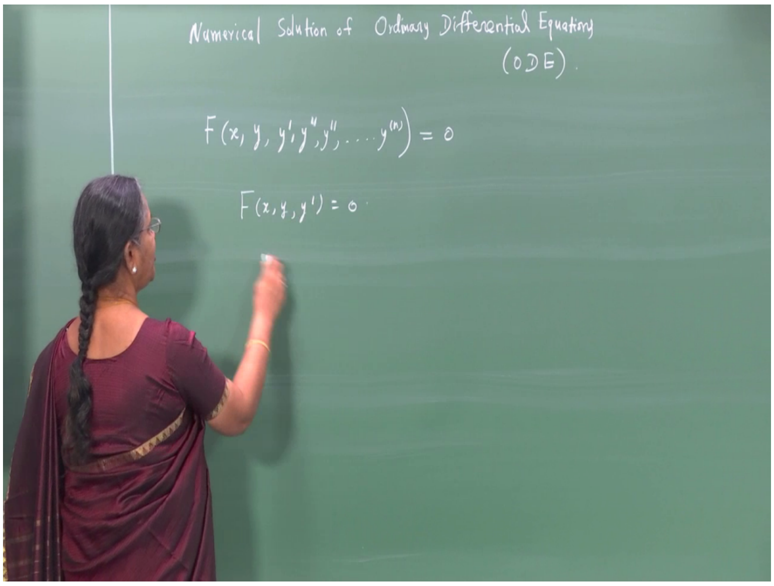
Numerical Analysis
Prof R Usha
Department of Mathematics
Indian Institute of Technology Madras
Lecture 18
Numerical Solution of Ordinary Differential Equations (ODE) 1

Good Morning, so in this class we shall consider Numerical solution of Ordinary differential equations. What are Ordinary Differential Equations?

(Refer Slide Time: 00:28)



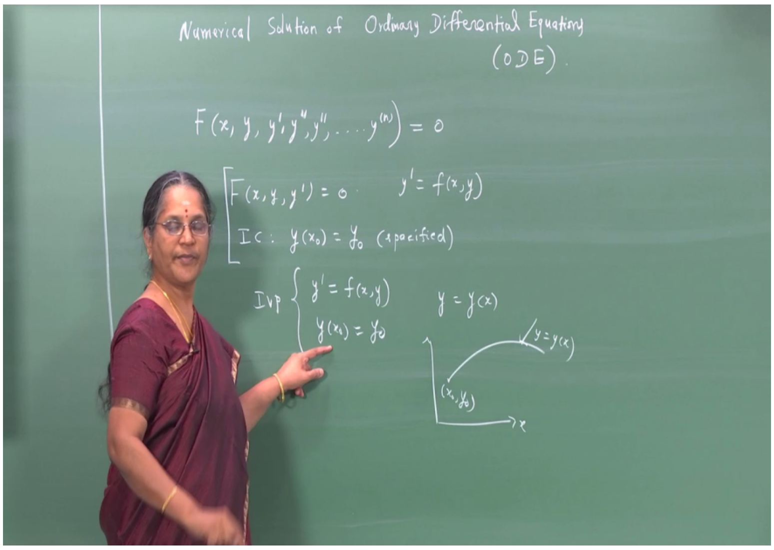
Suppose say we have a relation which connects an independent variable x with dependent variable y which depends on this independent variable x and then various order derivatives of the dependent variable, say I consider derivatives upto order n, so that I have a relation of this form where we say that we have an n th order Ordinary Differential Equation for the dependent variable or the unknown y.

Why is it a n th order differential equation? The highest order derivative that appears in this relation is the n th order derivative of y with respect to x and so we say that it is an n th order differential equation. So in this course our focus would be to learn numerical methods for solving first order differential equations. So they will be relationships of the form f(x,y,y prime) equal to 0.

The highest order derivative is the first order derivative of y with respect to x and so we would like to develop numerical methods by means of which we can solve differential equations of order 1. So you may ask me why do we require numerical solution because you already have learned certain methods by means of which you can solve first order differential equations namely if the first order differential equation is an exact differential equation or it is a variable separable form or it is such that it is a linear Lagrange differential equation or it may be a non exact first order differential equation. But you can multiply by an appropriate integrating factor and make it exact and then solve the differential equation.

So you have learned some first order differential equations which can be solved analytically and you know the methods of solving these differential equations. In practical applications you may not always get first order differential equations of the form that we have talked bout. So in such cases we need to have some methods by means of which we have to find the solution of such differential equations. So in this course we will focus in this topic on the numerical solution of first order differential equations.

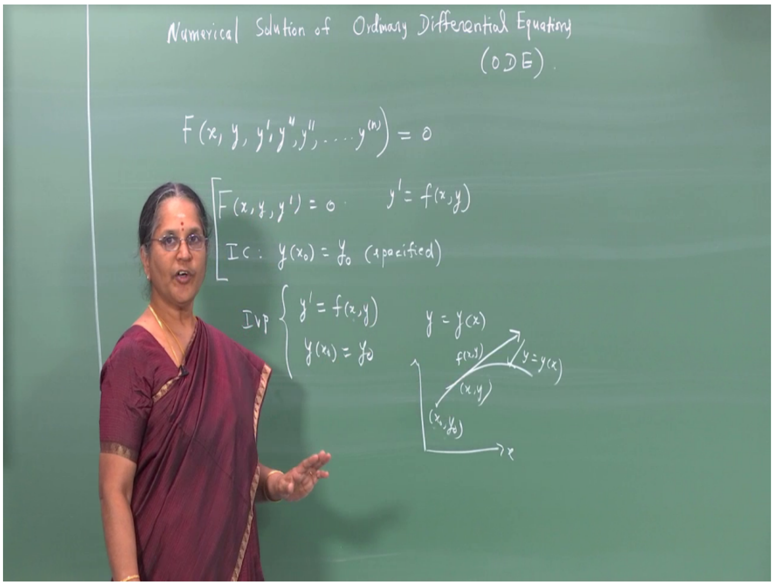
(Refer Slide Time: 03:24)



Usually these first order differential equations will come along with some initial conditions, namely you will be given y at some initial points say x 0 which is equal to y 0. So this y 0 will be specified or will be given to you. And one can write down this relation explicitly in the form y prime equal to the function of x,y. So our goal would be to develop numerical methods by means of which we solve an initial value problem of the form y prime equal to f(x,y) subject to y(x 0) equal to y(y 0).

The first question that comes to our mind is that, can we interpret it and understand that what is differential equation represents and what is the initial condition give us? So we know that what the solution of this differential equation is some function y is equal to y (x) which is a smooth function, such that in the xy plane I need to find out what this curve y equal to y(x) is such that the intial condition tells that it passes through the point x 0 y 0. So I need to find a function y is equal to y(x) which passes through the point x 0, y 0 at what more is given to us. We are given that the derivative of y is f(x,y) at any point x , y on that solution.

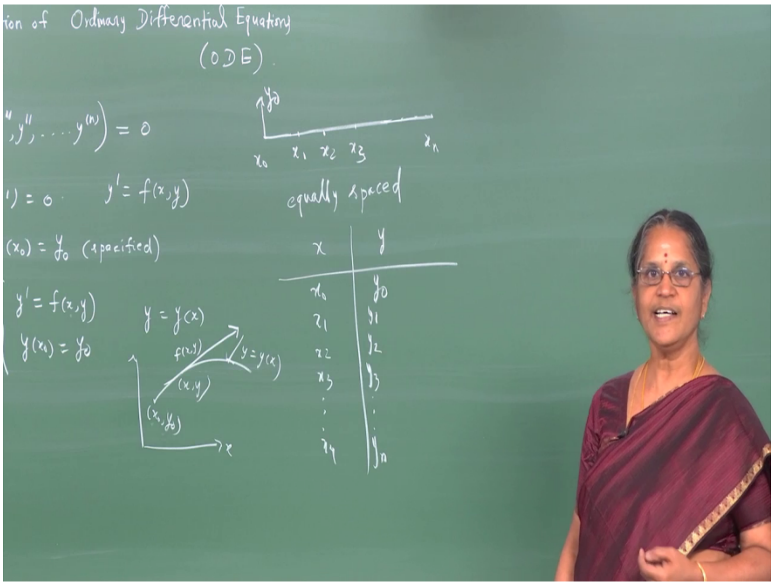
(Refer Slide Time: 04:58)



So if I take any typical point x,y then the derivative represents the slope of the tangent to the solution curve y equal to y(x) at that point x,y. So this is given by the slope of the tangent is given by x,y. So one ca geometrically interpret an initial value problem for the first order differential equation as determining a curve y equal to y (x) in the x y plane such that it satisfies the differential equation and the intial condition namely this curve is such that it passes through the point x 0 , y 0 and has at any point on it the slope of the tangent to be given by f(x,y).

So we look for such a solution of this differential equation. So as I said earlier we are going to look for numerical solution of this differential equation. So it is not possible to get a continuous solution by the methods that we are going to develop and we want to get the solution numerically.

(Refer Slide Time: 06:24)



Essentially we would be getting the solution at a set of discrete points say x 0 the solution is already known namely the curve passes through the point x 0, y 0. So I will be moving ahead and determining the solution at a set point say x 1 x 2 x 3 etc say x n where these points are equally spaced. So that we have a discrete set of values of x and the corresponding values of y namely at x 0 solution is y 0.

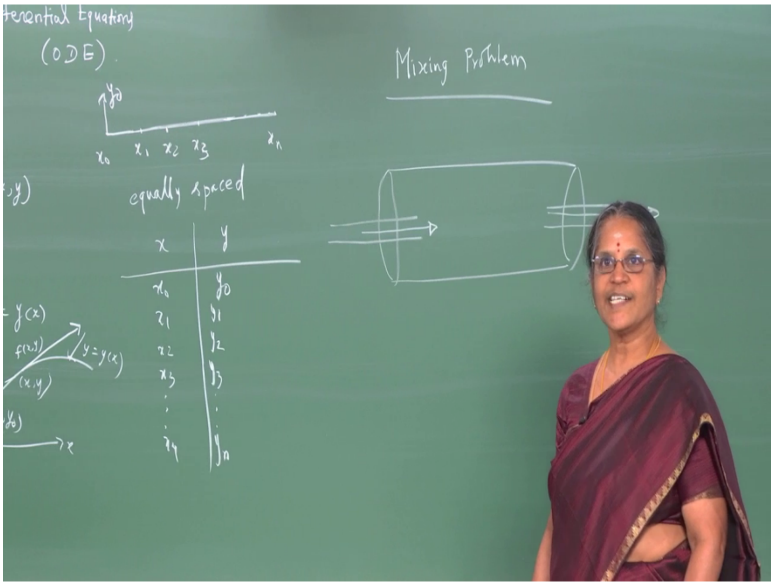
The methods that we are going to develop for solving this differential equation will enable us to get the values of y at x 1 which we call as y 1. The value of y at x 2 which is y 2 and so on. So we will be determining y i at the corresponding x i for i is equal to 1,2,3 etc upto n. Once we have a discrete set of values of x,y then we already have learned how we can reconstruct the function with the help of polynomial interpolation.

So we will obtain a smooth curve y is equal to y(x) which is an approximation of the exact solution of this differential equation so that this approximate solution has the property that it passes through the point x 0 , y 0 and it has at any point x i, y i its slope to be given by f(x i,y i). So it is with this goal that we are going to work and develop methods use them and see how numerical solution of such differential equations can be obtained.

And since we are getting approximate solutions we must be very interested in giving how much error that has been incurred in obtaining such an approximate solution so we need to perform error analysis. The next question that would come to your mind is where at all we come across such differential equations? What is the need for obtaining solution of such differential equations? In science and Engineering there are several real life situations and problems which can be modelled as first order differential equations.

So let us take a simple example of a mixing problem and then see how it is ordered into a first order differential equation and then any problem which occurs in Science or in engineering it is possible for you to model it in terms of a differential equation depending upon what the problem is. It may be modelled into a higher order differential equation also, but that higher order differential equation can be rewritten in terms of a system of first order equations so that methods that we develop for obtaining solutions of first order equations can be extended to solving higher order differential equations.

(Refer Slide Time: 09:58)



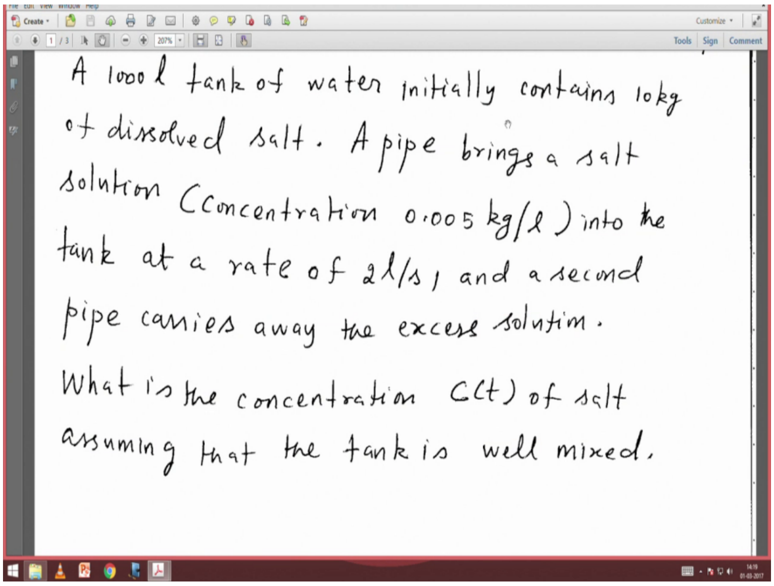
So let us take a simple example as I said in the case of a mixing problem. So what is a mixing problem? So I have say I have a tank right? I have a tank full of some salt solution at some concentration then I pumped into this tank through a pipe another salt solution which has a different concentration. And at a certain rate I pump this salt solution and then mix the salt solution within the tank. And then extract out the mixed salt solution when it is very well mixed in such a way that I extract out the solution at the same rate at which I pump in the solution.

So the question now is, can I determine the concentration of the salt assuming that the tank is well mixed. So we have this problem and we would like to model it first and we will show that the model results in the first order differential equation along with some initial condition.

So that we essentially have modelled this mixing problem into the solution of first order differential equation with an initial condition.

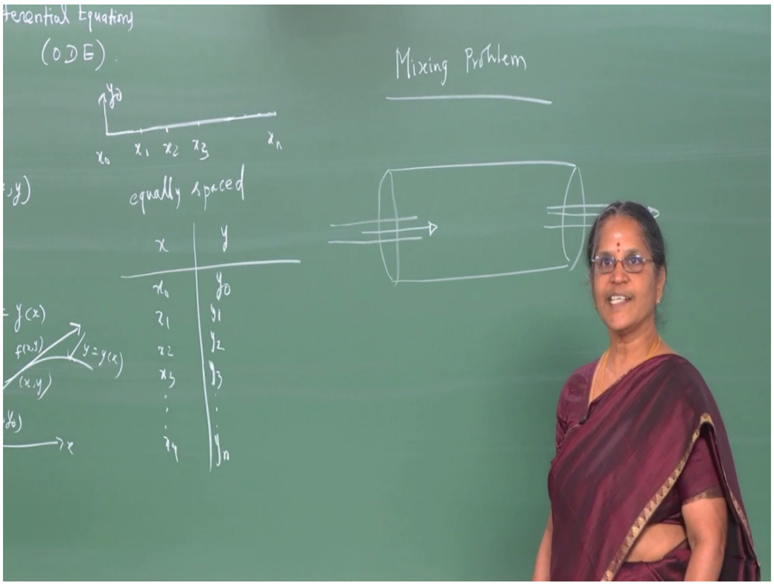
So that the modelling of this mixing problem results in an initial value problem for a first order differential equation. So let us elaborate on this problem and see how we can model. So please look at the monitor for the problem.

(Refer Slide Time: 12:10)



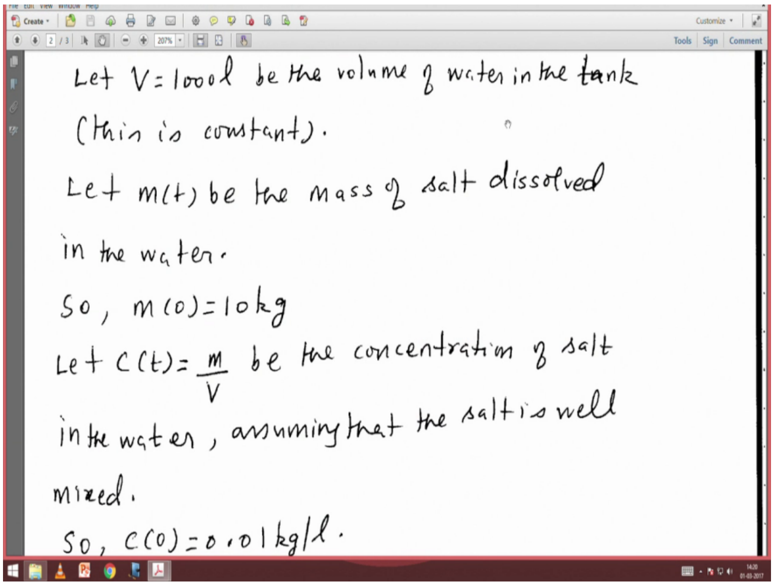
So let us take a thousand litre tank of water is initially contains 10 Kilograms of dissolved salts. A pipe brings a salt solutions, concentration is 0.005 kilograms per litre into the tank as I said at some rate. The rate at which it pumps in is say 2 litre per second.

(Refer Slide Time: 12:44)



And there is a second pipe here which carries away the access solution. So our problem is to find out the concentration, let us call it c as a function of say time t of salt assuming that the tank is well mixed. So our goal is to model this problem.

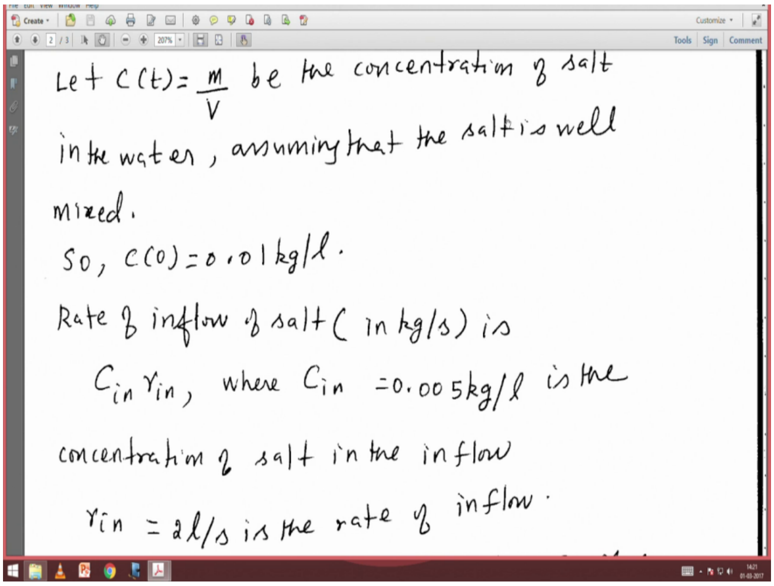
(Refer Slide Time: 13:11)



So let us take v to denote the volume of water in the tank since initially it contains 1000 litre, v is 1000 litres and it is constant. So let m(t) be the mass of the salt that is dissolved in water. So initially we said that it contains 10 kilograms of salt. so m at time t is equal to 0, so m(0) is 10 kilograms.

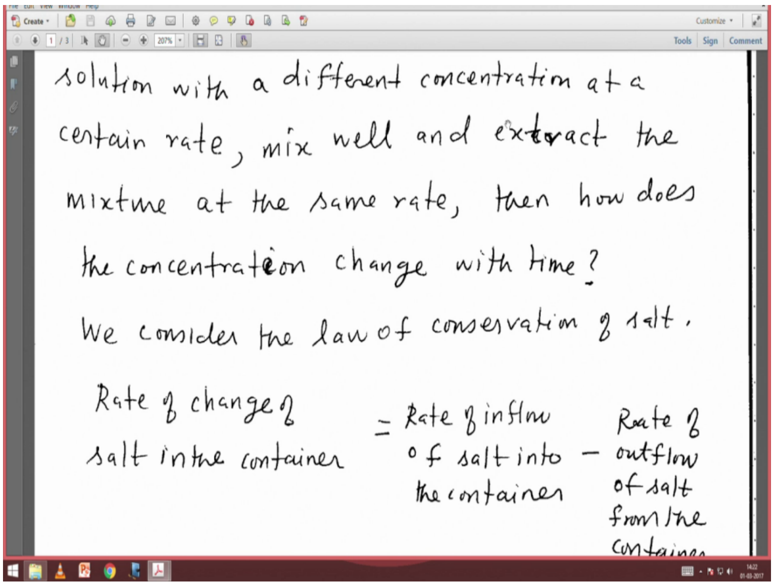
And what is the concentration of salt it is mass per volume so c(t) at any time t will be m by v. That will be the concentration of salt in the water assuming that the salt is well mixed.

(Refer Slide Time: 14:00)



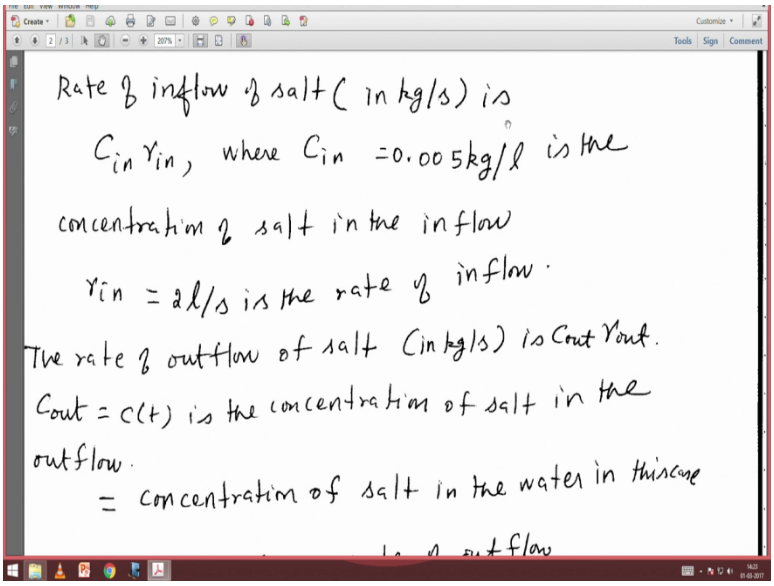
And what is do you have a knowledge of concentration at time t equal to 0 ? Yes we have in 1000 litres of water 10 kilograms of salt is dissolved. Initially the tank contains the solution and therefore c at time t equal to 0 is 10 by 1000 and that is going to be 0.01 kilograms per litre. So how are we going to model this? Right? What is the principle based on which this problem can be modelled?

(Refer Slide Time: 14:40)



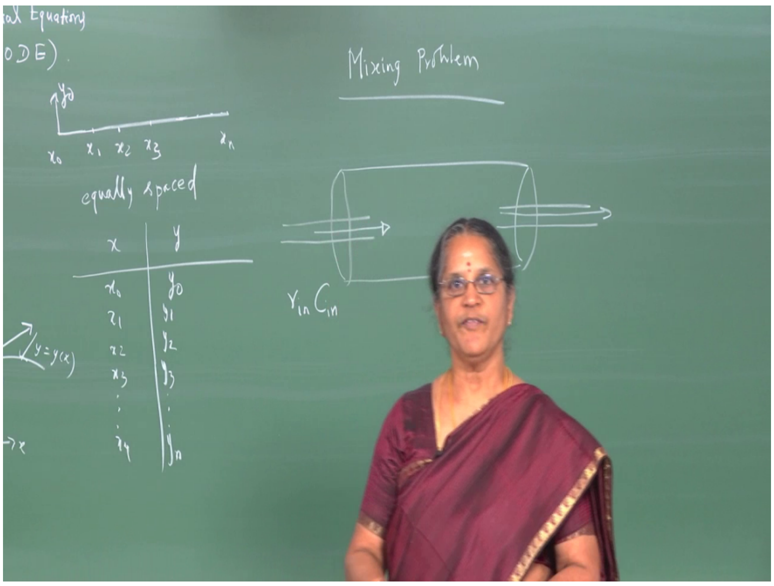
So we have to consider the law of conservation of concentration of salt namely the rate of change of salt in the container is nothing but the rate of inflow of salt into the container minus the rate of outflow of salt from the container. So this is the law of conservation of salt so the rate at which the salt content is changed in the container is equal to rate of inflow minus the rate of outflow.

(Refer Slide Time: 15:21)



So we make use of this law of conservation of concentration of salt and write down what it gives us. So let us compute the rate of inflow of salt in kilograms per second. So what is the rate of inflow of salt?

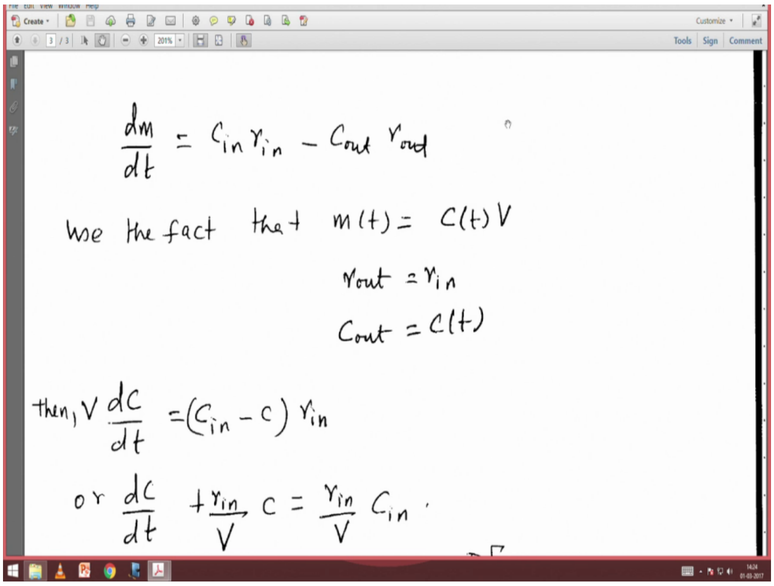
(Refer Slide Time: 15:42)



It is going to be the rate of inflow if I denote it by r inflow multiplied by the concentration of salt that is (C in) in the inflow. So the rate of inflow will be C in multiplied by r in, where C in is known to us, it is given to be 0.005 kilograms per litre that is the concentration of salt in the inflow.

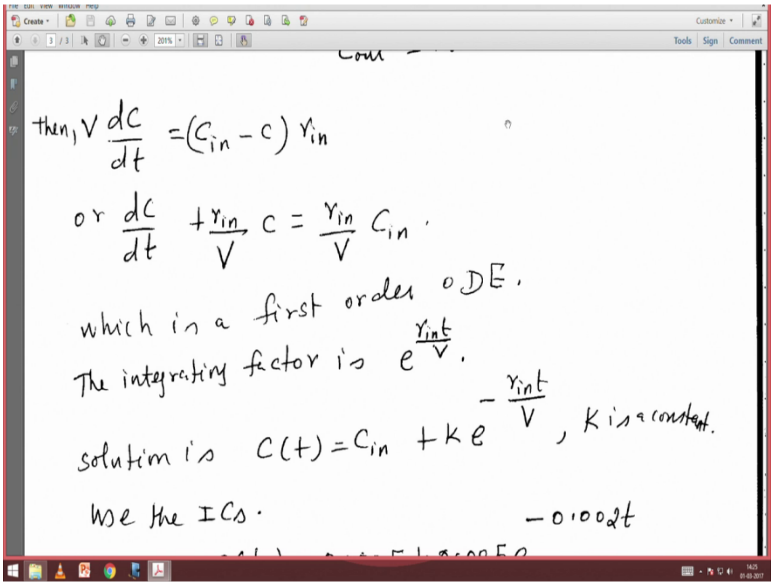
The rate at which this solution is pumped in is also given which is 2 litres per second. So r in is the rate of inflow. Now let us write down what is the rate of outflow. So the rate of outflow will be C out multiplied by r out where C out is nothing but C t which is the concentration of salt in the outflow and what about r out it is the rate of outflow. What is the problem says the problem says the rate at which the salt solution is pumped in is the same as the rate at which mixed salt solution is pumped out. So r out is going to be again 2 litre per second.

(Refer Slide Time: 17:12)



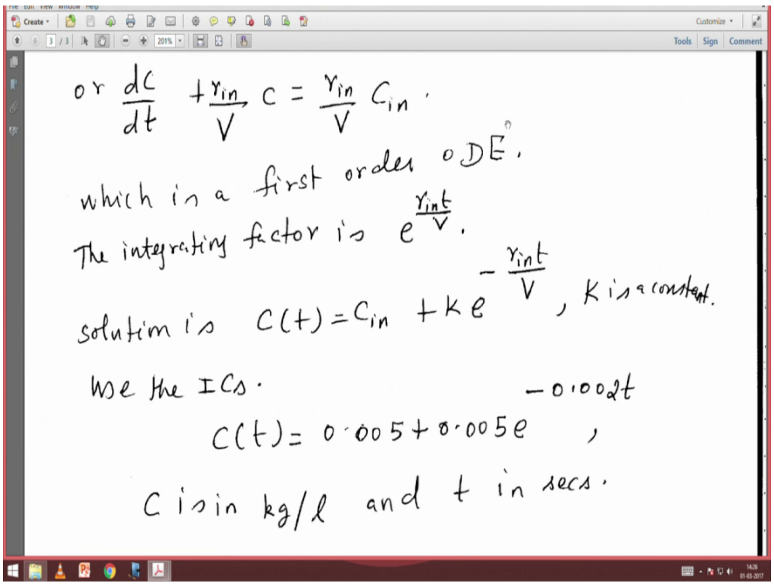
So we now substitute in the law of conservation of salt whatever that is available to us. So rate of change of salt content in the container is going to be d by dt of m what is it? It is rate of inflow minus rate of outflow. So it is going to be C in into r in minus c out into r out. So we use the fact that m(t) is C(t) into v, r out is r in and c out is denoted by C(t). So when we use that we get dv into dc by dt , v is constant. So d by dt of m where m is C (t) into v will give us v into dC by dt and that will be equal to C in minus C multiplied by r in

(Refer Slide Time: 18:19)



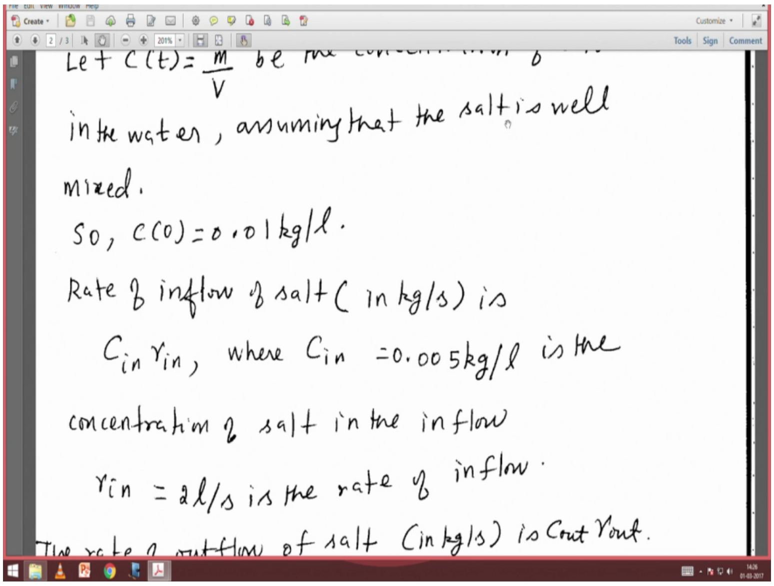
So just arrange these terms you end up with a first order differential equation for the concentration C namely dC by dt plus r in by v into C equal to r in by v into C in which is nothing but a first order differential equation. What is the initial condition? We have already seen what are we going to determine? What is our unknown?

(Refer Slide Time: 18:44)



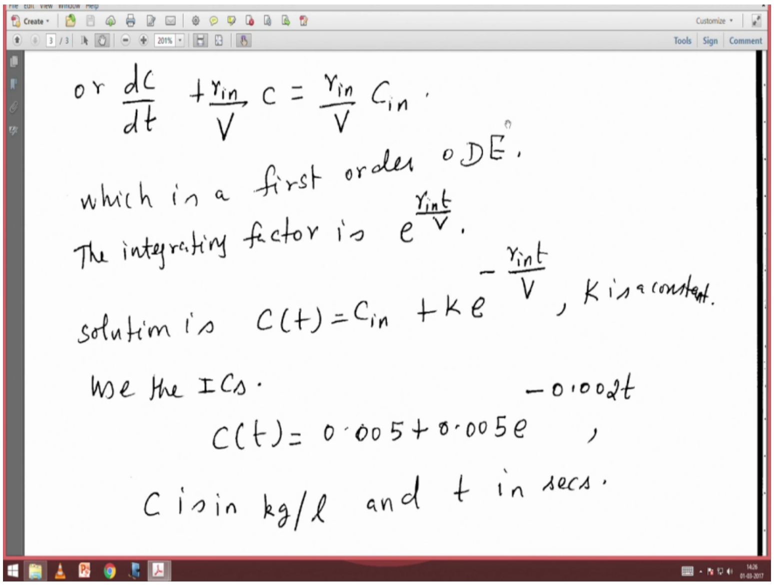
Our unknown is C (t), so the initial condition must be specified for C at time t equal to 0. So we must know what is C(0).

(Refer Slide Time: 19:00)



So we already have seen that C (0) is given by 0.01 kilograms per litre

(Refer Slide Time: 19:07)



And we have a differential equation which is of first order differential equation of C(t) which is dC by dt plus r in by v into C equal to r in by v into C in. So we have an initial value problem. Fortunately we see that this differential equation is of the linear Lagrange form. So we can analytically solve this differential equation by getting the integrating factor which is e to the power of r in into t divided by v.

So the solution immediately comes out to be C(t) equal to C in plus k into e power minus r in into t by v where k is a constant which is determined using the initial condition which is given for C(t) equal to 0. Then we immediately obtain the solution as given there and C is in kilograms per litre and t is in seconds. So we observe that this simple mixing problem which makes use of the law of conservation of salt within the container results in an intial value problem for a first order differential equations along with the initial condition.

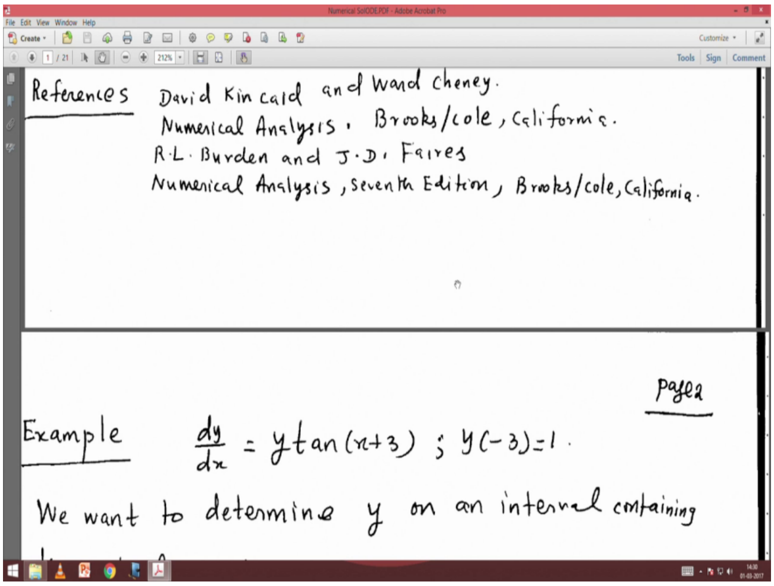
And therefore there are many such practical situations and problems in Science and Engineering which can be modelled in terms of first order or higher order differential equations which can be rewritten in terms of a system of first order equations which cannot be solved analytically by the available known techniques or the methods that we already have learned.

And in such cases we need to obtain a numerical solution of such differential equation by which we mean we require solution at a set of discrete equally spaced points. So that we end up with a table of values for the independent variable and the correspondent dependent variable which we use to reconstruct the function y as a function of x then we end up with a solution of the initial value problem of the first order equation.

So we would like to now develop such methods but before going over to the development of such methods we require some preliminary results on the existence and the uniqueness of solution of such first order differential equations. And at what conditions on the function f we may get a solution which is unique and which exists and where does it exist all these questions have to be answered. So we shall look into some preliminary results on the theory of ordinary differential equations.

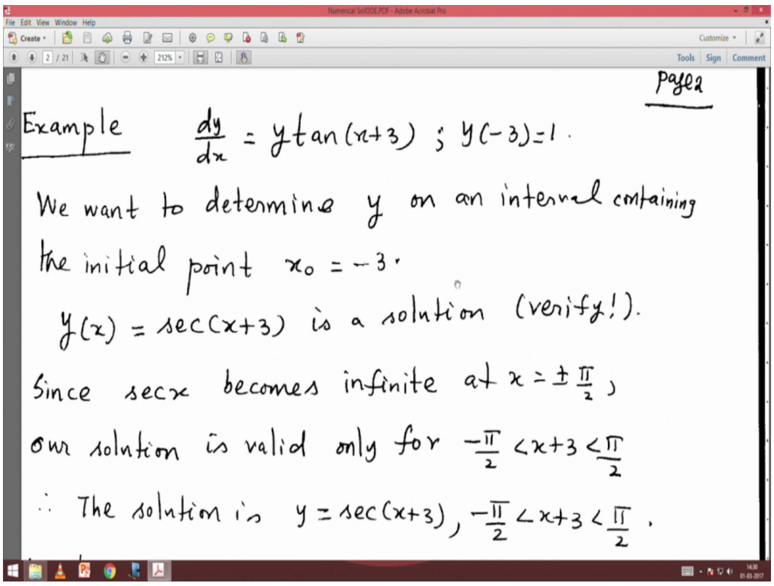
And then move on to the development of the numerical methods by means of which we can solve differential equation.

(Refer Slide Time: 22:32)



I have referred to the two books by David Kincaid and Cheney on Numerical Analysis and Burden and Fairs on Numerical Analysis for these preliminary results.

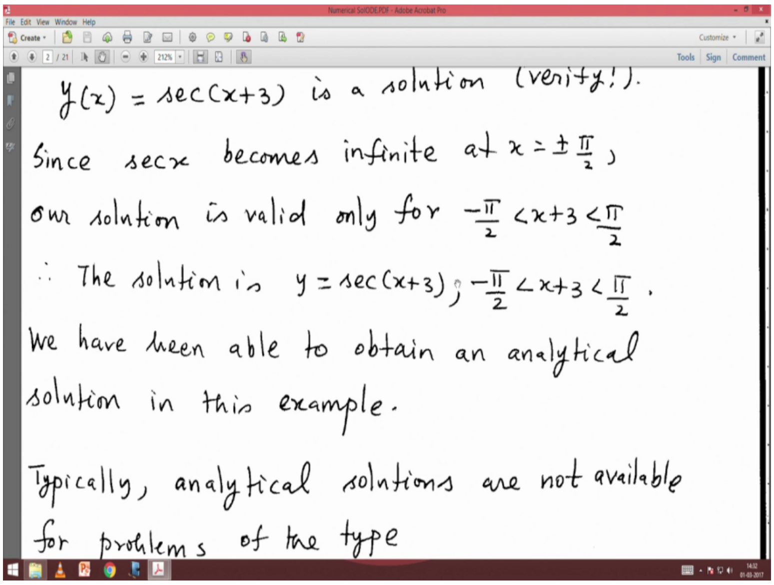
(Refer Slide Time: 22:50)



So let us first take an example. So please watch the monitor for the results that I am going to present. So consider a first order differential equation dy by dx is y tan (x plus 3); y(minus 3) is 1. So we want to determine y on an interval containing the initial point x 0 which is minus 3.

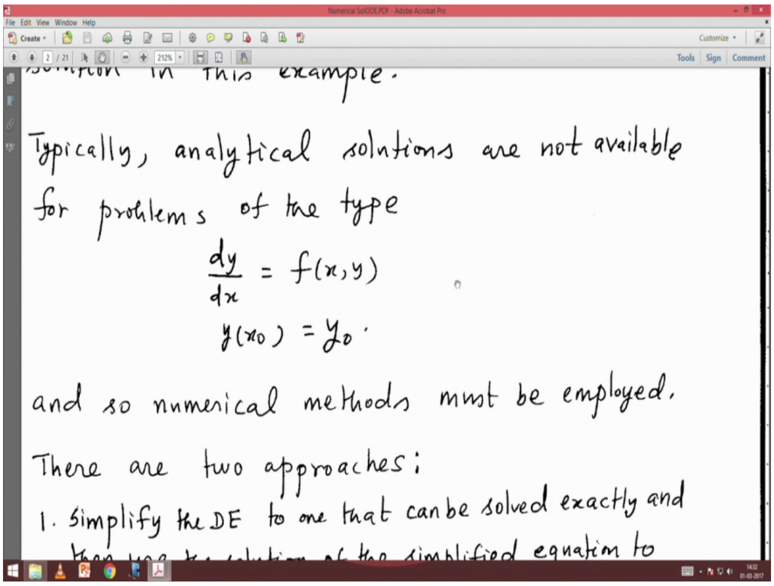
Please see the initial condition is y(minus 3) is 1, so the intial point x 0 is minus 3. So we require a solution of this differential equation passing through the point minus 3 , 1. So we want to determine y on an interval which encloses this initial point x 0 which is minus 3 and it is obvious that y equal to c can take x plus 3 as a solution. You just take the derivative of sec(x plus 3) you immediately get dy by dx to be given by y into tan (x plus 3) and the condition y(minus 3) equal to 1 is satisfied.

(Refer Slide Time: 24:10)



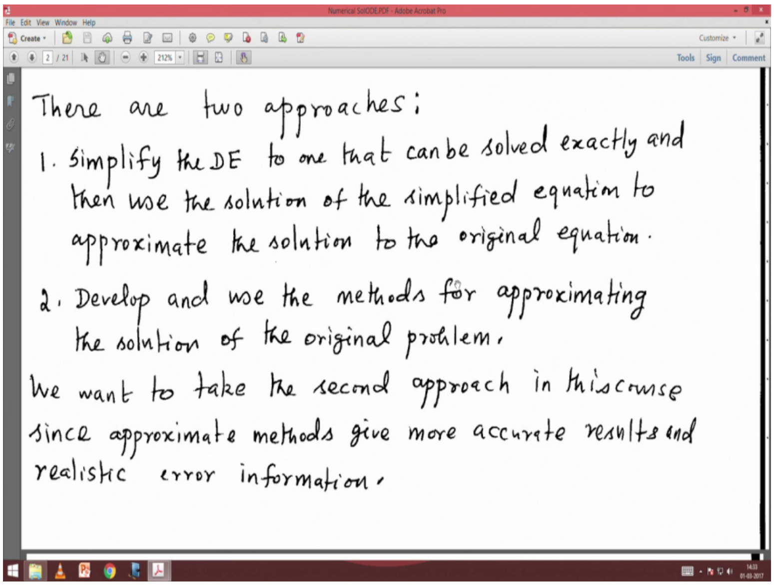
So this is an analytical solution we see that sec(x plus 3) is a solution and sec(x) become infinite at x is equal to plus or minus Pi by 2. So our solution sec (x plus 3) is valid only for those values of x which satisfy the condition that x plus 3 lies between minus Pi by 2 and plus Pi by 2. So the solution is y equal to sec(x plus 3) such that x plus 3 lies between minus Pi by 2 to plus Pi by 2. And in this case we have been able to obtain an analytical solution.

(Refer Slide Time: 24:50)



But typically the solutions are not available for initial value problems of the form dy by dx is equal to f(x,y) and y(x 0) equal to y 0. So we have to adopt numerical techniques of which these differential equations can be solved.

(Refer Slide Time: 25:11)

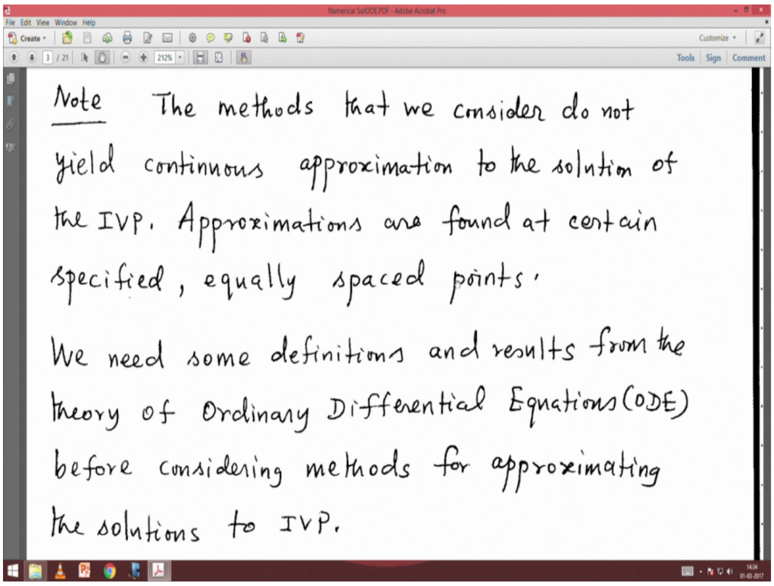


There are two approaches to this. The first approach is 1 where you simplify the given differential equation so that you will be able to analytically solve the simplified equation and take that as an approximate solution of the original problem. That is one approach, the second approach is try to develop and use the methods for approximating the solution of the original problem itself.

These are the two possibilities and we would like to use the second approach namely we want to now develop numerical methods and use those numerical methods in getting approximate solution of the original initial value problem that is given to us. Why are we very particular about the second approach?

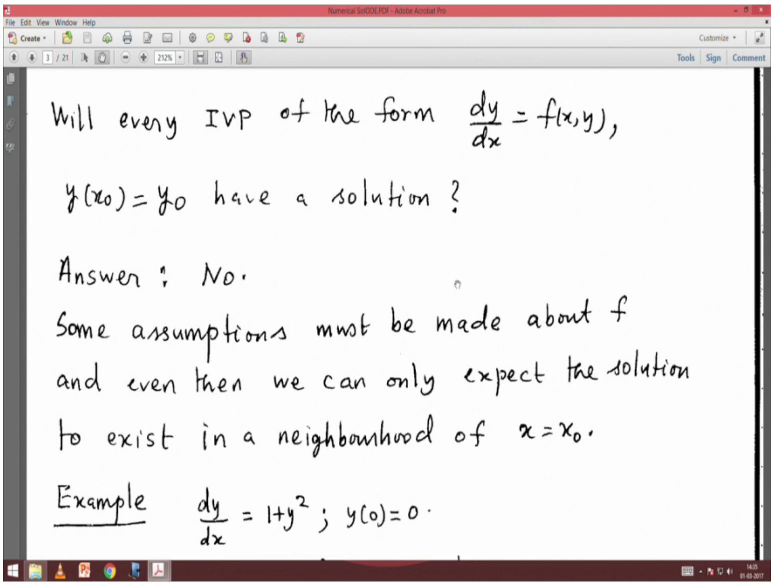
The reason is that it is going to be an approximate solution that is one and secondly it would be possible for us to specify some information about the error that is incurred in getting this approximate solution which has been obtained through the numerical method that we have developed. So it is for this reason that we would like to approach through the second method.

(Refer Slide Time: 26:42)



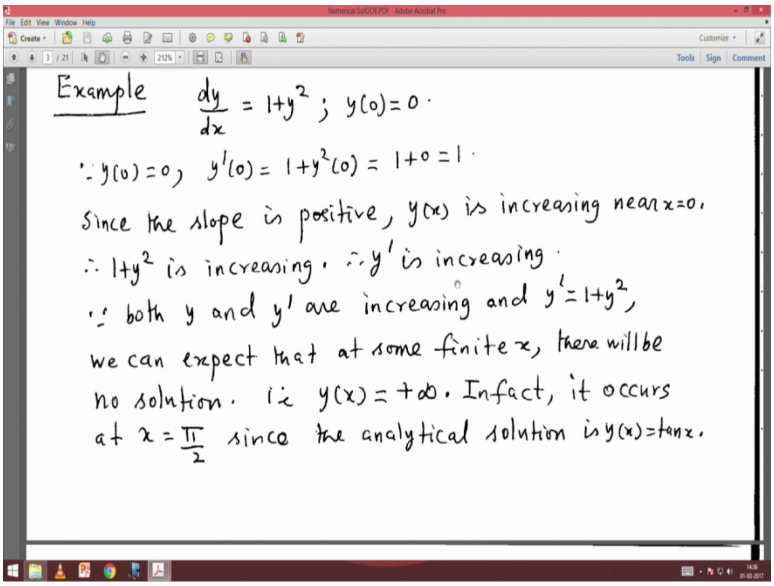
So as I already mentioned keep in mind that the methods that we are going to develop will not yield as continuous approximations to the solution of initial value problems. These approximations are going to be obtained at a certain set of specified equally spaced points. So as I remarked earlier we require some definitions and results on the theory of ordinary differential equations.

(Refer Slide Time: 27:20)



So let us look into those details. So first question that we want to post is this. Does every initial value problem of the form dy by dx is equal to f(x,y) with y(x 0) equal to y 0 possess such a solution? Does it have a solution? That is the first question. The Answer is no we have to have some conditions and assumptions that will have to be satisfied by f so that a solution of this initial value problem may be shown to exist in a neighbourhood of the initial point x 0.

(Refer Slide Time: 28:09)

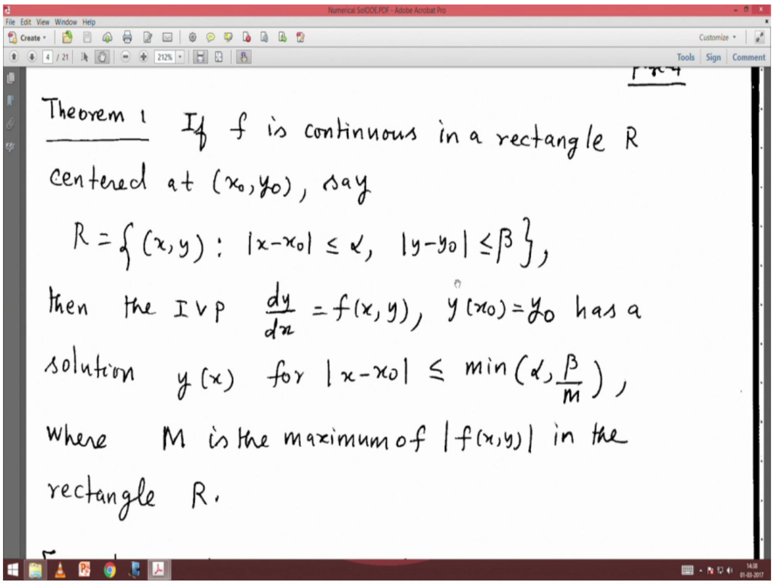


So let us see what this following example gives us. So consider this initial value problem dy by dx is equal to 1 plus y square y(0) is 0. so y(0) is 0 that is an initial condition. So look at the function f(x,y) what is it ? It is 1 plus y square. So y prime (0) is 1 plus y square(0) which is 1 plus 0 so which is 1. So y prime (0) is 1 so the slope is positive. What does it mean? y(x) is increasing where near x is equal to 0.

And 1 plus y square is increasing that is y prime equal to 1 plus y square is increasing so y prime is increasing. What is it that we have shown from the information which is given to us namely the intial value problem we are able to see that both y and y prime are increasing and y prime is 1 plus y square so we can expect that at some finite x there is going to be no solution because both y and y prime are increasing.

So there will be some finite x at which y(x) will become infinity, infact can you tell where it becomes infinity in fact you can solve the equation dy by dx is 1 plus y square. So it is of the variable separable form, so dy by 1 plus y square is dx. So tan inverse y will be equal to x plus a constant used is initial condition so you immediately observe that `x is equal to tan inverse y or y is equal to tan x or x is equal to Pi by 2 your analytical solution is such that it is going to become infinite.

(Refer Slide Time: 30:08)

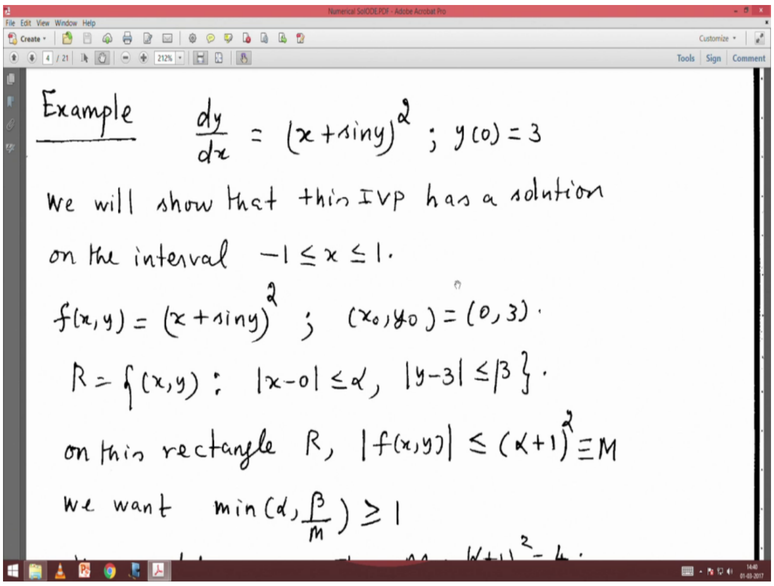


So you observe that you have from this example that your solution is such that at some point x is equal to Pi by 2 it becomes infinite, it does not exist. So let us keep this example in mind and see what does the next result tell us? It says or it gives us the condition under which the solution of an initial value problem exists in a neighbourhood of that initial point. So it says if dy by dx is equal to f(x,y) is first order differential equation given to you subject to an initial condition y(x 0) equal to y 0, then if that function f(x y) is continuous in a rectangle capital R which is centred at the point x 0 , y 0. So how are you going to describe this rectangle R, this rectangle R is the set of all points x,y such that mod x minus x 0 is less than or equal to alpha and mod y minus y 0 is less than or equal to beta.

So the centre of the rectangles x 0 , y 0. What is this x 0, y 0? Our initial condition says y (x 0) equal to y 0 so the centre is this point x 0 , y 0 at which the initial condition will be specified. So there result says if f is continuous in this rectangle then the initial value problem has a solution y(x). What does it mean? It means the solution exists where for mod x minus x 0 less than or equal to minimum of alpha, beta by m that is a solution exists in an interval surrounding this point x 0 that is a solution exists in a neighbourhood of the point x 0 at which the initial condition is specified.

What is capital M? M is the maximum of absolute value of f (x,y) in the rectangle R. so what is the condition of that f has to satisfy? f has to be continuous at rectangle R. Then a solution exists for this initial value problem in an interval which is a neighbourhood of the initial point x 0. So let us consider an example and try to understand the result given in the theorem.

(Refer Slide Time: 32:48)



So if you take dy by dx is equal to (x plus sin y) the whole square which y(0) equal to 3 we will show that this initial value problem has a solution on the interval (minus 1 to 1) that encloses this initial point x 0 which is 0. So what is f, f is (x plus sin y) the whole square, what is (x 0, y 0) (0,3)? And what is the rectangle mod x minus x 0) so mod x minus 0 less than or equal to some alpha, mod y minus y 0 that is mod y minus 3 is less than or equal to beta that is the rectangle having centre at (0,3)

So let us see what is modulus of (x,y)? f(x y) is (x plus sin y) the whole square, so since mod x is less than or equal to alpha for all xy in this rectangle modulus of f(x,y) is going to be less than or equal to (alpha plus 1) the whole square. We have call this or denoted this by capital M. And what is it that we want? We want the minimum of alpha , gamma , beta by m to be greater than or equal to 1 because what is the problem say show that the solution exists in an interval minus 1 to 1. So we want minimum of alpha, beta by m to be greater than or equal to 1.

So we can let alpha to be equal to 1 and since capital M is alpha plus 1 the whole square which is 4 in this rectangle R, minimum of alpha, beta by 4 is greater than or equal to 1 can be satisfied by taking beta to be greater than or equal to 4. So we have been able to show by applying the existence theorem to this example.

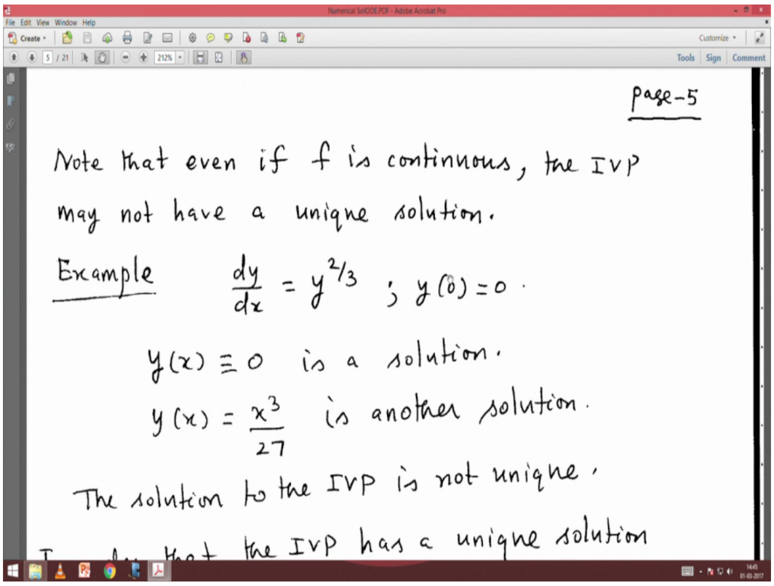
Since f is continuous in that example and we have been able to show that there is an interval in a neighbourhood of x is equal to 0. Namely minus 1 to 1 the condition is satisfied namely what does that condition say, the theorem says the initial value problem has a solution y(x) for mod x minus x 0 less than or equal to minimum of alpha, beta by m.

So in our case y(0) equal to 3 so it has a solution y(x) for mod x minus 0 less than or equal to minimum of 1, beta by m where beta is greater than or equal to 4 and M is equal to 4.Namely the solution exists on the interval mod x which is less than or equal to minimum of alpha, beta by M which is 1. Namely the solution exists in an interval of the form minus 1 to 1 which encloses this initial point x 0 is equal to 0.

So what is the condition the function f(x,y) must be continuous in that rectangle R which is centred at the point x 0 , y 0.So we note that even if the function f is continuous the solution to an intial value problem need not be unique isn't it we only showed using the result of the first theorem that solution to an initial value problem exists provided x satisfies some conditions namely f is continuous in a rectangle R centred at the point x 0, y 0.

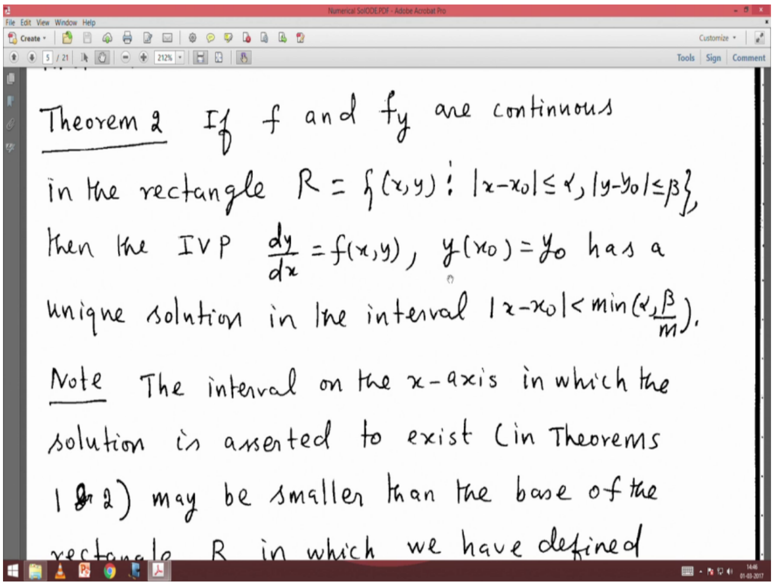
And in that case a solution exists in an interval enclosing the origin. Is the solution unique? is the next question to be answered. So even if f is continuous inside that rectangle centred at x 0 , y 0 it may happen that the solution need not be unique. So let us see an example.

(Refer Slide Time: 37:18)



Consider dy by dx equal to y power 2 by 3 with y(0) equal to 0. So we observe immediately that y(x) equal to 0 is one of the solutions which is a trivial solution. In addition y(x) is equal to x cube by 27 is another solution y just verify by taking the derivative of y dy by dx will come out to be 3x square by 27 that is x square by 9 which is nothing but y power 2 by 3. So the second solution that I have written down also satisfies the differential equation, so we have both y(x) is equal to 0 y(x) is equal to x cube by 27 to satisfy this initial value problem.

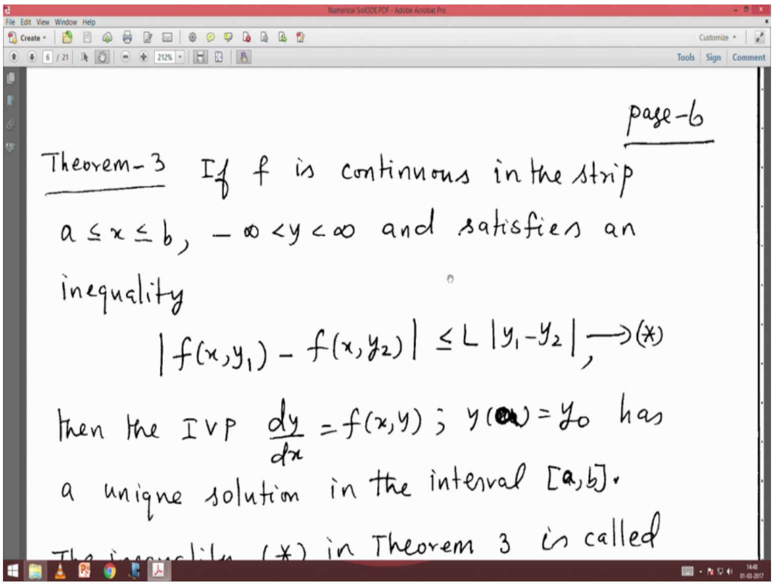
(Refer Slide Time: 38:08)



So in order that this intial value problem has a unique solution in a neighbour hood of x is equal to x 0 it is necessary that f satisfies some more conditions, so that is given by the next theorem so let us look into those conditions. So if f and f y are continuous in the rectangle which is centred at the point x 0, y 0 then the initial value problem has a unique solution in the interval mod x minus x 0 less than minimum of alpha, beta by m.

So both f and the partial derivative of f with respect to y must be continuous in the rectangle centred at x 0, y 0 then the initial value problem has a unique solutions. So initial value problem has a solution and that solution is unique where in an interval which is such that it encloses the initial point x 0.

(Refer Slide Time: 39:25)

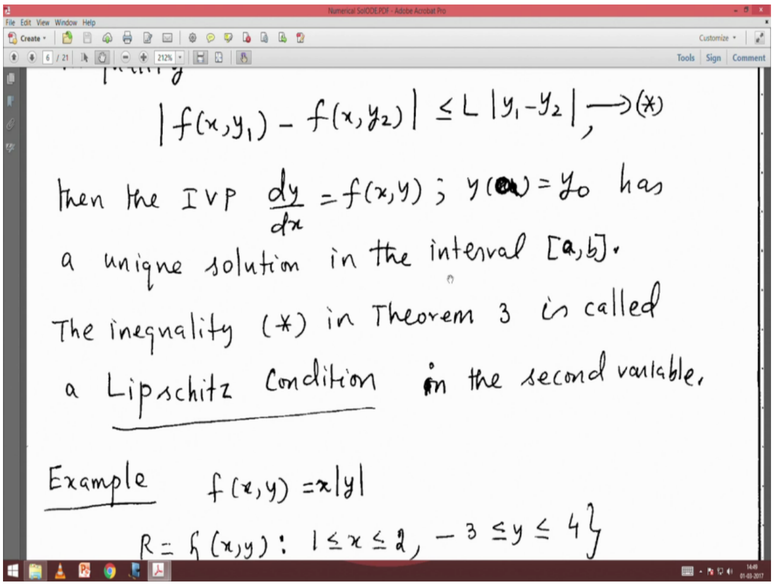


So it may also happen that the interval on the x axis in which the solution is asserted to exist in the previous two theorems may be such that it is smaller than the base of the rectangle R in which we have defined the function f(x,y). So we have a next theorem which gives the existence and uniqueness of solution of an initial value problem on an interval say of the form a,b which is a closed interval.

So the result says if f is continuous in the strip a less than or equal to x less than or equal to b and minus infinity less than y less than infinity, so if you consider a strip bounded by x is equal to a and x is equal to b and y runs from minus infinity to infinity. And if f is continuous in this strip and in addition f satisfies an inequality modulus of f(x,y 1) minus f(x , y 2) is less than or equal to l into mod y 1 minus y 2. If this condition is also satisfied then the initial value problem for the first order equation has a unique solution in the entire interval [a,b].

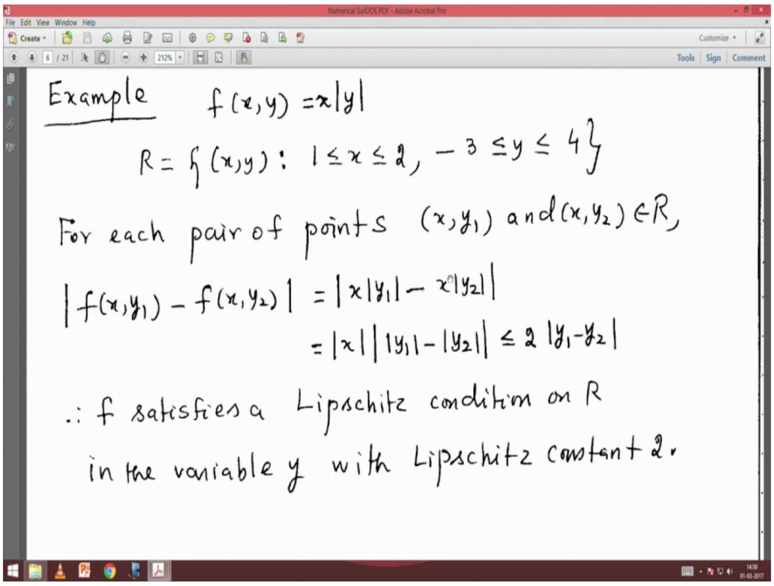
So this theorem asserts the existence and uniqueness of a solution of an initial value problem in an interval [a,b] and says the function f must be continuous in the strip and it should satisfy this inequality.

(Refer Slide Time: 41:10)



This inequality is called a Lipchitz condition in the second variable y. So function f must be continuous inside the rectangle centred at x 0, y 0 and f might satisfy a Lipchitz condition in the second variable then theorem asserts that the initial value problem has the unique solution in an interval for the form a to b. Just observe that the initial condition is specified at the left end point a of the interval a,b y(a) is y 0.

(Refer Slide Time: 41:53)

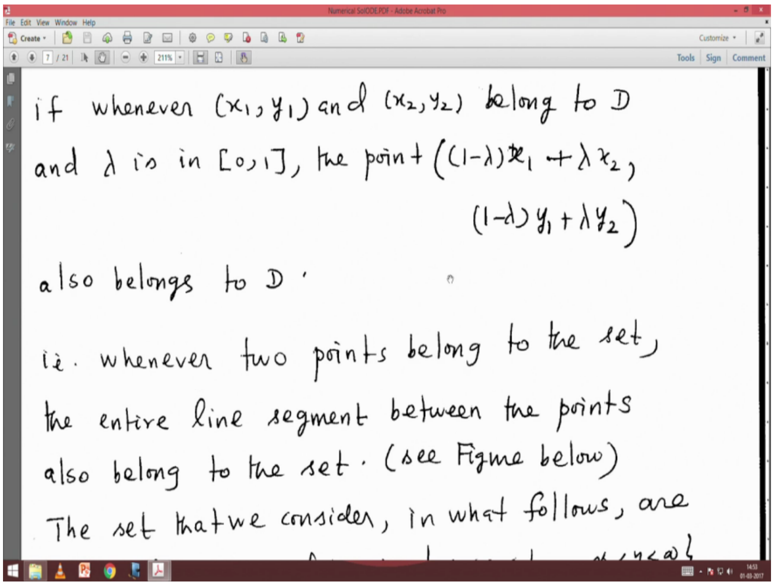


So let us consider an example. If suppose I have an initial value problem dy by dx is equal to f(x,y) and y(a) is given to be y 0 where f(x,y) is f mod y. And let the rectangle R be the set of all x,y such that x lies between 1 and 2 and y lies between minus 3 and 4. So the function is continuous in this rectangle that is clear

We should see whether f satisfies a Lipschitz condition in the second variable y. So let us consider two points x 1 y 1 and x 2 y 2 belonging to the rectangle. So let us consider the absolute value of the difference between f(x, y 1) and f(x, y 2). We know that f(x , y) is x mod y. So the right hand side will be the absolute value of x mod y 1 minus x mod y 2 which is mod x into modulus of mod y 1 minus mod y 2 and that would be less than or equal to twice mod y 1 minus y 2 because x is less than or equal to 2 inside that rectangle

So we observe that f satisfies a Lipchitz condition in the second variable y on the rectangle R and f is also continuous. So by the conditions of the theorem and the initial value problem of the form dy by dx is equal to f(x,y) subject to the condition y (1) equal to y 0 will have a unique solution in an interval that encloses the initial point a which is 1.

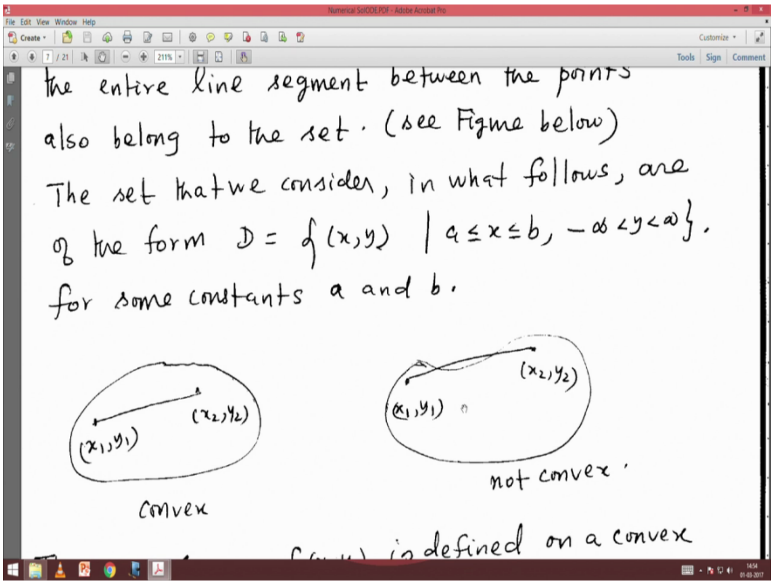
(Refer Slide Time: 44:00)



Let us now consider this definition and give another result through which again the existence and uniqueness of solution of an initial value problem can be guaranteed to exist. So we define a set D as subset of R2 to be convex. If whenever x 1 y 1 and x 2 y 2 belong to the set and lambda ranges between 0 and 1 the point 1 minus lambda x 1 plus lambda x 2, 1 minus lambda y 1 plus y 2 also belongs to d.

What does that mean? Whenever two points x 1 y 1 x 2 y 2 lie inside the a line segment joining x 1 y 1, x 2 y 2 is considered then every point on this line segment joining x 1 y 1 x 2 y 2 also will lie within D.

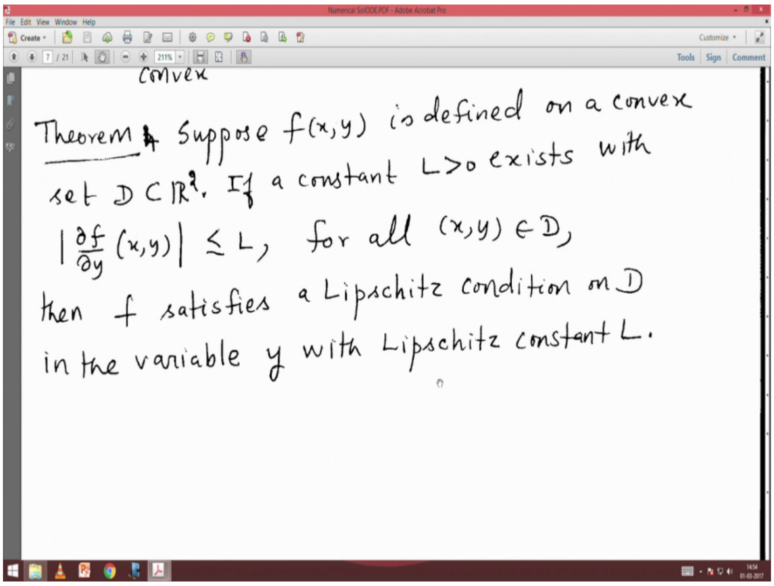
(Refer Slide Time: 45:06)



So just look at this figure that has been given. So take any two points x 1 y 1 x 2 y 2 draw a line segment if D is such that a line segment lies within D then the set is said to be convex.

In the other figure that does not happen a part of the line segment lies outside the set D so this set is not a convex set.

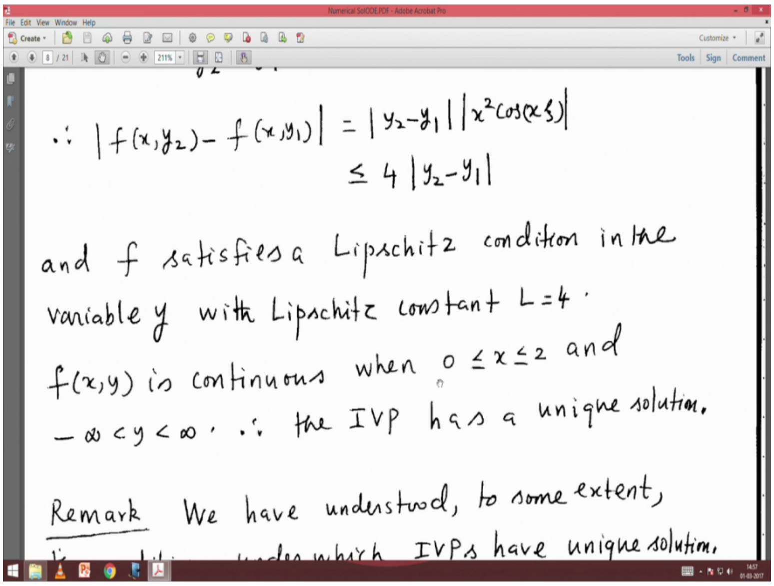
(Refer Slide Time: 45:37)



So consider a set which is convex. In what follows our discussion is based on such a set which is a convex set. So let us present this result. If suppose the function f(x,y) that appears on the right hand side of a first order equation dy by dx is equal to f(x,y) is defined on a convex set D which is a subset of R 2. If you can find a constant L which is a positive such that modulus of partial derivative of f with respect to y is less than or equal to this constant L for every xy in that convex set D then it means that f satisfies the Lipchitz condition on the set D in the second variable y with a Lipchitz constant L.

So given a function f(x,y) if defined on a set D you can check whether mod f y is less than or equal to L for all x in D namely you should be able to find out a capital L positive such that for all xy in D the partial derivative of f with respect to y in absolute value should be less than or equal to capital L. If this condition is satisfied then the theorem guarantees that the function f satisfies the Lipchitz condition in the second variable y with a Lipchitz constant L.

(Refer Slide Time: 47:17)



So let us consider an example. Say dy by dx is 1 plus x into sin x y for x lying between 0 and 2 with y(0) equal to 0. I would like to check whether the function f(x,y) satisfies a Lipchitz condition. So I would like to find what is f y suppose I hold x constant and apply mean value theorem to this function f(x,y) then we see that whenever y 1 is less than y 2 we can psi belonging to the interval y 1 to y 2 such that f(x, y 1) minus f(x, y 2) by y 2 minus y 1 is the partial derivative of f with respect to y evaluated at x , psi where psi lies between y 1 and y 2.

So if you rewrite that then you get modulus of f(x , y 2) minus f(x , y 1) is mod y 2 minus y 1 into mod x square into cos (x psi). But that is less than or equal to 4 times mod y 1 minus y 2 because your x is such that x lies between 0 and 2. So what is it that you have shown? You have shown that f satisfies a Lipchitz condition with Lipchitz constant equal to 4.

You have been able to show that there exists a constant L which is equal to 4 such that f satisfies a Lipchitz condition what does that mean? If f is continuous on that strip then the initial value problem has a unique solution. So let us see what is f? F is 1 plus x sin (xy) so it is a continuous function on the interval 0 less than or equal to x less than or equal to 2 and minus infinity less than y less than infinity and therefore the initial value problems has a unique solution, so in our discussion.

So far we have been able to understand to some extents the condition under which the initial value problem will have a unique solution there is one more thing that has to be addressed namely what happens if I give small changes to the initial value problem and the initial condition. Will there be large deviations in the resulting solution essentially we are checking up with our solution remains stable.

Namely will the small changes in the problem as well as the initial condition lead to small changes in the resulting solution of the initial value problem or in which case we say that we have a stable solution or does it lead to large deviations in which case we cannot rely upon the numerical method that we are going to employ to get the solution of the initial value problem. So we shall address this issue in the next class.