Numerical Analysis
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Lecture 19
Numerical Solution of ODE 2
Stability Single Step Methods 1
Taylor Series Method

Good Morning, in the previous class we considered some preliminary results in the theory of differential equations for the first order equations we also remarked at the end that we will have to look into the issue namely whether small changes in the statement of the problem leads to small changes in the resulting solution. This is an important issue because at every step of our numerical solution when we apply these numerical techniques, there will be round off errors which will slightly perturb the problem and therefore we are not actually solving the original problem but we are solving a perturbed problem.

Which means there are small changes which are incorporated in the statement of the problem and we are solving the perturbed problem and obtaining a solution. So we would like to see whether this small perturbation given in the statement of the problem has resulted in small changes in the corresponding solutions of the original problem and the perturbed problem.

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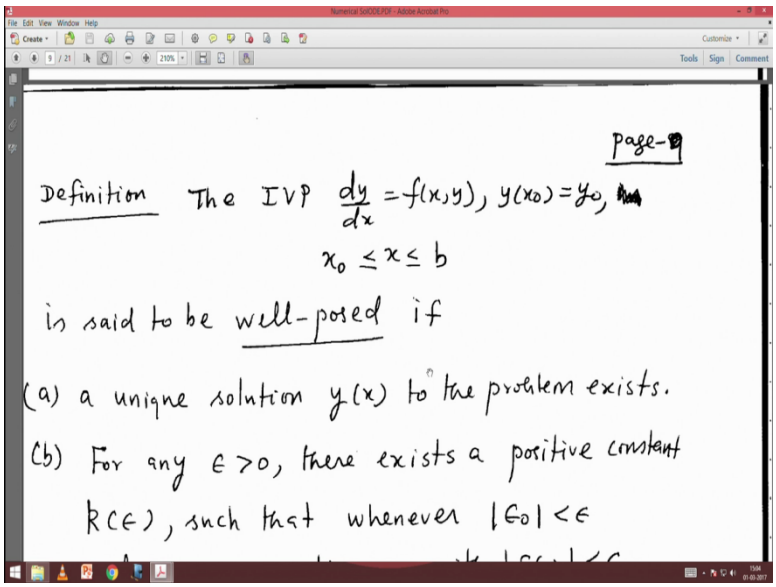

If that is so then it is fine. We have absolutely no problem and we can trust the numerical solution that we have got by using the numerical technique which we have applied to the original problem. If this does not happen then it is simply not possible to guarantee that the

solution that we get represents realistic solution of the original problem. So let us define some concepts and then look into results which will guarantee that small changes in the problem results in small changes in the corresponding solutions of the problems.

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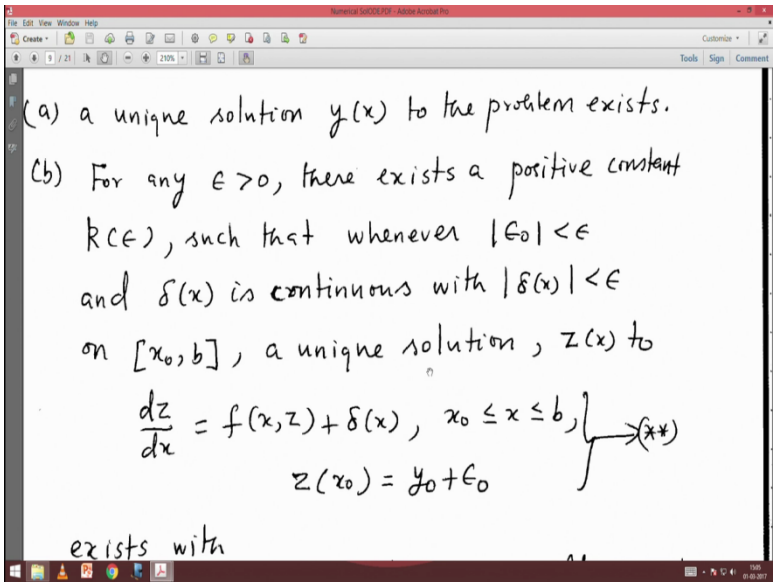

The initial value problem for a first order differential equation is said to be well post if the following conditions are satisfied namely the intial value problem must have a unique solution y(x) and for any Epsilon positive there exist a positive constant k (Epsilon) such that whenever mod Epsilon 0 is less than Epsilon and delta (x) is continuous with mod |delta (x)| less than Epsilon on the interval x 0 to be where x 0 is an initial point at which the condition is specified.

A unique solution z(x 2) a new problem dz by dx is equal to f(x,z) plus delta (x) for x lying between x 0 to b with z(x 0) equal to y 0 plus Epsilon 0 exists with the difference between the solution to the new problem namely z(x) and solution to the old problem namely y(x) can be made less than k(epsilon) for all x 0 less than or equal to x less than or equal to b. I hope you have understood the definition.

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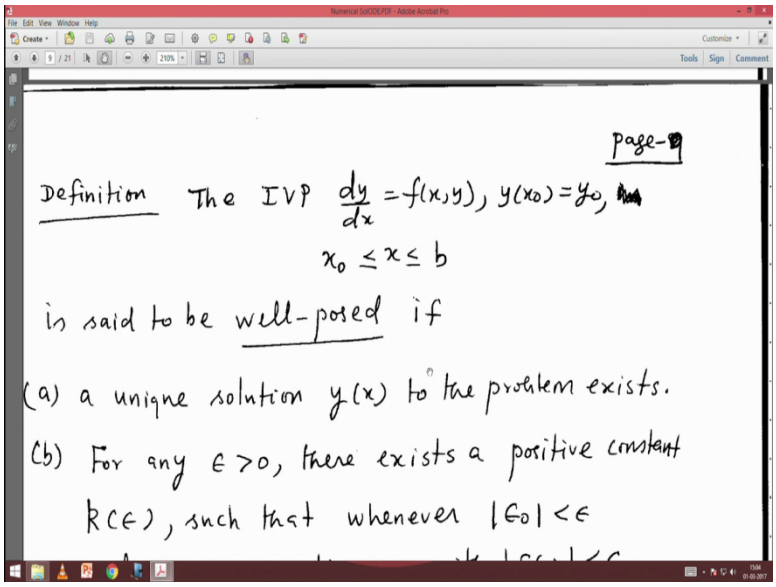

I have considered the original problem what is the original problem dy by dx is equal to f(x, y) subject to the condition y(x 0) equal to y 0 where x lies between x 0 and b.

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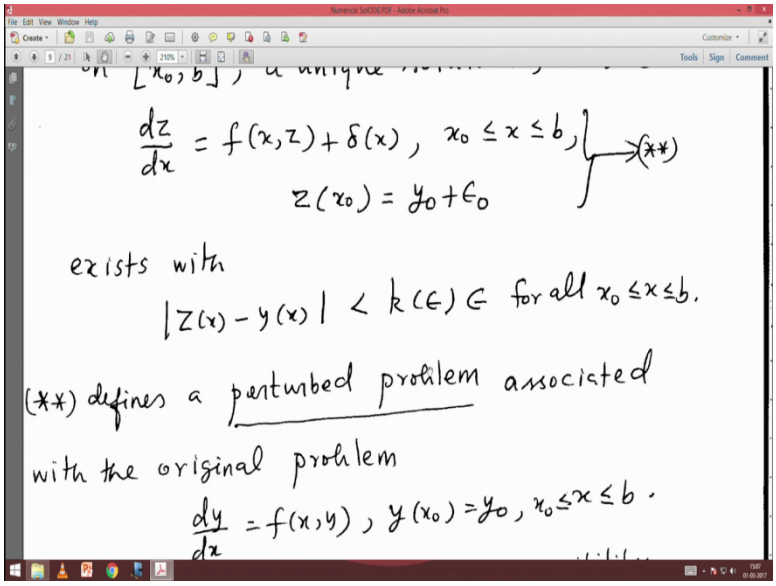

I have also considered another problem what is that problem? dz by dx is equal to it is the same f , f(x,z) it is a new problem because I have an additional term here this delta (x) is the small perturbation which I give to f. So that I have a resulting new problem for the unknown z in the same interval where x lies between x 0 and b.

In addition not only the differential equation is perturbed there is also a change in the initial condition, z(x 0) satisfies the condition that it is y 0 which is y(x 0) plus a small increment which is Epsilon 0. So now have two problems the original problem and another problem which is a perturbed problem of the original problem where there is a perturbation delta (x to f) and epsilon 0 to y 0 in the initial condition.

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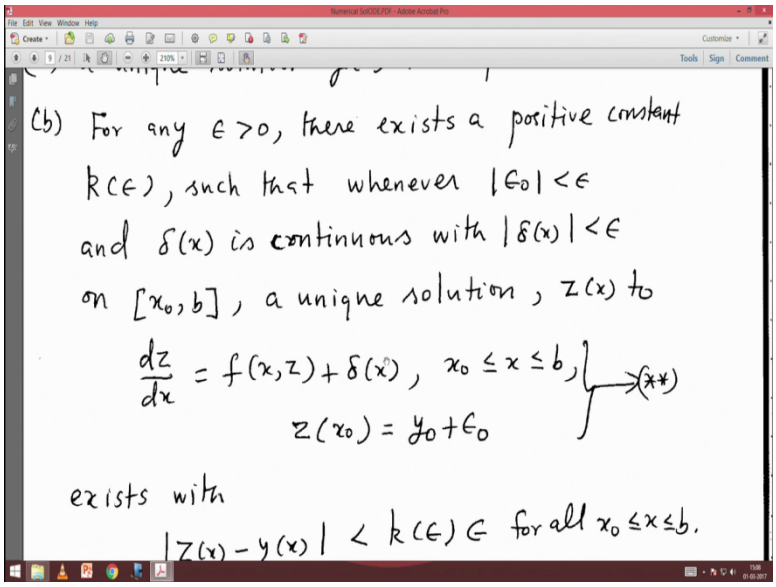

So what does the definition say? The initial value problem the original problem is well post when it is well post if the original problem has a unique solution number 1 that should be satisfied in addition the second perturbed problem that we have stated. If such that, that has a unique solution given by z(x) such that the difference in absolute value between z(x) and y(x) can be made less than k(Epsilon) into Epsilon for all x 0 less than or equal to x less than or equal to b where this k(epsilon) delta(x) epsilon 0 are all given here.

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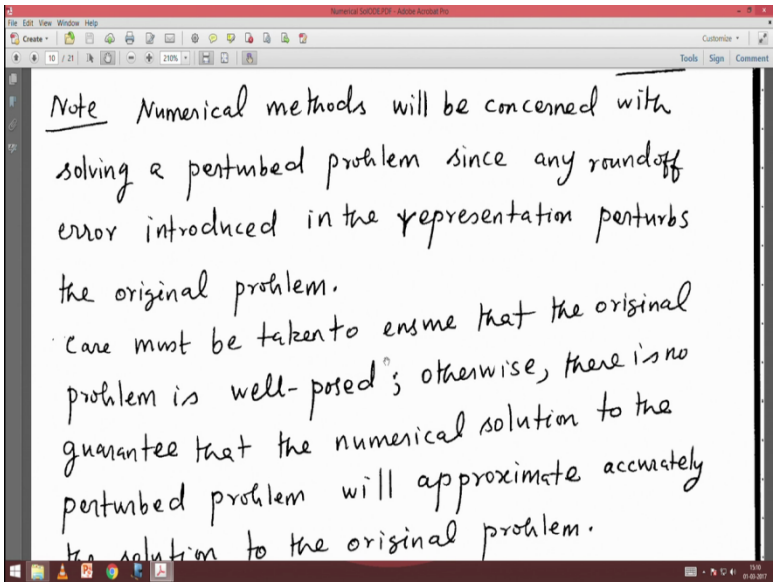

So the second problem that we have defined is essentially called a perturbed problem which is associated with the original problem . So the second perturbed problem assumed the possibility of an error delta (x) which can be incorporated in the differential equation and epsilon 0 in the initial condition shows that there is a possibility of incorporating a small error in the initial condition. So the resulting problem is a perturbed problem of the original initial value problem.

So why should we worried about the perturbed problem when we are discussing the numerical methods for solution of obtaining the original intial value problem. The reason is the following: namely any round off error that is introduced in the representation perturbs the original problem and therefore as I remarked earlier at any step where we apply the numerical method that we are going to develop we are not solving the original problem but we are solving a perturbed problem which results due to round off errors.

So we must take care to ensure that the original problem is well post namely the definition that we have given earlier is for the original problem to be well post. So we check that the original problem is well post , otherwise we cannot guarantee that the numerical solution that we get by solving the perturbed problem will give us a solution which represents realistically solution of the original problem.

So the conditions that we have given shows that the initial value problem is well post and specified in the following theorem. So let us see what the result is.

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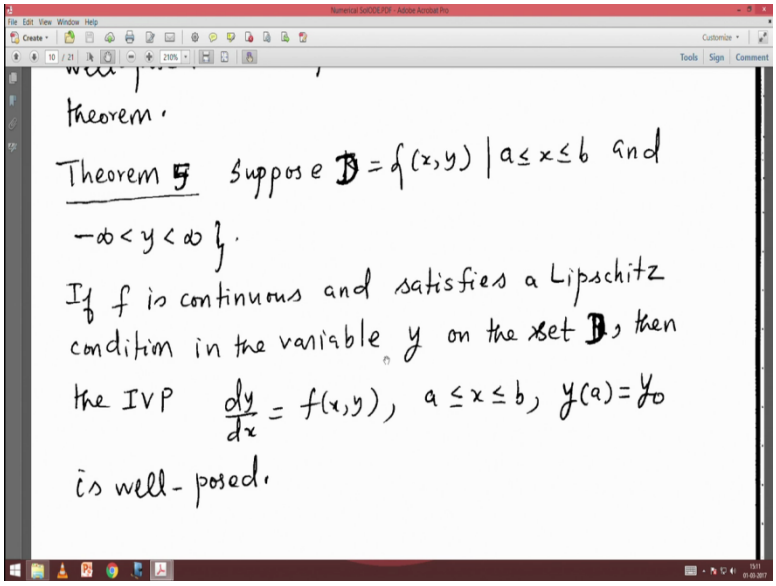

Suppose we have a set D which is a set of all xy such that a less than or equal to x less than or equal to b and y lies between minus infinity and infinity. If f is continuous and it satisfies the Lipschitz condition in the variable y on the set D then the initial value problem dy by dx is equal to f(x,y) y(a) equal to y 0 where x lies between a and b is well post.

So we do not have to check by taking a small perturbation to the problem as well as the initial condition and see whether the new perturbed problem has a unique solution and check the condition that marks z(x) minus y(x) is less than k(epsilon into epsilon) and so on.

We do not have to do all that we only have to ensure whether the conditions given in the theorem are satisfied? And the theorem guarantees that if the conditions are satisfied then the initial value problem is well post so that the numerical methods that we develop for such a well post initial value problem when they are used to solve such well post initial value problem will yield as solution which are reliable and will be obtained correct to the desired degree of accuracy.

So what are the conditions, f must be contiinuous and f must satisfy a Lipschitz condition in the second variable y where on the set D what is D? D is the set of all xy such that x lies between a and b and y lies between minus infinity and infinity. This initial value problem is then going to be a well post initial value problem so that small changes in the original problem will result in small changes in the solutions that we obtain so that the solution is a stable solution.

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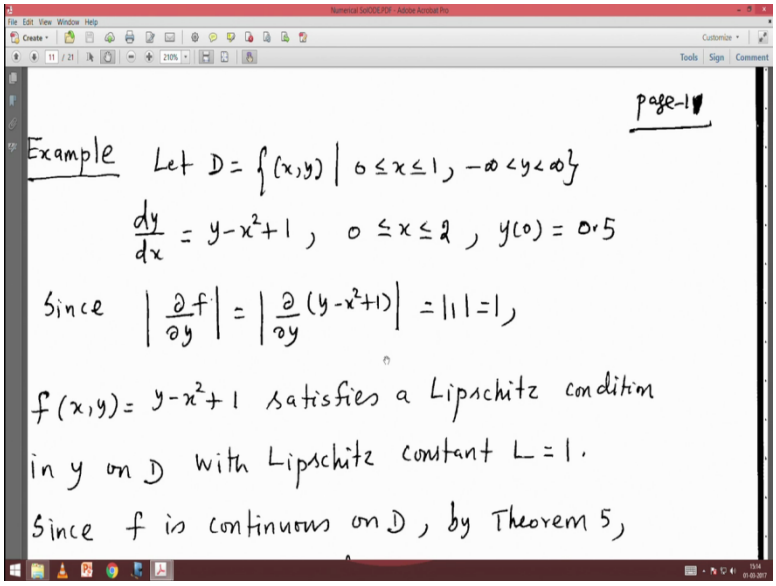

So we now have results which enable us to apply the numerical methods that we are going to develop to initial value problems, so that when we check the conditions that we have given in those theorems have satisfied by the functions f(x, y) in the initial value problems that we consider on appropriate sets then we are guaranteed that we are solving a well post problem and the resulting problem will have a unique solution in an appropriate interval so that our numerical solution that we obtain are reliable solutions obtained correct to the desired degree of accuracy.

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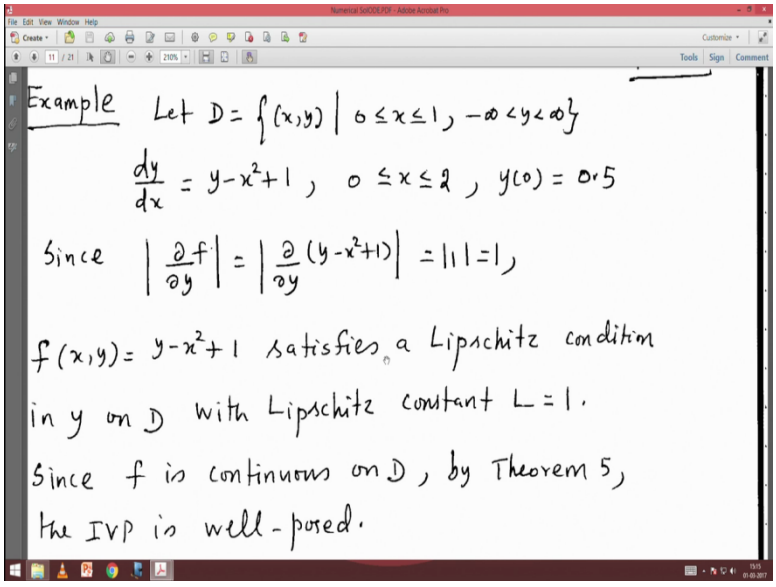

So let us take an example and complete this discussion. So let D be the set of all (x, y) such that x lies between 0 and 1 and y lies between minus infinity and infinity. Let us consider the initial value problem given by dy by dx is equal to y minus x square plus 1, for x lying between o and 2, y(0) is equal to 0.5.

So let us find what is modulus of |f y| so it is modulus of | d by dy of f | which is (y minus x square plus 1) and that will give you 1. So f(x,y) therefore satisfies a Lipschitz condition, why? because we have been able to find out capital L which is 1 which is true valid for xy in that set D. And therefore we have f(x,y) to satisfy a Lipschitz condition in the second variable y on the set D with Lipschitz constant L is equal to 1.

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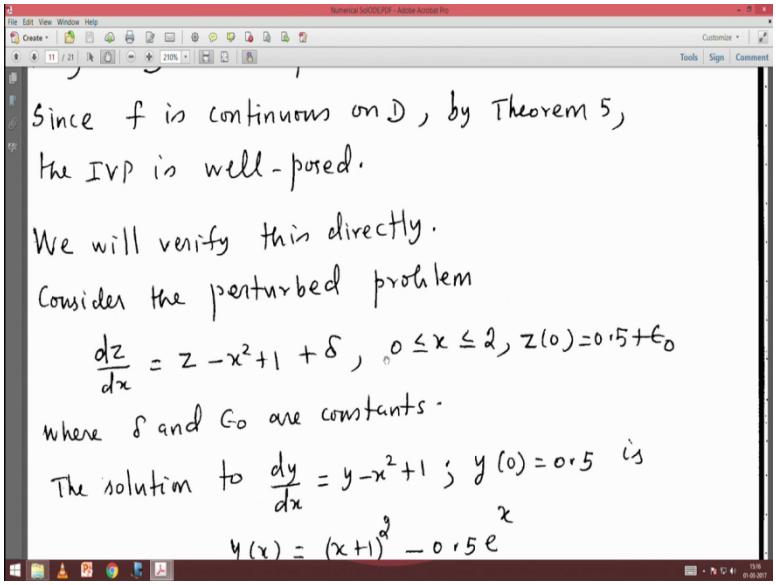

In addition the function f is a continuous function on D. And therefore by the theorem that we have considered the initial value problem is well post, right? So we now verify whatever we have said directly. So what do we have to do? We should consider a perturbed problem. This is not necessary because we have already said that the original problem is a well post problem, however we would like to start with a perturbed problem and then show that the original problem is a well post problem.

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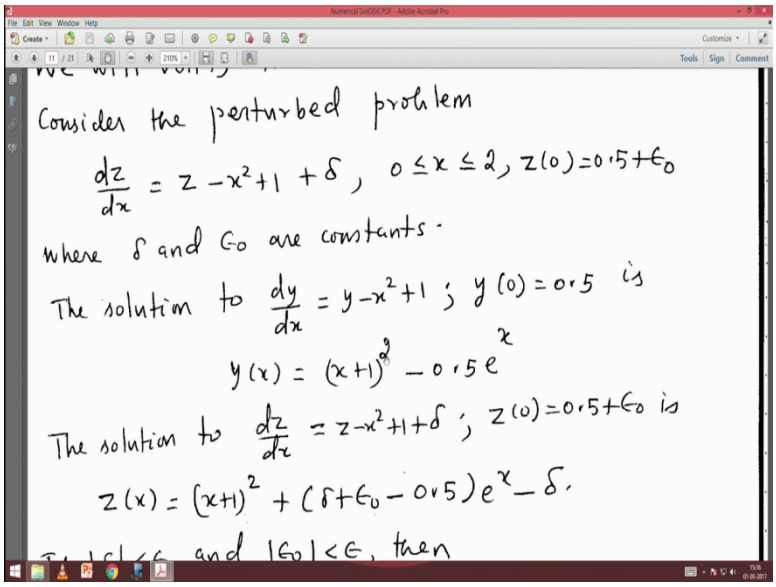

So let us consider a perturbed problem which is dz by dx is equal to the same f as the function x and z. I give a small perturbation say delta to the right hand side. And z (0) initial condition is y(0) which is 0.5 I give a small perturbation say Epsilon 0 to be initial condition. So delta and Epsilon 0 are some constant.

So when is it well post condition 1 the original problem has a unique solution. So let us solve dy by dx is equal to y minus x square plus 1; y (0) equal to 0.5. It is clear that this equation is again a linear Lagrange equation subject to this initial condition and therefore it has a solution which is given by this. One can also directly verify by differentiating it and showing that you obtain this differential equation.

So the first condition is satisfied namely the original problem has a unique solution in this interval. Now let us consider a perturbed problem we should show that the perturbed problem also has a unique solution given by z(x) so the perturbed problem has right hand side z minus x square plus 1 plus delta with z(0) given by 0.5 plus epsilon 0.

So again it is a linear Lagrange problem its initial condition when you solve you get z(x) to be equal to this. So if mod |delta |is less than epsilon so delta is an increment given to the right hand side, it is a perturbation given to a statement of the problem.

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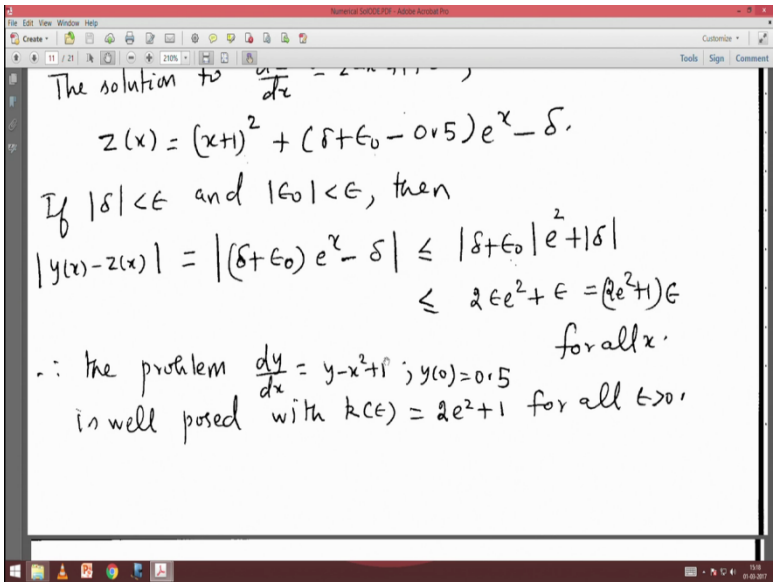

So if you take mod |delta| to be less than Epsilon as well as mod |Epsilon 0| to be less than Epsilon and find out the difference in the solutions of the original problem and the perturbed problem. Well it gives you modulus of |(delta plus epsilon 0) into e power x minus delta| which is nothing but modulus of | (delta plus epsilon 0)| into e square because our x is less than or equal to 2 plus mod |delta|, So this is less than or equal to twice epsilon into e square because mod | delta | is taken to be less than epsilon mod | epsilon 0| is less than epsilon.

So 2 epsilon e square plus epsilon which is (2 e square plus 1) into epsilon. Recall we should be able to find out a k as epsilon right? Let us look at the second condition that we have given for a problem to be well post.

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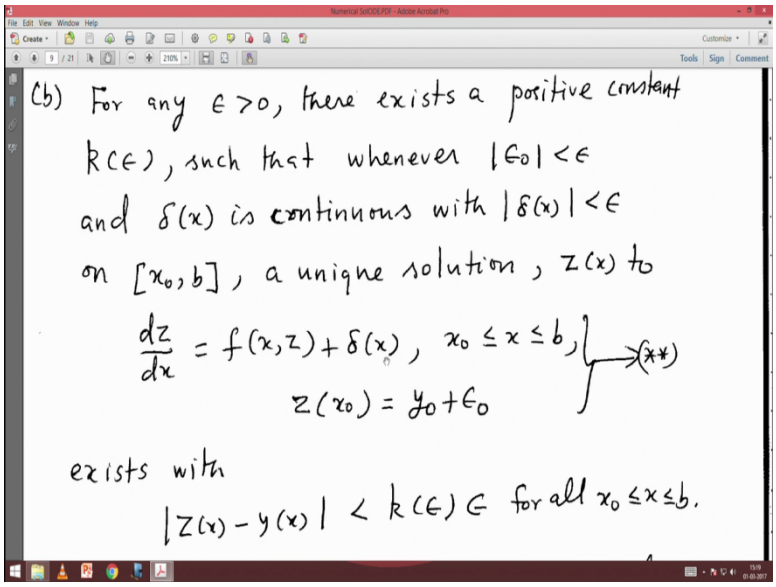

Second condition says that for any Epsilon greater than 0 there exists a positive constant k (epsilon) such that whenever mod |Epsilon 0| less than Epsilon and mod | delta | less than Epsilon on this interval x 0 to b a unique solution z(x) to the perturbed problem exists yes we have shown that. And modulus of | z minus y| must be less than k(epsilon) into epsilon for all x 0 less than or equal to x less than or equal to b.

So you should be equal to find k (epsilon) yes in the example that we have considered we have been able to show that mod |y(x) minus z(x)| is less than or equal to 2 e square plus 1 multiplied by Epsilon, so this 2 e square plus 1 is the k(epsilon) that we are looking for.

And this is so for all x and therefore the problem that we have considered is such that the original initial value problem has a unique solution in that interval and we have been able to find out a k(epsilon) such that the absolute difference between y and z can be made less than k (epsilon) into epsilon for all x in that interval.

So both the conditions are satisfied and therefore by a definition of a well post problem, the problem that we have considered is a well post problem.

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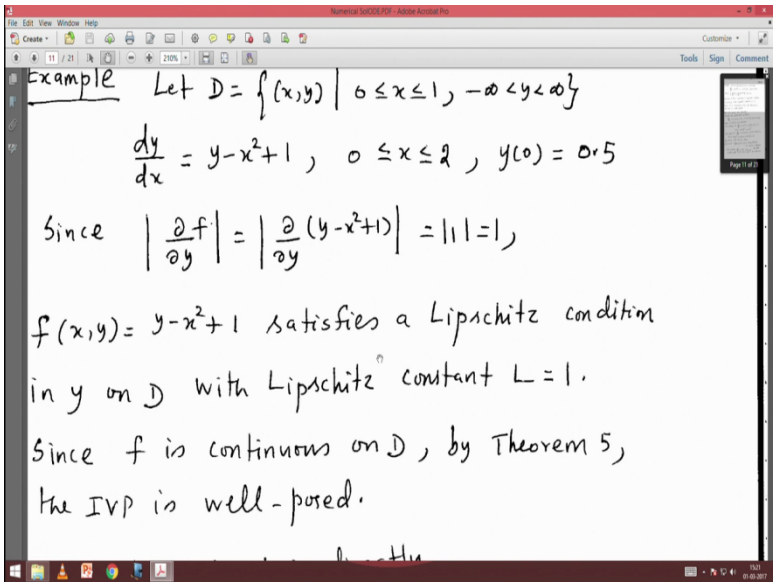

So we actually have demonstrated how confined that k (epsilon) and show that the conditions given in the definition are verified so that the original problem is a well post problem, we also have shown earlier that the conditions specified in the theorem are satisfied and therefore it is a well post problem in both ways we have been able to estabilish the well post property of the corresponding initial value problem that we have considered.

So far we have looked into the important results which are required so that we can move further to develop numerical methods that will guarantee that they are reliable. So we now have a knowledge of what are all the factors we should keep in mind, so that we have a reliable solution obtained using the numerical techniques that we have developed. So we are going to now develop a number of numerical techniques with the help of which we can solve an initial value problem governed by the first order differential equation.

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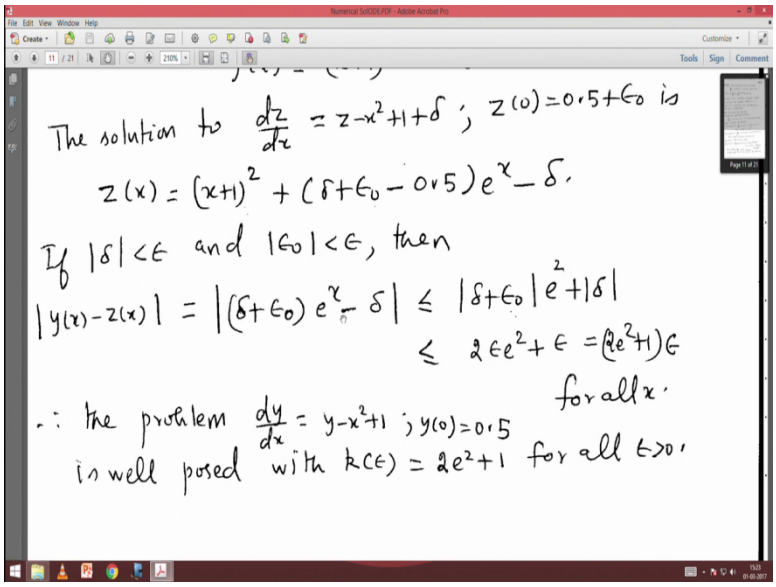

To begin with we are going to consider methods which are called single step methods which are also referred to as explicit methods. you will understand the meaning of what a single step method is? Why is it an explicit method? When we develop these methods? A very powerful method which is a single step method which is used in obtaining solution of a first order differential equation is Taylor series method.

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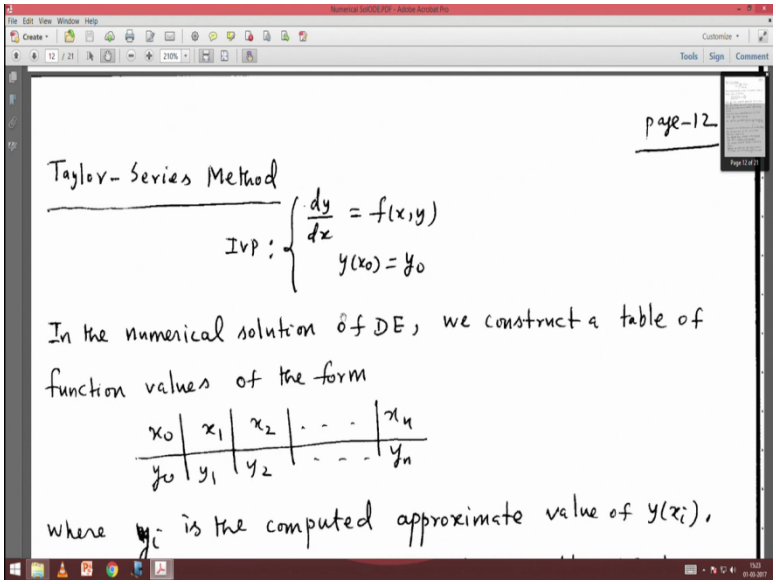

So let us see what this method is? So in a Taylor series method our goal is to solve an initial value problem dy by dx is equal to f(x,y) subject to the condition that y(x 0) is y 0. As we already remarked we are trying to get a numerical solution of this differential equation.

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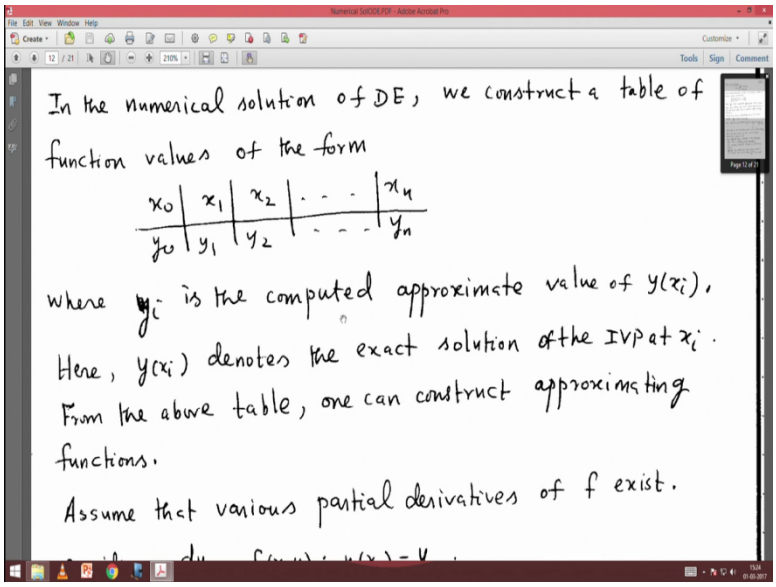

So we are going to construct a table of function values of the form x 0 y 0 x 1 y 1 etc x n y n where y i represents the computed approximate value of y(x i) where y(x i) is the exact solution of the initial value problem at any x i. So as we said already once we know the table of values we can reconstruct this function y is equal to y(x).

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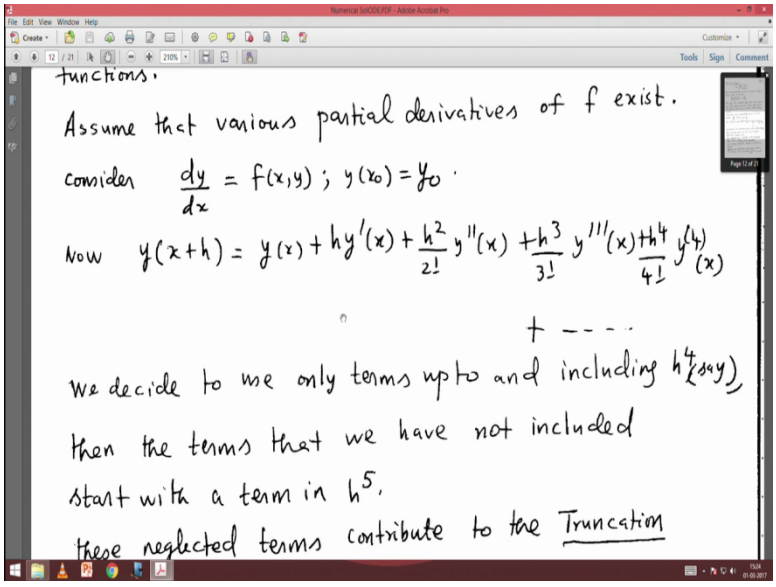

So Taylor series method requires the following assumptions namely we assume that various order partial derivatives of f(x,y) exists. So let us consider the initial value problem. So if I start at any x and if I want to find out a solution at say x plus h then I use a Taylor series expansion of y so that y(x plus h) is y(x) plus h y dash(x) plus h square by factorial 2 y double dash(x) plus h cube by factorial 3 y triple dash (x) plus h power 4 by factorial 4 fourth derivative of x first derivative of y (x) plus etc.

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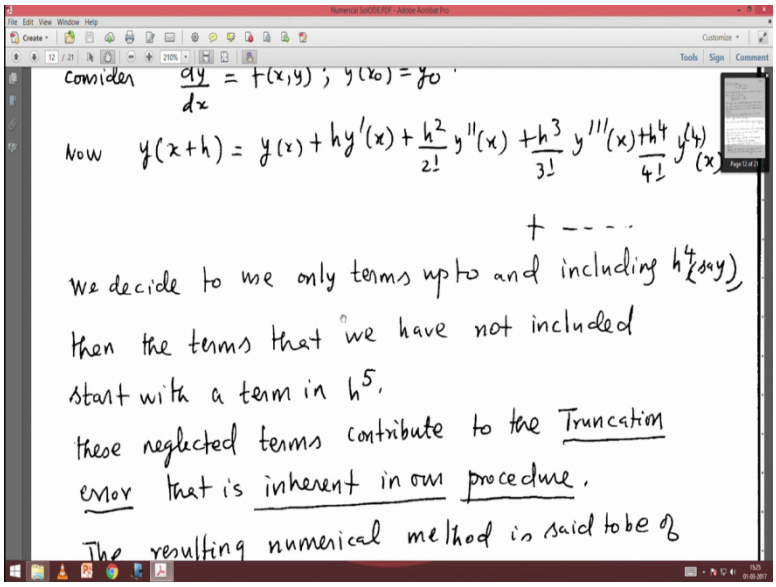

Suppose say in this expansion we decide to take only terms including h to the power of 4 by 4 factorial into fourth derivative of y then the terms that we have neglected start from h power 5 onwards. And these are the terms that contribute to what is called the truncation error, that is inderent in our problem.

What is a truncation error? The truncation error is an error due to truncating an infinite process to a finite process. So in this case we had a Taylor expansion of y(x plus h) which contains infinite terms. But we truncate it say at the term which contains h to the power of 4 by factorial 4 into first derivative.

So an infinite process is truncated and only finite number of terms is taken into consideration. When this is done we come across what is known as truncation error. And because of what we are doing this truncation error is inherent in our procedure itself. And it is independent of the round off error that may occur.

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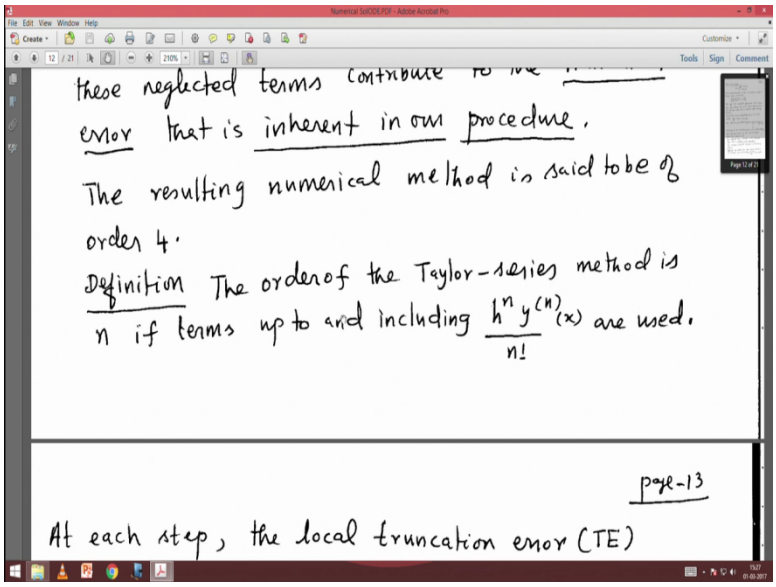

So the resulting numerical method that we have after truncating say at terms h to the power of 4 by factorial 4into the fourth derivative of y it is called the Taylor series method of order 4. So it includes in the Taylor series terms upto h to the power of 4 and terms of order of h to the power of 5 and higher orders are neglected. Then we have Taylor series method of order 4.

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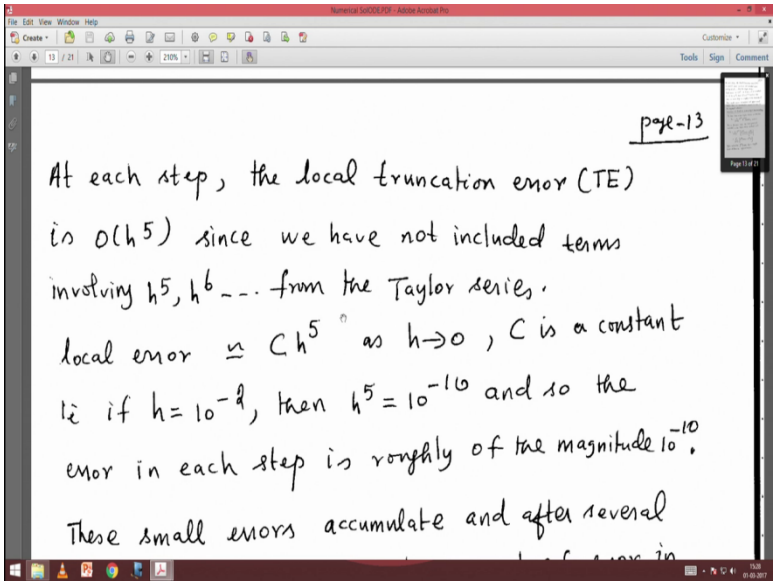

So the order of the Taylor series method is said to be n if terms upto and including h power n by n factorial into n th derivative of y are used in the computations. So we observe that at each step we have local truncation error. What is it? It is of order of h to the power of 5

because we are omitting terms of order of h power 5, h power 6 etc. The first neglected term is of order of h to the power of 5.

So the local error behaves like a constant time h to the power of 5 as h goes to 0 and so if h is taken to be say 0.01 that is h is taken to be 10 to the minus 2 then h power 5 is 10 to the minus 10. So you may think that the error that is incurred namely that local truncation error is of the order of 10 to the power of minus 10 at any step which is very very small.

Which can be neglected because it is of order of magnitude 10 to the power of minus 10 at any step, but these small errors get accumulated as we move on from one step to another step an there is a considerable amount of error that is incorporated during the computation of the solution.

So the question is can you estimate this local truncation error at 8 th step of your numerical computation? If it is possible so let us see what is the estimate.

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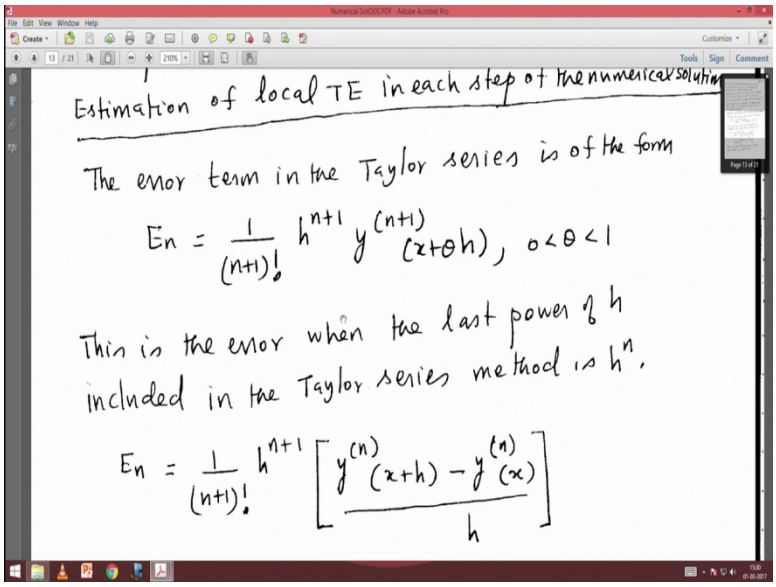

So we know that the error term in the Taylor series of order n is of the form say E n which is equal to the first neglected term , what is it? We have included terms upto h to the power of n in our computation. The first neglected term is 1 by (n plus 1) factorial h power n plus 1 into n plus 1 th derivative of y evaluated at (x plus theta h), where theta lies between 0 and 1.

So this is the local truncation error when we use Taylor series method of order n in computing the solution where at x plus h when we have information about y(x). So can you give an estimate? Answer is yes.

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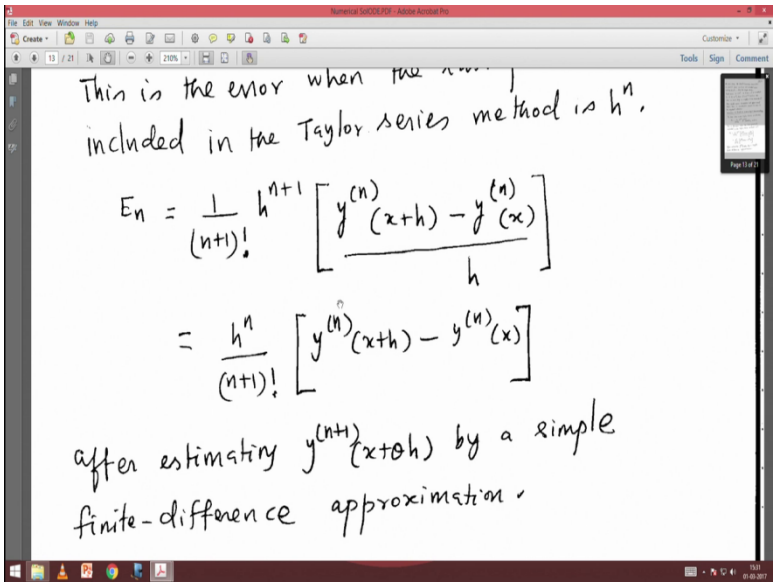

What do we do? We write down approximate this n plus 1 derivative of y at (x plus theta h) in terms of the finite difference namely E n will be 1 by (n plus 1) factorial h power n plus 1 into the n plus 1 th derivative is replaced by a forward difference approximation namely it is the n th derivative at x plus h minus n th derivative at x divided by h.

So that will be equal to h power (n by n plus 1) factorial multiplied by the difference in the n th derivative of the function at x plus h and x. So this is done after estimating the n plus 1 th derivative of y (x plus theta h). So we have an estimate of this error when we use Taylor series method of order n. So we observe that E n h power n by (n plus 1) factorial multiplied by some quantity this is the estimate of the local truncation error.

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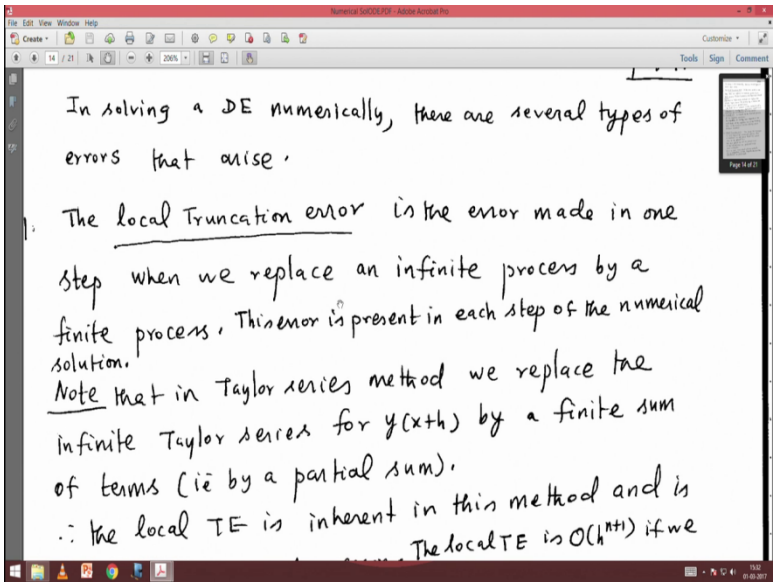

So we now have to see is this the only error that we come across during our computations. Now we have different types of errors which we come across as we move on with the numerical solution of the initial value problem. The first type of error is what we have seen, namely the local truncation error what is it? It is the error that is made in one step when we replace an infinite process by means of a finite process.

So as we have remarked this truncation error is present at each step of our computation and it is inherent in the method itself and note that in Taylor series method this is what we do, we replace an infinite Taylor series for y(x plus h) by a finite sum of terms namely by a partial sum.

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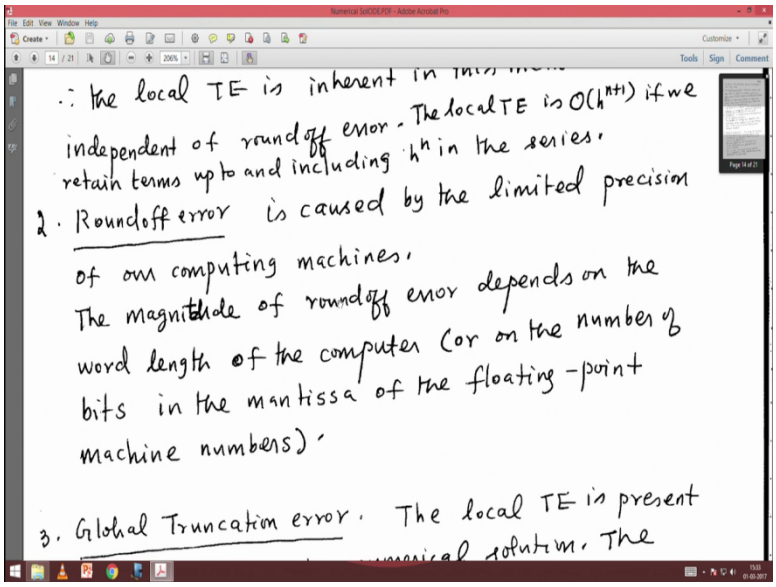

And it is independent of the round off error and the local truncation error is said to be of order h power n plus 1 when we retain terms upto and including h power n in the series. This is the first type of error that we come across when we apply this Taylor series method to solve our initial value problem.

Another type of error is what is called the Round off error. Why does it occur? It occurs because of the limited precision of the computer that we have. So the magnitude of the round off error depends upon the word length of the computer that we use. Or it depends on the number of bits in the mantissa of the floating point machine numbers.

So it occurs due to the machine that we use to compute the numerical solution.

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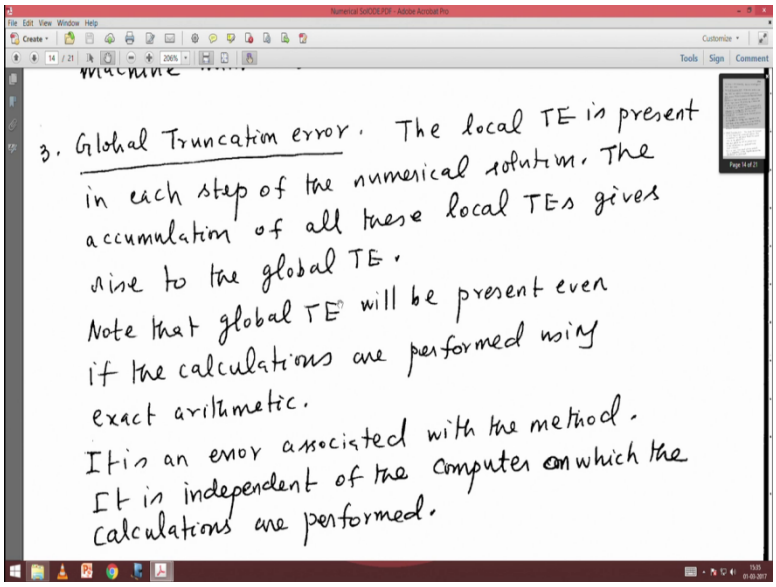

And the third type of error that occurs is Global truncation error. Why does it arise? So we said that the local truncation error occurs at each step of our computation, so the accumulation of all these local truncation errors is the global truncation error. So even if you perform your computations using exact arithmetic, right? This global truncation error will be present in your numerical solution because it occurs due to the accumulation of the local truncation error.

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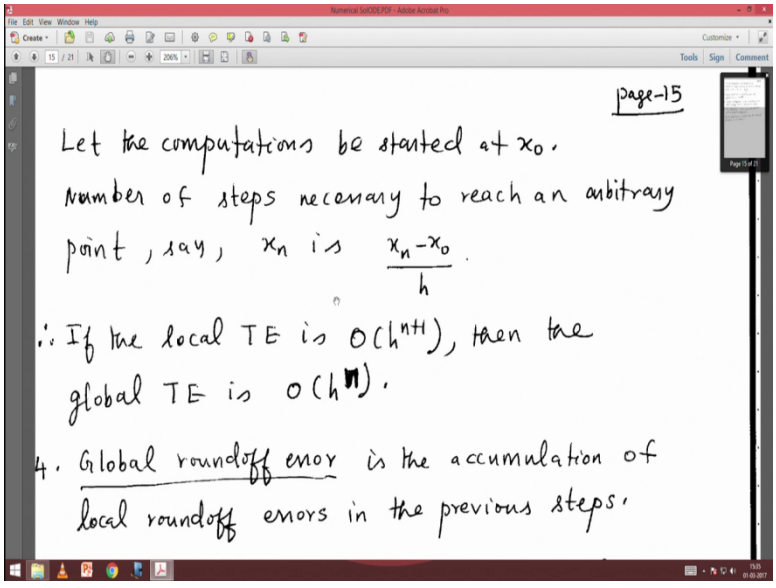

And again this global truncation error is independent of the computer that you use to obtain a numerical solution. So where do you start your computation, you start your computation at some initial point x 0 while you want your solution curve to pass through the point x 0 , y 0.

So you start your computation at this point because your curve passes through this point y(x 0) equal to y 0 and now you move to the next point which is x 0 plus h. Alright so you have moved to one step and at this step you have a local truncation error. Suppose you are interested in getting the solution to this initial value problem in an interval of the form x 0 to some b, so you will divide this interval x 0 to b into a number of sub intervals.

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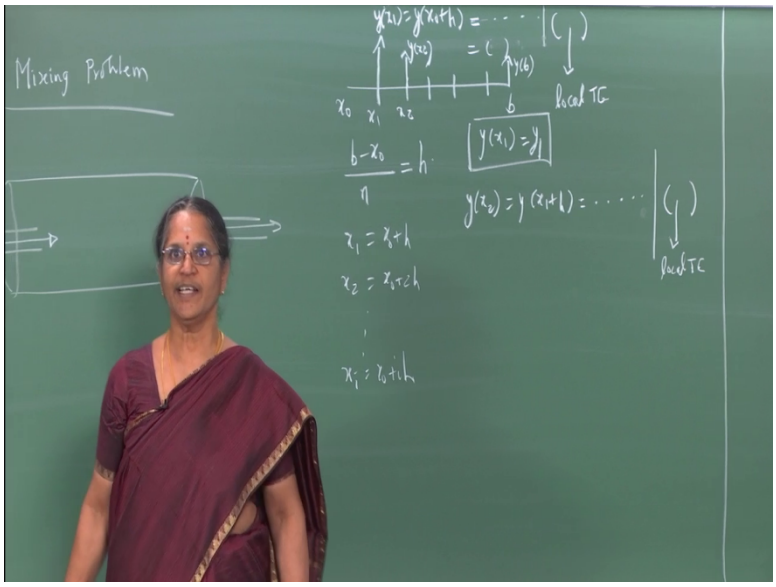

So let us see how we do it? So we will divide the interval x 0 to be into a number of smaller sub intervals of the form b minus x 0 by n equal to h so that x 1 will be equal to x 0 plus h; x 2 is x 0 plus 2h; and so on x i is x 0 plus i h. So you will use your Taylor's method and compute what is y (x 1) that is y(x 0 plus h) which will be involving some finite number of terms and you omit this term which contribute to the local truncation error.

So you arrive at its value. So now you have y(x 1) to be say from y 1 using this method. Now you use this information and move to this point x 2 which is the next point. So you compute y(x 2) here again what will you do you will say y(x 2) is y(x 1 plus h) you apply Taylor series truncated upto a finite number of terms say upto terms involving h power n you have omitted this term that will contribute to the local truncation error at this step.

So every time you move ahead by one step till you reach the point at which you will determine your solution using Taylor series method. So number of steps necessary to reach that point b is going to be if I call it x n b is what I have call as x n.

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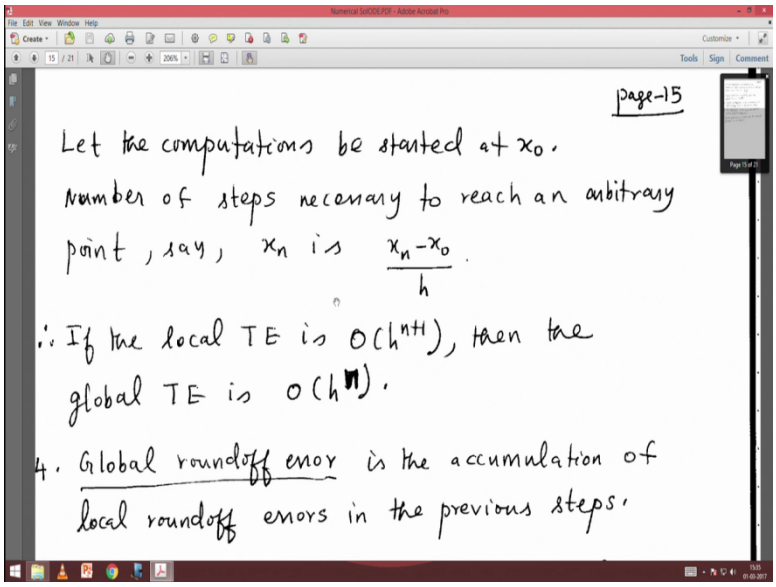

Please look at the monitor then the number of steps necessary to reach this point x n which is b is x n minus x 0 by h. You have moved so many steps the number of steps is given by x n minus x o by h. So if the local truncation error at each step is of order of h power n plus 1 why you have used the Taylor series method of order n by retaining terms upto h to the power of n.

So your local truncation error is of order of h power n plus 1. And number of steps to reach this point x n is x n minus x 0 by h and at each step you have accumualted the local truncation error and therefore what is the global truncation error it is going to be proportional to 1 by h into order of h power n plus 1 namely it is going to be of order of h power n.

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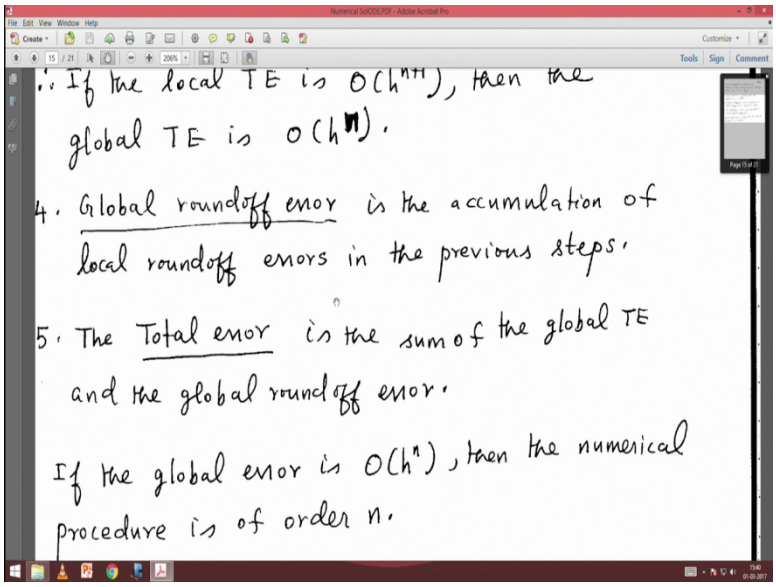

So if the local truncation error in Taylor series method is of order of h to the power of n plus 1 then the global truncation error is of order of h to the power of n. So if the global truncation error of a method is order of h power n then the numerical method is said to be of order n. Then what about the global round off error is what is it? It is the accumulation of the local round off error in the previous steps.

And then what is the Total error is the sum of the Global round off error and the global truncation error. So as I already have said if the gobal error is of order of h power n then we say that the numerical procedure is of order n. So we now know what we mean by Taylor series method of order n right? which we want to apply to an initial value problem and see how the solution can be obtained.

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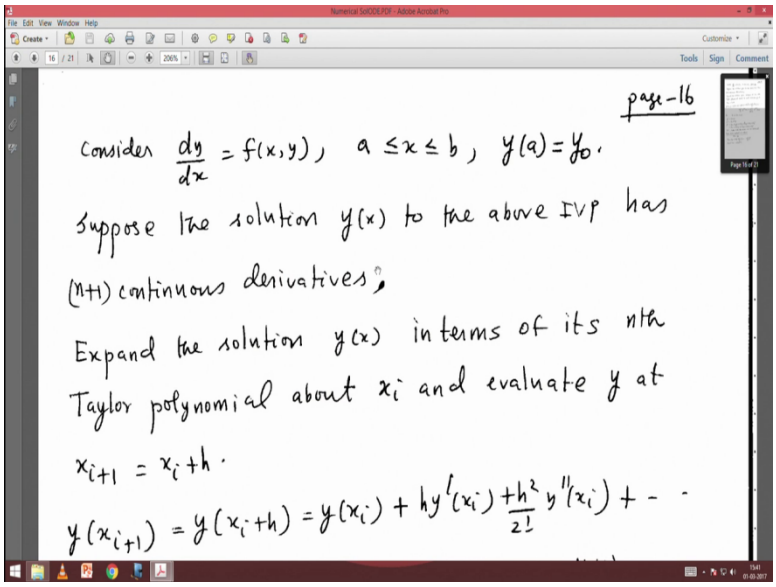

So let us consider the initial value problem dy by dx is equal to f(x,y) a less than or equal to x less than or equal to b is the interval in which I want to get the solution condition is y (a) equal to y 0. So suppose the solution to the problem has n plus 1 continuous derivative that is y can be differentiated n plus 1 times the derivatives exists and they are also continuous.

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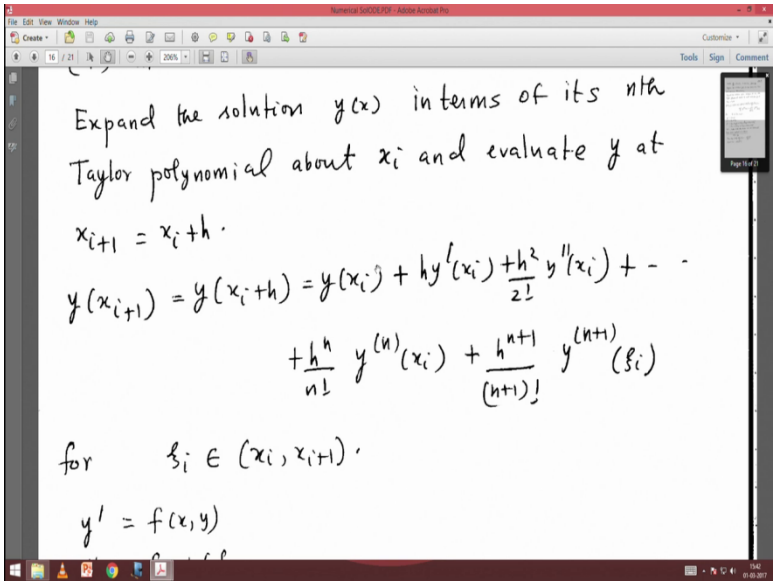

Then I expand using Taylor series right? y (x i plus 1) which is y (x i plus h) that is y (x i) plus h y dash (x i)and so on for the last term which I want to neglect which is the local

truncation error is h power n plus 1 by factorial n plus 1 into (n plus 1) th derivative (psi i) where Psi i belongs to x i , x i plus 1.

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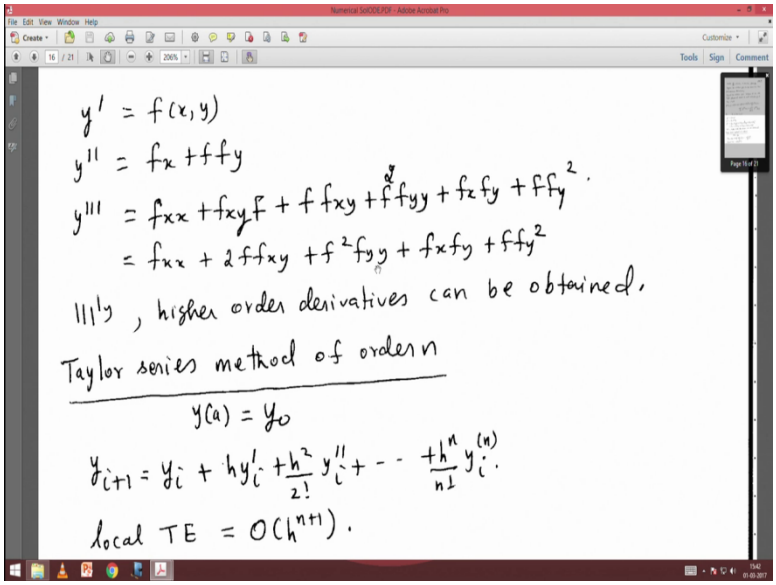

So I compute the derivatives which appear there as terms of the Taylor series as y prime y double prime etc. y prime is f(x y). What is y double prime it is d by dx (y prime) but what is d by dx (y prime) so let us see what it is?

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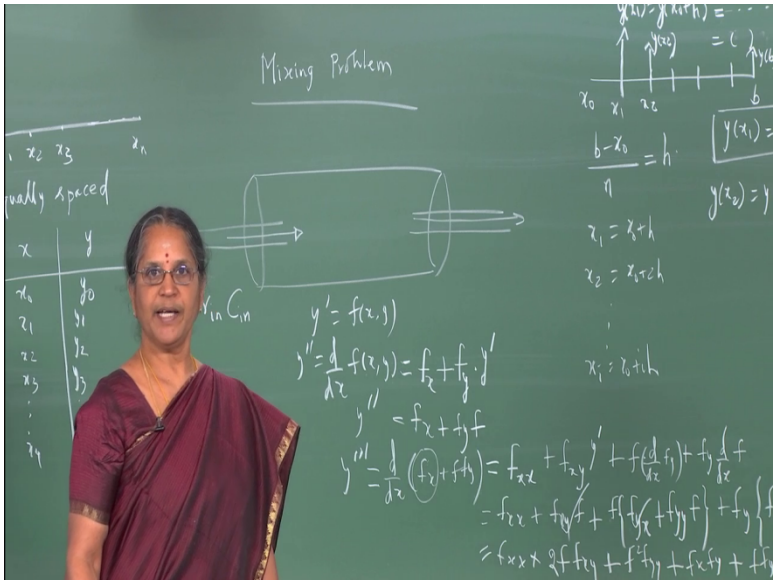

So y prime is f(x,y) so I require y double prime which is d by dx f(x , y), What is d by dx f(x , y)? It is d by dx of a function of two variables x and y where y itself is a function of x.

So it is nothing but partial derivaitve of f with respect to the first argument plus the partial derivative of f with respect to the second argument into y with respect to x.

So this is f x plus f y into what is y prime it is f. So you have got your second derivative ffx plus ffy. So let us compute the third derivative so it is d by dx (f x plus f into f y). So we have a function of two variables here and I require d by dx of that function of two variables. So it must be equal to click this first term so the derivative of the first term with respect to the first argument x plus derivative of the first term with respect to the second argument namely y into y with respect to x.

So we have obtained d by dx of this, we now move on to d by dx of this term. We see that it is a product of f and fy, So we must apply the product rule while taking the derivative. So it is going to be f into d by dx of f y that is the first term plus the second term is fy into d by dx (f). So this will be ffx plus fxy into what is y prime f and again f into I require d by dx (fy) but fy is a function of x and y, so treat it is a function of two variables so it must be fy with respect to first argument plus fy with respect to the second argument into y with respect to x that is y prime which is f itself.

Then I go to the next term fy into d by dx f it is a function of two variables. So f with respect to the first argument plus f with respect to the second argument into y with respect to x which is f itself. And so the third derivative of y is fxx plus you observe that you have a fxy and you have a into f and you have a f into f yx , function f is a continuous function and so the mixed partial derivatives are equal so we have f into f xy plus f into f yx to contribute to twice f into fxy.

Then the next term is f square into fyy and then we have fx into fy then we have f into f y square so this is our third derivaitve y double prime. Similarly the higher order derivatives can be computed and they involve the partial derivatives of various orders and these have to be evaluated where at x i and then substitute on the right hand side in the Taylor series expansion and then the value of y (x i plus 1) must be obtained.

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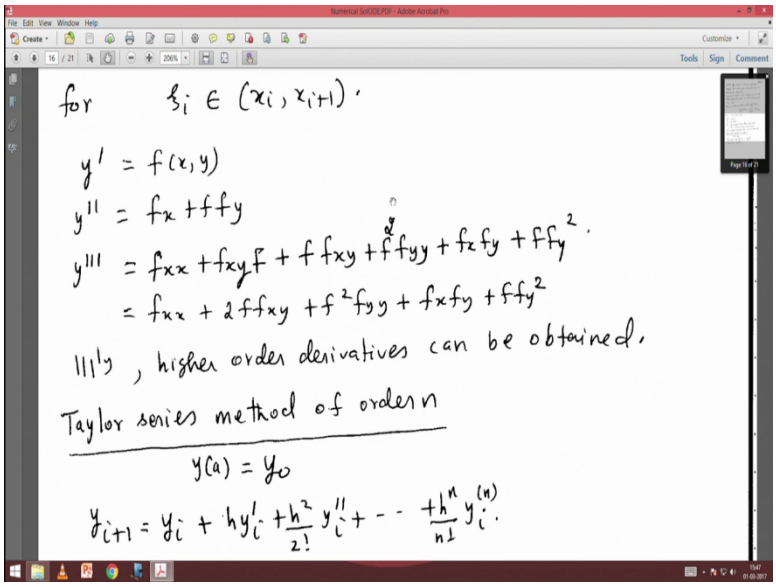

So if I use the Taylor series method of order n then how do I get it y(a) is y 0 and y (i plus 1) what is the notation that I use? If I compute y(x i plus 1) namely y(x i plus 1) then when I use the Taylor expansion or Taylor series method of order n I obtain an approximate solution I shall denote that solution by y (i plus 1) so that is the notation that we have used here so y(i plus 1) will be y i plus h y dash i plus etc upto h power n b factorial n into n th derivative of y evaluated at x i.

I have neglected the higher order terms first neglected term is of order of h to the power of n plus 1. So the local truncation error that is incurred by using a Taylor's method of order n is order of h to the power of n plus 1. So we does have understood what Taylor series method of order n is and what are the conditions that have to be satisfied by the function f in order that we have a reliable numerical solution say at points which are given by x 1 x 2 etc say x n.

So we can move ahead from one step to another step using the information available to us at the previous step and compute the solution at this step namely if I have information at x 0 which is given by the initial condition I use Taylor series method of order n get the solution at x 1. Now that I have the solution at x 1 namely y 1 I solve a new initial value problem what is it? dy by dx is f(x, y) subject to the initial condition y(x 1) is y 1. So I can get the solution at the next point namely what is y (x 2) again by Taylor series method of order n so I get y(x 2) which is y 2. So I move every time by moving over to the next point which is instance of h

where x i is x 0 plus i h and reach the end point of the interval where I require numerical solution.

So I now have solution at a set of discrete points x 0, x 1, x 2 etc x n which are denoted by y 0 , y 1, y 2 etc y n. So we have a solution at a set of discrete specified equally spaced points and this is what we want this what we call numerical solution of an initial value problem. So we now have learnt Taylor series method of order n by means of which we can solve an initial value problem. So in the next class we shall take up some examples to illustrate this method and then see the range of values of x for which such an approximation will be a valid one. So we will continue in the next class.