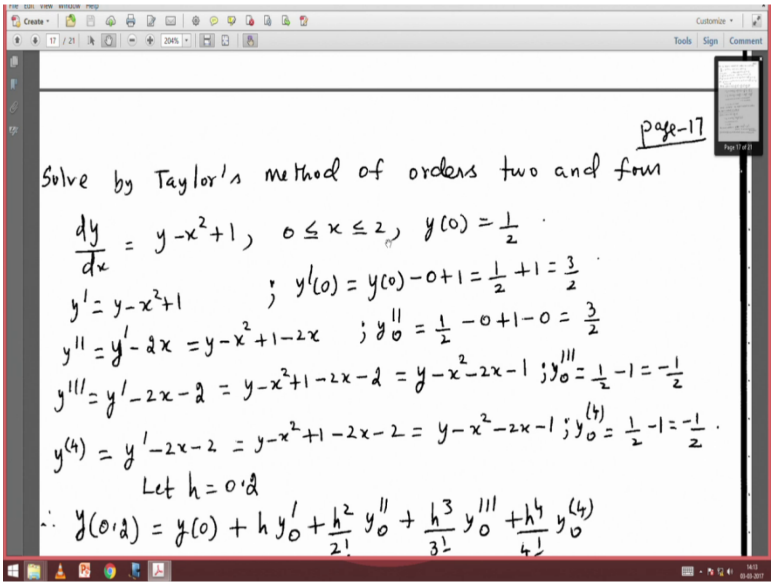

So we are required to compute the solution in this interval 0 to 2 or take the step size to be 0.2. So we have worked out the details by computing the solution at 0.2. So now solve the new initial value problem which is dy by dx is equal to y minus x square plus 1 subject to the condition that y(0.2) is 0.8293 by Taylor series method of order 4.

So we have a new initial value problem so we would like to march one step ahead. Since h is 0.2 we would compute the solution at x is equal to 0.4 and obtain y(0.4). And we continue this process till we reach the end point of the interval and the end point of the interval is x is equal to 2.

So we can march step by step with a step size h is equal to 0.2 and determine the solution by Taylor's method of third or fourth and similarly use Taylor's method of order 2 and compute the solution at discrete points which are equally spaced with step size h is equal to 0.2 and move on upto the end point of interval 2 and determine the solution.

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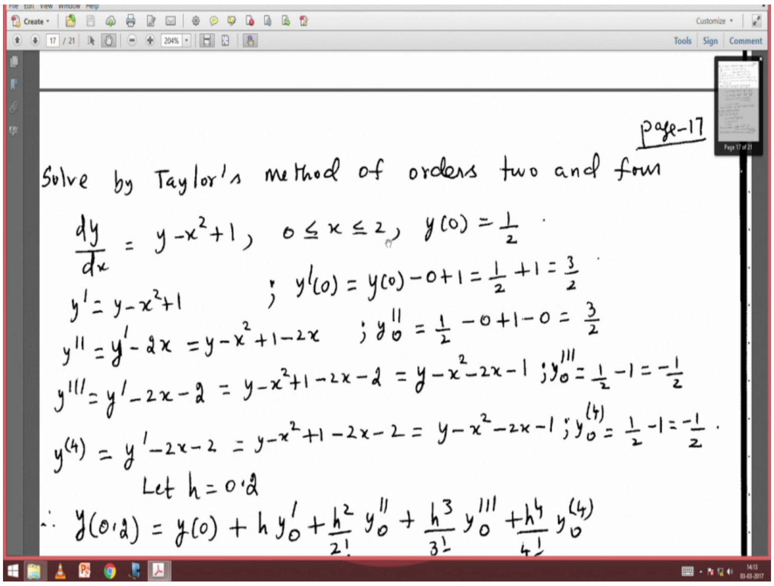

And since we march at any time one step ahead to determine the solution of the initial value problem we say that this method is a single step method. In addition it is also an explicit method because if we know the solution at y i then we compute the solution explicitly at the point x i plus 1 namely if we know the solution at x i which is y i then we compute the solution at the next point x i plus 1 which is y i plus 1.

So we also call this method as an explicit method, so it is a single step explicit method which can be used to solve the initial value problem governed by first order differential equation. And depending upon the accuracy that we want we can include as many terms as we want so that the desired degree of accuracy is reached. So let us consider another example.

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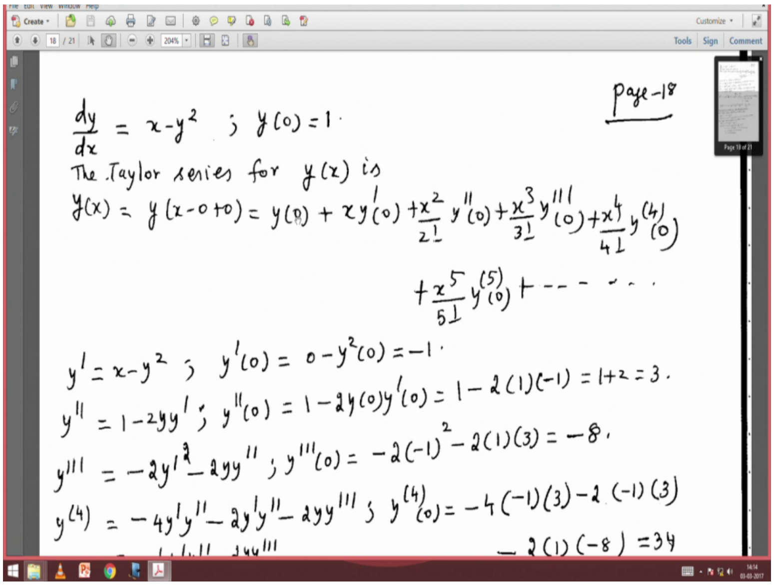

So I want to solve this time the differential equation dy by dx is equal to x minus y square; initial condition y(0) equal to 1. So Taylor series for y(x) because initial point is 0 so Taylor series for y(x) will y(x minus 0 plus 0) so it is y(0) plus x y prime (0) plus x square by 2y double prime (0) x cube by 3y triple prime (0) and so on.

Depending upon the number of terms that we want to include we can consider these terms. So let us compute these derivatives we see that y prime, y double prime, y triple prime etc say upto fifth derivative have been computed and the values of these derivatives at 0 also have been computed. So that y prime (0) is minus 1 y double prime (0) is 3 and so on. So the values of the derivatives are also evaluated at this point.

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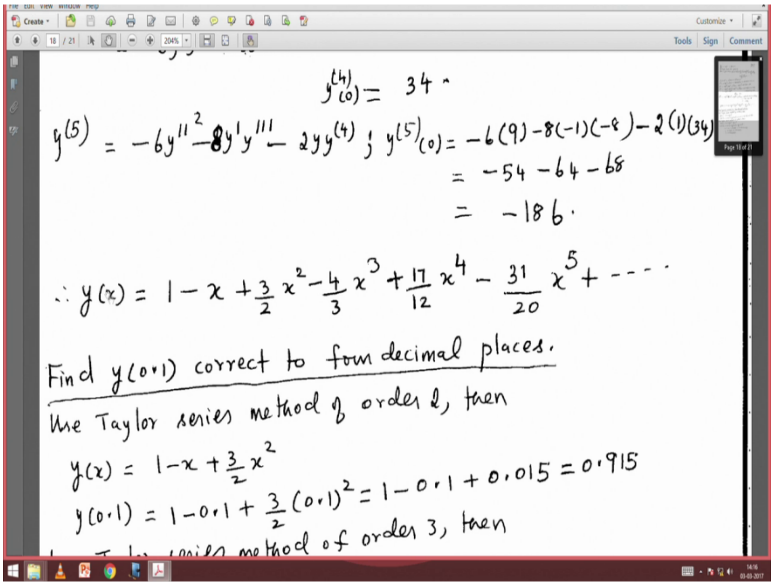

So we write down at any x the solution is given by y(x) is equal to 1 minus x plus etc where x is greater than 0 because the initial point is x is equal to 0. So starting from 0 we marched ahead and then determine the solution by this method. Or if you want to go march backward and get the solution say minus 0.2, minus 0.4 etc that can also be done using this expansion y (x) is equal to 1 minus x plus etc which is given by this.

Now the question is suppose we want to compute y (0.1) taking the step size h to be 0.1 right? and determine the solution correct to 4 decimal places at that point, so we do not know how many terms we have to add in the Taylor series, so to begin with I take the first three terms I can start with any number of terms to begin with.

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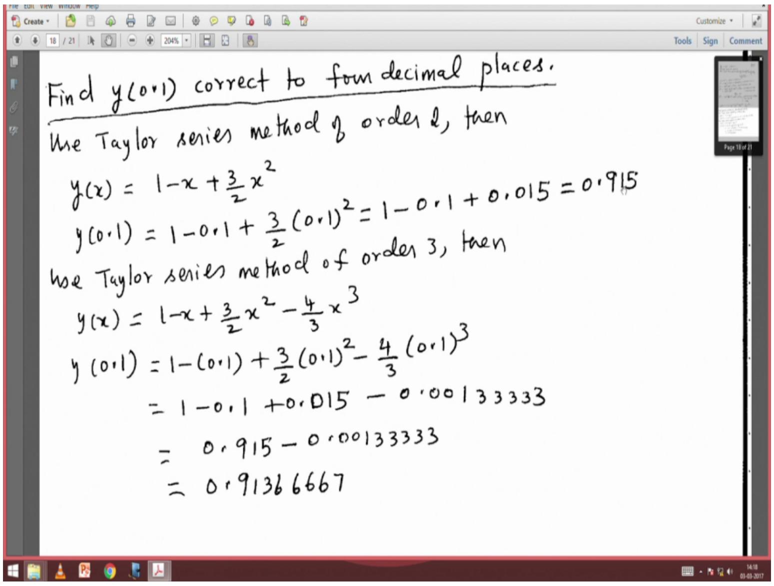

Let me start with the first three terms which is 1 minus x plus 3 by 2 x square and compute y(0.1) so I substitute these values and determine y (0.1) to be 0.915. Alright , since I want solution correct to 4 decimal places let me use a Taylor series method of order 3 now. Namely I include the next term and then substitute x is equal to 0.1 and evaluate y(0.1) by taking x as 0.1.

Then I end up with a value which is 0.91366667 so with third order method it is this value, with the second order method the value of y(0.1) is 0.915 I observe that they differ even at the third decimal place.

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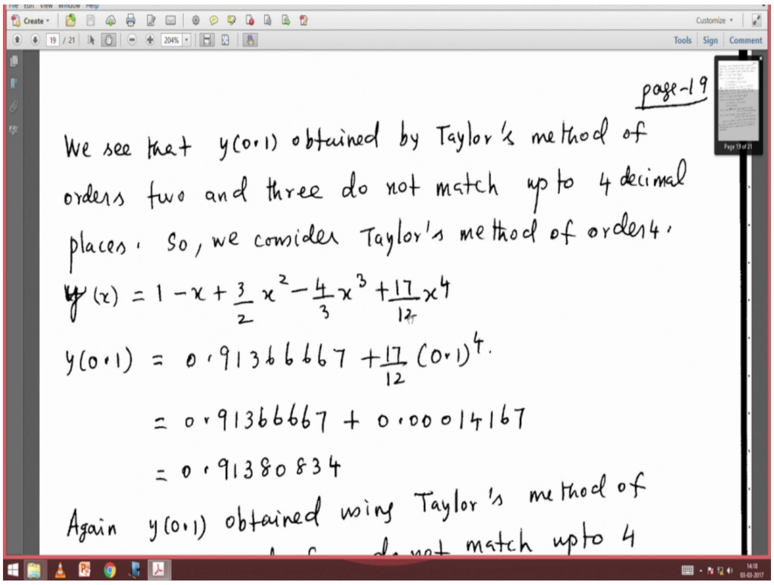

So I cannot stop so Ii have to compute and so I consider the Taylor series method of order 4 so include the fourth order terms also. Already I have completed the solution by taking into account upto third order terms so I know the value I simple have to add the fourth term and evaluate it at x is equal to 0.4.

When I do that I see that it is 0.93180834 when I round it off to 4 decimal places this gives me 0.9138 that is from the fourth order method. Whereas by a third order method I obtain this solution which when rounded off to 4 decimal places will be 0.9137 so they do not match at the fourth decimal place.So I cannot stop and so i will have to add say another term and check whether I get the desired accuracy.

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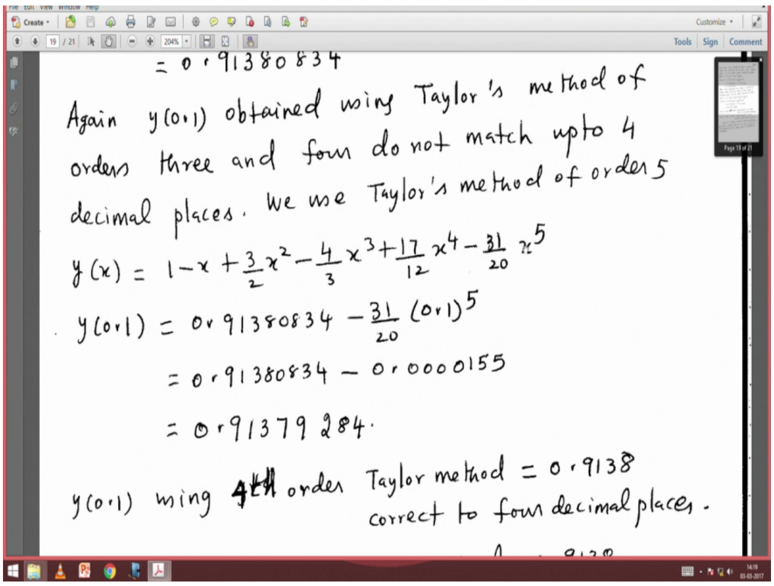

So I write down the solution y(x) will incorporating terms upto order of h to the power of 5. So I have the values upto this by my previous computation I only have to add this term to it and when I do that I end up this solution and I observe that correct to 4 decimal places this gives me 0.9138 and correct to 4 decimal places the solution obtained by adding just the terms upto h to the power of 4 also gave me 0.9138,

so by including the fifth order term I am able to get a solution which matches with the previous solution obtained by Taylor series method of order 4 correct to 4 decimal places and therefore my solution is 0.9138 correct `to 4 decimal places. And this is obtained by using terms upto x to the power of 4 in the Taylor expansion.

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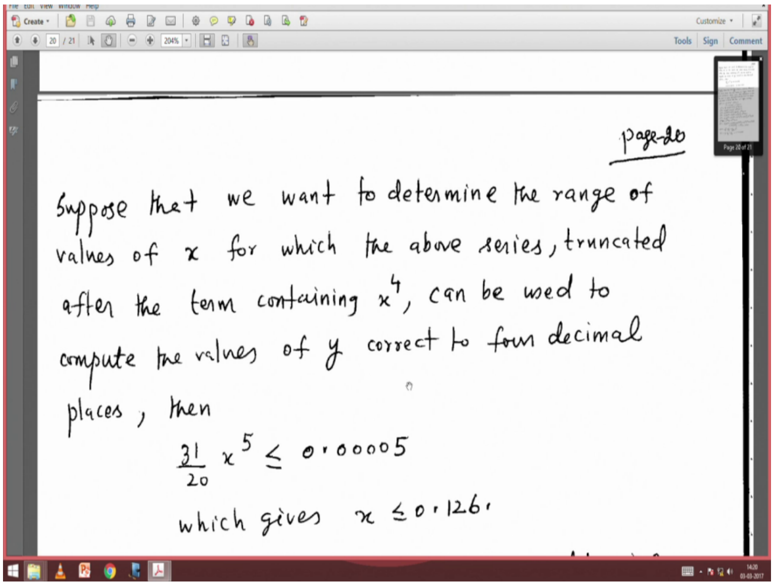

So now suppose say I would like to check upto what values of x can I use Taylor expansion upto terms containing x to the power of 4 such that the solution that I get from that will be correct to 4 decimal places.

So is that solution valid for all values of x or upto what values of x is it going to be a solution which gives me the desired degree of accuracy. So we look at the error term and say that the error must be such that it should be less than right, 0.00005 because I want the solution correct to 4 decimal places.

So what is the neglected term? The neglected term is the term containing x to the power of 5 because the term upto terms containing x to the power of 4 have been taken and the solutions have been obtained at 0.1 and we have been able to satisfy our requirement that it satisfies the desired accuracy. Namely we have been able to get solution correct to 4 decimal places.

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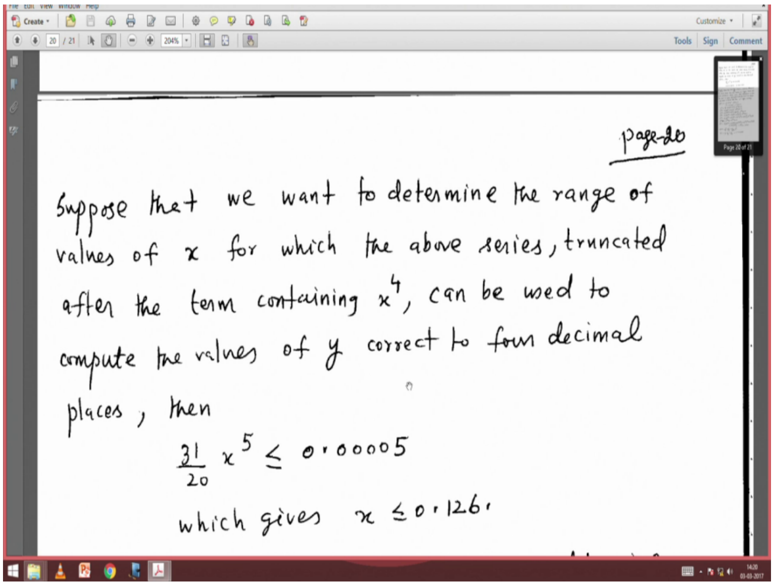

So let us see for what values of x is this correct which means the neglected term namely the error term must be less than or equal to 0.00005. So this gives you x must be less than or equal to 0.126 so all values of x upto 0.126 wherever you want to obtain solution using Taylor series expansion of order 4 you will get your solution correct to 4 decimal accuracy. That is what this information gives us.

So this is one way of approaching namely making the Taylor series method to be of higher order so that incorporating more and more terms and then checking whether the solution obtained by a method of order n plus 1 matches with the solution obtained by using a method of order n correct to the desired degree of accuracy.

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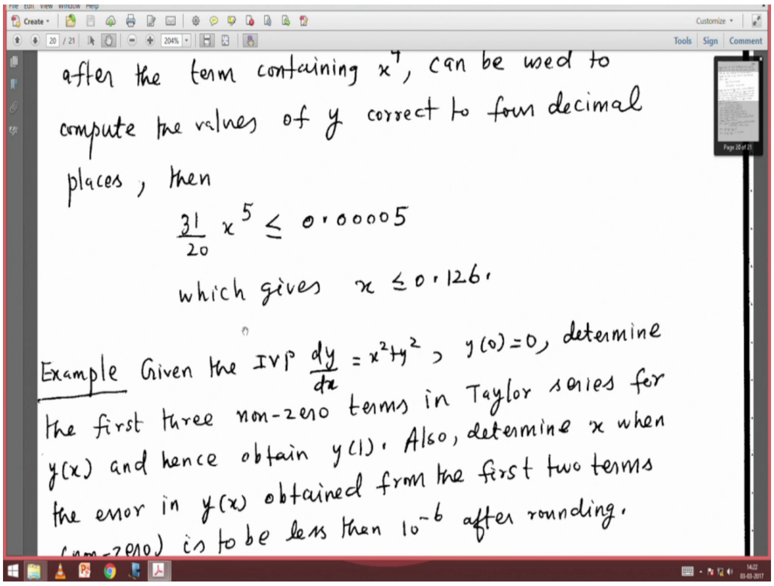

Then one can also do the one can also compute the solution by taking the step size to be smaller and smaller. Here we have taken h to be 0.1 and then evaluated the solution at 0.1 by using different Taylor series method namely the Taylor series method of order 2 order 3 order 4 and so on and then we stopped only when the desired degree of accuracy was attained.

On the other hand we can choose a smallest step size h. Instead of h as been taken as 0.1 start with h to be equal to say 0.05 and check whether your solution obtained by Taylor series method say of order 2 or of order 3 gives you the solution at x is equal to 0.1 correct to the desired degree of accuracy.

If it is not reduce your h for that and then work out the detail step by step and then finally obtain the solution at 0.1 and check whether your desired accuracy is attained. So that is another way of approaching the same problem namely reducing the step size h or taking the step size h to be very small and then checking out the solution at the point at which we desire the solution and then if it is satisfied we stop our computations that is one approach.

The other approach would be to take Taylor series method of different orders take the step size h to be say 0. 1 and then compute the solution at 0.1 by different Taylor series method of different orders and then check whether our accuracy is attained at some stage right?

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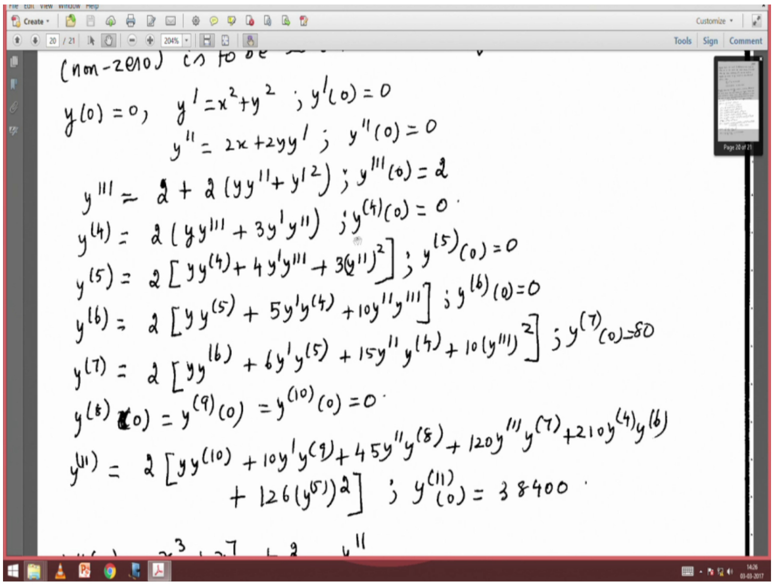

So let us now consider another example. So give this initial value problem dy by dx square plus y square, y(0) is equal to 0 we determine the first non zero term in Taylor series for y(x) and hence also determine x when the error in y(x) is obtained from the first two terms which are non zero should be such that it is less than 10 to the power of minus 6 after rounding. So there are two parts to it.

One is take the first three terms in Taylor series for y(x) and obtain the value of y(1). So let us first look into the details. The initial point is 0 so we expand Taylor series about 0 so we require the derivatives to be evaluated at 0, so we first compute the derivatives. So y prime is f that is x square plus y square but y prime (0) is 0, right? So the first term how does it look like y(x plus 0 minus 0) which is y(0) plus x into y prime 0 that y prime 0 turns out to be 0.

So we essentially have only on term which is y(0) and what about that y(0) is given to be 0. So if we have a Taylor series method of order 1 y(0) plus h y prime 0 then we essentially have nothing on the right hand side because y prime 0 is 0 y(0) is also 0. We compute y double prime and that evaluated at 0 also turns out to be 0. So terms upto order of x square all vanish. So we go to Taylor series method of order 3 compute y triple prime 0 that turns out to be non zero. So this is the first non zero term in the Taylor expansion, right?

So we get a non zero value of y at any x by using a Taylor's method of order 3 so that the contribution comes from y triple prime 0 and it is a non zero value. So we move ahead compute fourth derivative at 0 that turns out to be 0 the fifth derivative the sixth derivative also turn out to be 0. So far we only have 1 non zero term coming from this y triple prime o into h cube by factorial 3. So we compute the seventh derivative and evaluate it at 0.

We see that it is a non zero quantity namely at. The second non zero term in our Taylor expansion, so all along we have considered terms upto h power 7 by factorial 7 into 7 th derivative at 0 which is non zero. But on the whole we have only 2 non zero terms in the Taylor expansion.

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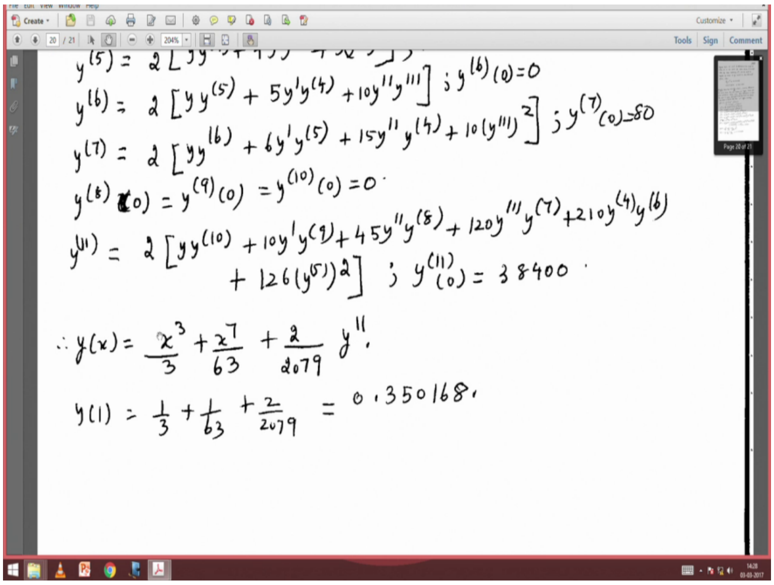

We go ahead because we are asked to take the first three non zero terms right? in the Taylor expansion. So we go further ahead to compute the next non zero term We observe that the nineth derivative at 0 is 0, the tenth derivative at 0 is 0, so we compute the elevent derivative and evaluate it at 0 and that comes out to be this which is 38400 which is non zero.

So we write out what y(x) is using these first three non zero terms which is nothing but x cube by factorial three into y triple prime 0 plus x power 7 by factorial 7 into 7 th derivative of y(0) plus 2 by 2 plus h power 11 by 11 factorial into 11 th derivative of y (0).

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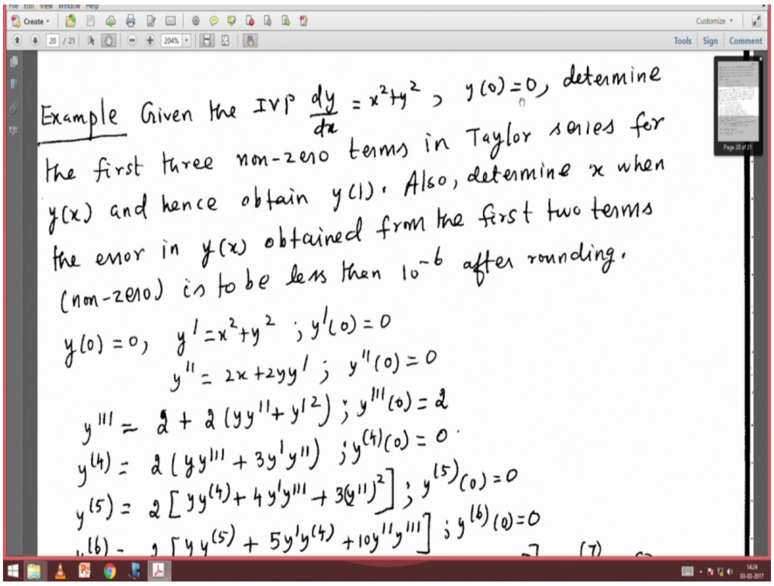

So that gives you the value that gives you the solution of y at any x. So we want the solution at 1, that is the problem take the first non zero terms the first three non zero terms in Taylor expansion obtained y(1). So we evaluate y(1) and that turns out to be say this quantity. So we have completed the first part.

The next part says values of x then error in y(x) obtained from the first two terms which are non zero to be less than minus 6 after rounding. So what does it say? It says use a Taylor series method and include upto the first two non zero terms in it , whatever be the error that is incurred in the computation should be such that, that must be less than 10 to the minus 6 after rounding.

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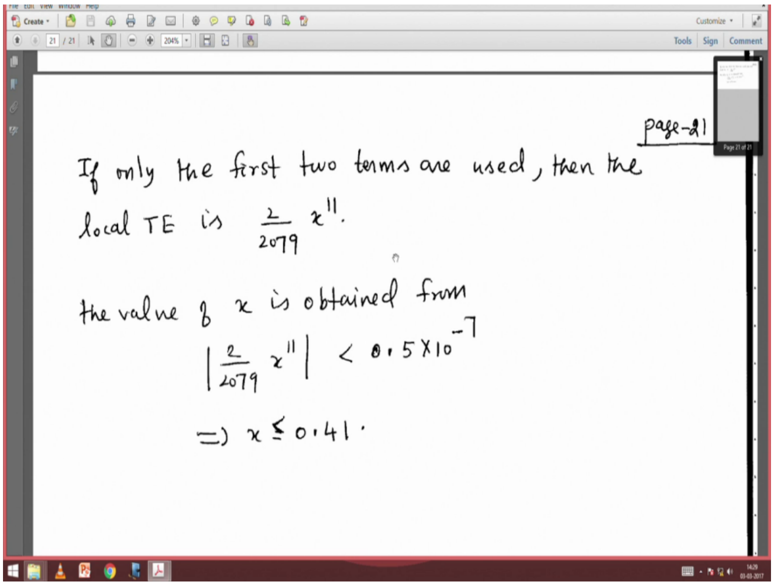

So we only are asked to take the first two terms, so the third non zero term in our expansion is the first neglected term which contributes to the truncation error. So what is y (x) it consists of these three terms. So we are asked to include the first two terms. So this term will be the third term which contributes to the truncation error. So the local truncation error will be 2 by 2079 into x power 11.

So I would like to find out the values of x for which if I include the first two terms alone in computing the solution at any point such that the local truncation error is going to be less than 0.5 into 10 to the minus 7.

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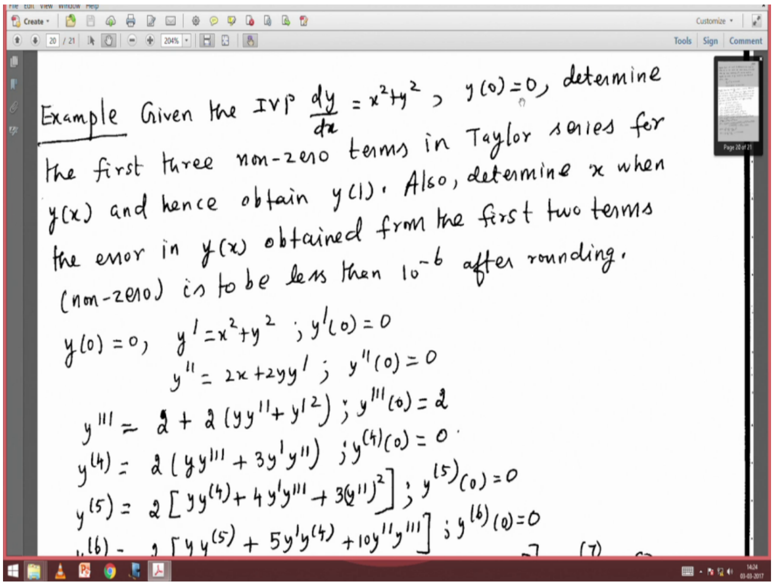

So I compute values of x for which this condition is satisfied and we see that x must be less than 0.041. So for all values of x take less than 0.041. You can use the Taylor expansion only upto these terms namely x cube by 3 plus x power 7 by 63 and write down the solution of this initial value problem which we have considered such that the error will be less than 10 to the minus 6 after routing.

So we have been able to get the value of x in this case to be less than 0.41. So for all these values of x we require accuracy will be obtained by just including the first two non zero terms in the Taylor expansion. So we have been able to illustrate Taylor series method by some examples about.

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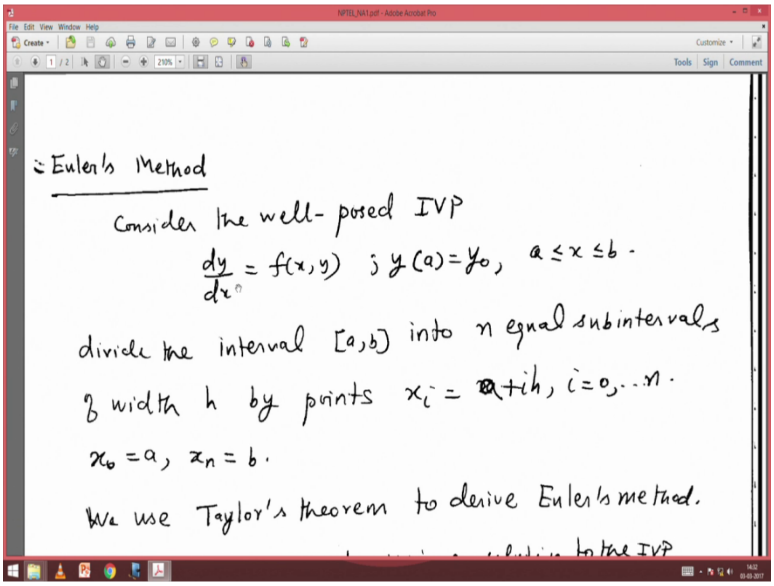

So we move on to the next method for solving inital value problems governed by first order differential equation. So we consider a well posed initial value problem governed by the first order equation dy by dx is equal to f(x,y) y(a) is y(0) and we want to determine the solution in the interval [a,b]. So what should we do in a numerical method we should divide the interval a to b into n equal sub intervals width say h by means of points x i which are given by a plus ih , i equal to 0 to n, such that x 0 is a and x n is going to be b.

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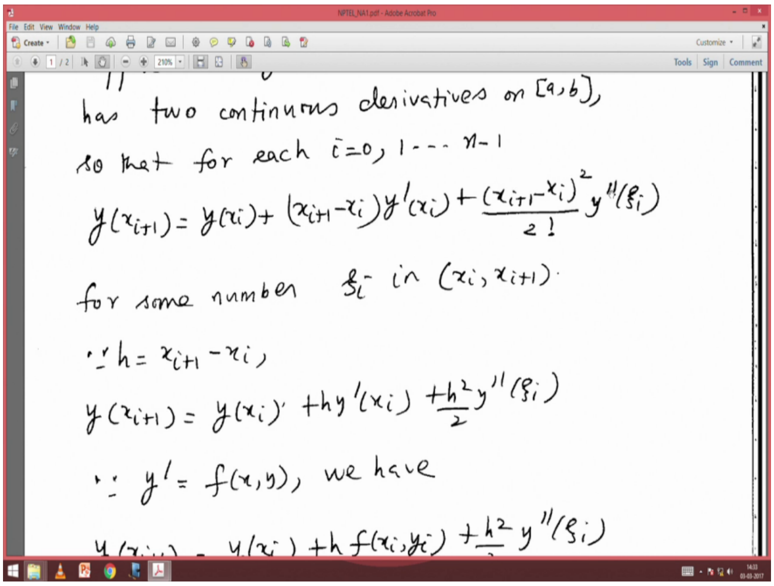

So let us use Taylor's theorem to derive Euler's method. Suppose say y(x) which is the unique solution to this initial value problem has two continuous derivatives on the closed interval [a,b] then by Taylor's theorem we know y(x i) plus 1) is y(x i plus x i plus 1 minus x i) into y prime(x i) plus (x i plus 1 minus x i) the whole square 2 factorial into y double prime (psi i) . For some numbers psi i which lies between (x i and x i plus 1).

So this x i plus 1 minus x i is the step size h. So I substitute that and then we see that y(x i) plus 1 is y(x i plus h) y dash(x i plus h square by 2) y double prime (psi i). So since y prime is f so I can substitute for y prime here as f. So we have this method.

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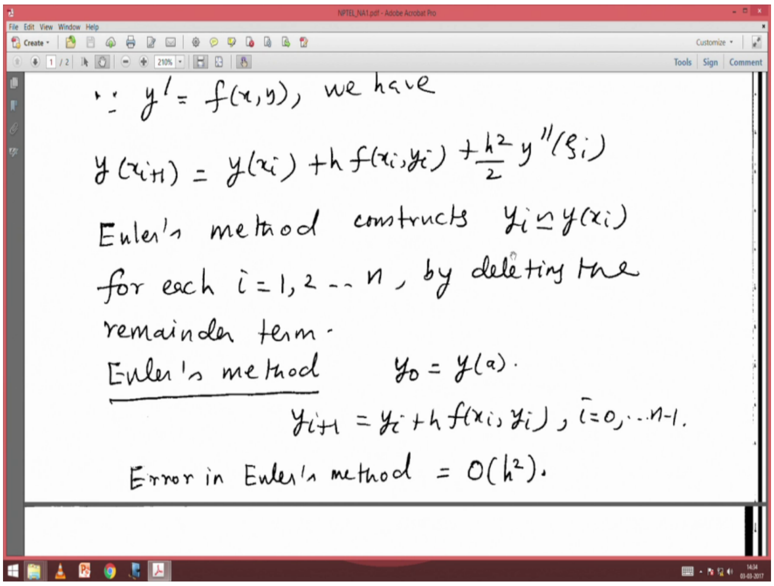

What does Euler's method do? Euler's method constructs an approximation for y(x i) call it y i at the point x i for each i which takes values 1 to n by deleting the remainder term, what is the remainder term this is the remainder term so I delete this term and write down the resulting method. So what is Euler's method , the initial condition given by y 0 equal to y (a) and after deleting the error term the method gives you y i plus 1 equal to y i plus h into f (x i, y i)for i is equal to 0 to n minus 1 when i is 0 you have the information that y(x 0) is y(a) which is y 0.

And so if you put i equal to 0 you will get y 1 to be y 0 plus h f(x 0 y 0). Once you get y 1 y 2 obtained by taking y is equal to 1 is y 1 plus h f(x 1 by 1) so continue this way till you reach the last point which is x n namely b so that it is y n is equal to y n minus 1 plus h into f(x n minus 1) by n minus 1. And you observe that Euler's method agrees with the Taylor series method of order 1 so error in Euler's method is of order of h square.

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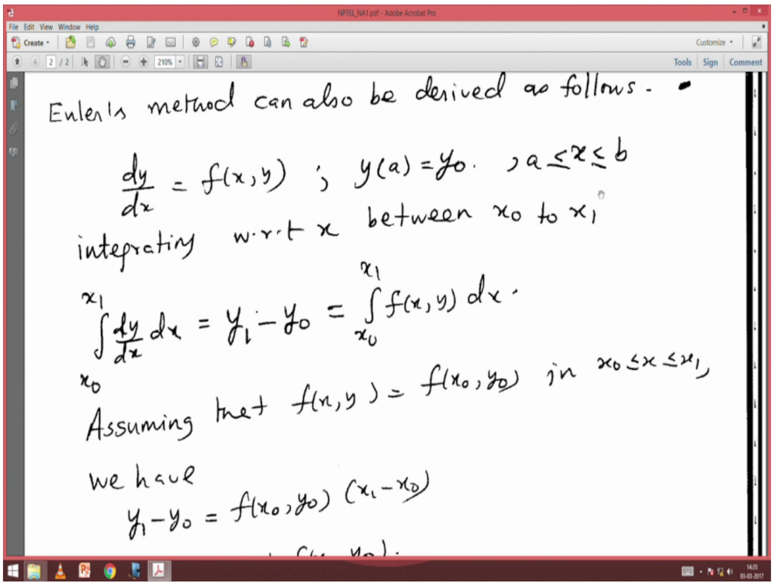

And Euler's method can also be derived by another way namely I take the differential equation dy by dx is equal to f(x,y) and the initial condition we want to obtain the solution in the interval [a to b]. So I integrate both sides with respect to x between x 0 and say x 1. How do I get this x 1 I divide the interval [a,b] into number of smaller sub interval by means of points x i, where x i is x 0 plus ih, i runs from 0 to n.

So I start at the x 0 where the initial condition is given I move to the next point which is at a distance of h units from here call it x 1. So i integrate the given differential equation between x 0 and x 1. So it is dy by dx into dx so that will give you y between x 0 and x 1, so it is y 1 minus y 0 and that is equal to integral of the right hand side which is x 0 to x 1 f(x,y) dx.

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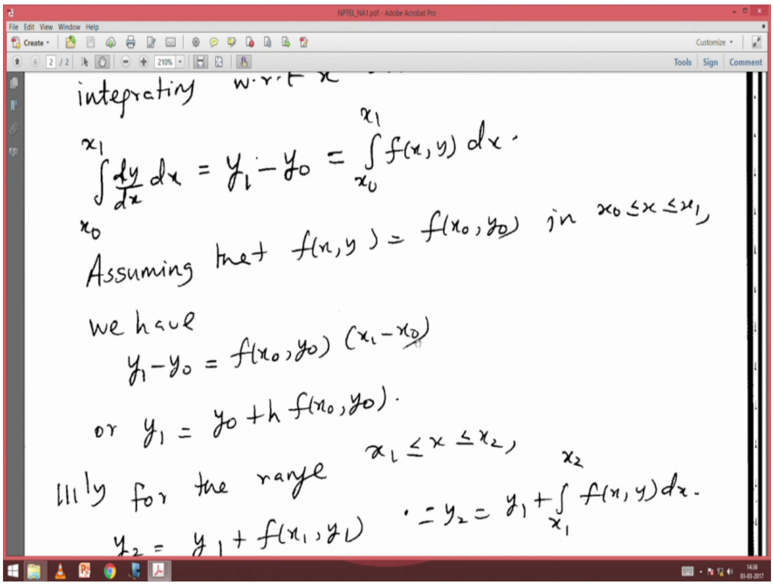

So we assume at this stage that we want to approximate f(x,y) by a constant polynomial namely the value of the function at x 0 , y 0. So take f(x,y) to be f(x 0, y 0) in the interval x 0 to x 1. So we end up with y1 minus y 0 equal to f (x 0 y 0) into (x 1 minus x 0) when we perform the integration. But x 1 minus x 0 is h, so we have y 1 equal to y 0 plus h into f (x 0 y 0) once we know y 1 I move on to y 2 what is it ? It is y 1 plus f(x 1 y 1) How do I get it? I integrate the differential equation between x 1 and x 2 and use the initial condition as y and x 1 is y 1, so I can get y 2 immediately as y 1 plus f(x 1 y 1).

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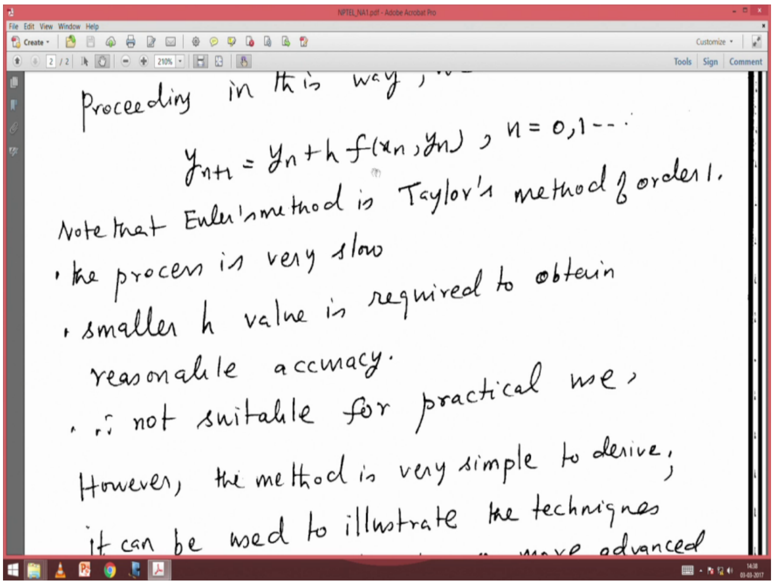

So move ahead similarly and then till you reach x n plus 1 at which the solution is y n plus 1 given by y n plus h f(x n, y n) for n running from 0,1,2,3 so that you end up with the solution at all the points starting from x 0 you got at x 1 then at x 2 x 3 etc x n and now you have gone to the point x n plus 1 at which the solution is given by this.

So you observe essentially that Euler's method is Taylor series method of order 1. And therefore what can you conclude about Euler's method the process will be a very slow process. You require smaller and smaller values of h to obtain desirable accuracy.

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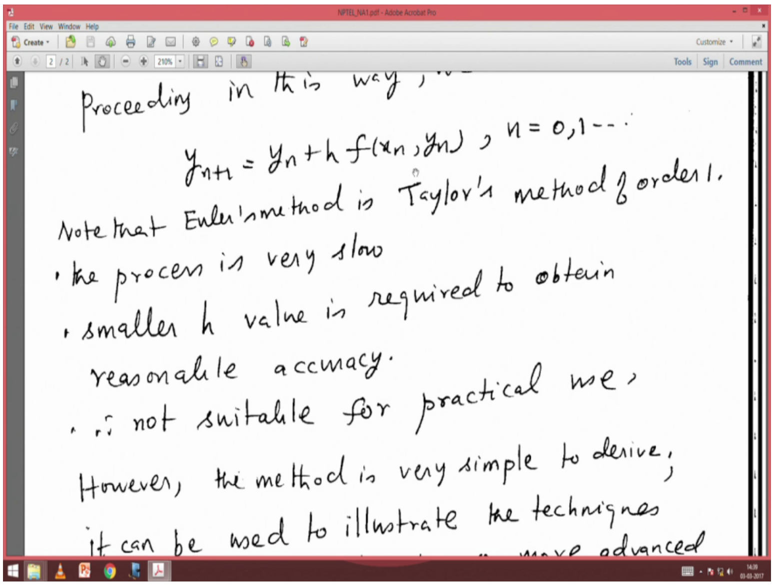

And the method is not suitable for any practical use , because the process is a very slow process and it requires the step size h to be very very small and lot of computational efforts needed to get the solution correct to the desired degree of accuracy. And therefore it is not suitable for practical use however the method is a very simple method to derive number 1 and secondly it can be illustrated to use the techniques that are involved in the construction of more detailed methods which we consider later on.

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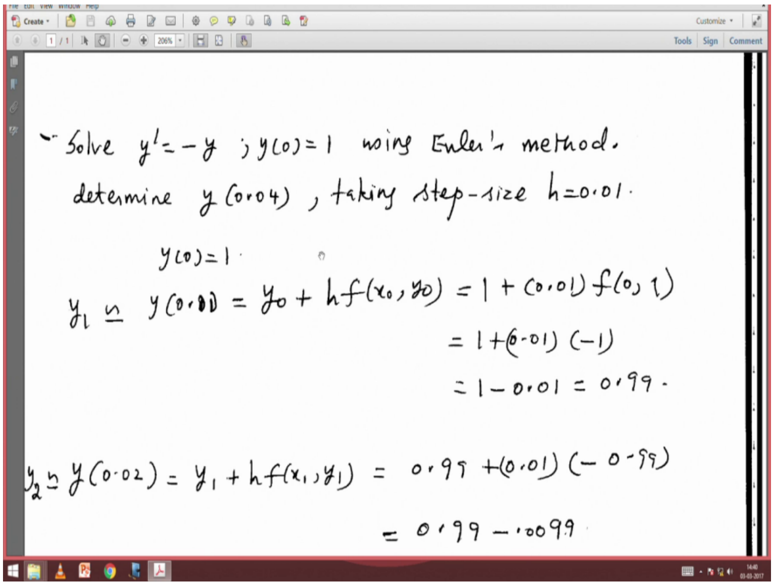

We shall now illustrate Euler's method by means of some example. So let us take y prime to be minus y, y(0) equal to 1 and the problem is to determine y (0.04) taking step size h to be 0.01. So y(0) is given to be 1. What y 1 it is y (0) plus h into f(x 0 y 0). So it is y 0 which is 1 h is 0.01 into f(x 0) is 0 and y 0 is 1. So I substitute f(x 0 y 0) as f(0,1) and what is its value f (x 0, y 0) is minus y 0 because f(x,y) is minus y. So it is minus 1, so I substitute here and then simplify and I end up with the value of y(0.01) as 0.99.

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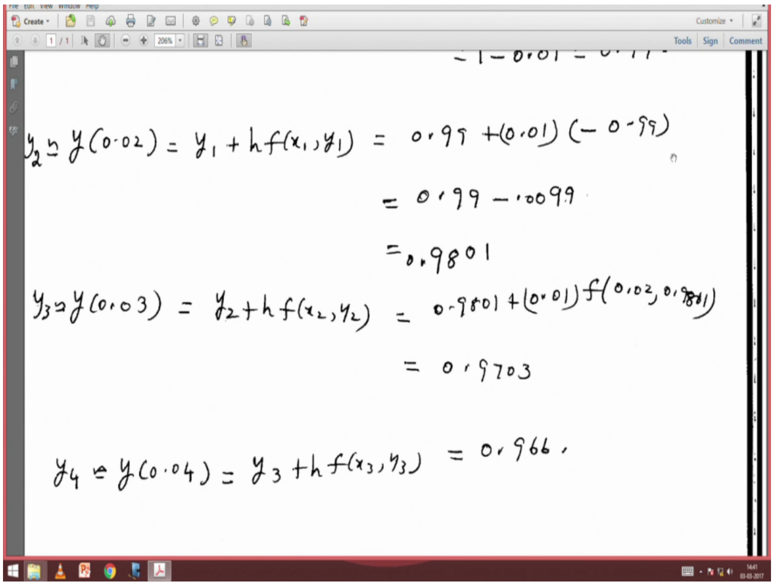

So now I go to the next step namely compute y(0.02), it is y 1 plus h f(x 1,y 1). So I feed in those values f(x 1 y 1) is minus y 1 so substitute and simplify you get the solution at 0.02. Proceed this way and you get the solution 0.03 and finally at 0.04 as this. So when it comes to Euler's method we observe that it is a Taylor series method of order 1 there was no need for computing any derivative the derivative which occurred there was the first order derivative, we replaced it by the function value namely f(x,y).

So this suggest can we develop single step methods which do not involve the computation of the various order derivatives as required in Taylor series method of order n, but it involves the information about the function f(x,y) at a number of intermediate points say x i to x i plus 1. When we march from x i to obtain solution at x i plus 1 namely let us take a number of points between x i and x i plus 1 and evaluate f at these intermediate points. f(x,y) represents the slope dy by dx.

So let us consider slopes just not at x i, y i but at a number of intermediate points between x i and x i plus 1 and take a linear combinations of these slopes and then see whether we can develop methods which are single step methods so that our computation does not require evaluation of the derivatives but it only involves evaluation of certain function namely f(x , y) at some points.

The answer is yes and such methods were developed by Runge and Kutta and they are referred to as Runge-Kutta explicit methods.So in the next class we shall discuss these single step explicit methods which are developed by Runge kutta are known as Runge kutta methods.