Numerical Analysis
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Lecture 16
Numerical integration 5
Gaussian Quadrature (Two point Method)

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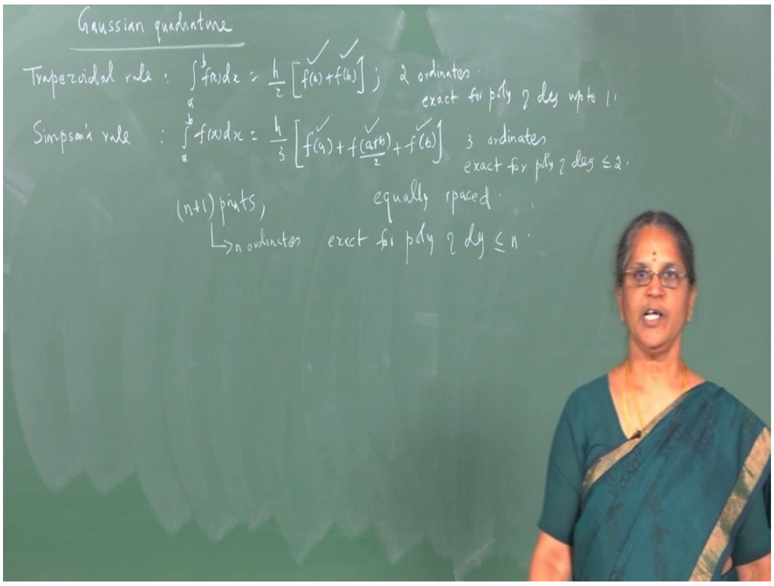

So let us now discuss another numerical method which is called Gaussian Quadrature. In deriving Trapezoidal rule and Simpson's rule we observed that Trapezoidal rule involved function evaluation at two points. So there are two ordinates which have to be computed. And the method is exact for polynomial of degree upto 1.

So you took two points and the method is exact for polynomials of degree upto 1. In the case of Simpson's rule it involves evaluation of function at three points. So three ordinates have to be obtained and the method is exact for polynomials of degree less than or equal to 2. In addition we know that these methods have been derived using equally spaced points.

And we observe that the Newton codes method which are derived with equally spaced points are such that if you are given information at n plus 1 points then you are able to obtain an integration method involving N ordinates and the method is exact for polynomials of degree less than or equal to n.

So one wanted to see can one derive an integration method such that the method is exact for polynomials of degree greater than n Gauss said that yes it is possible if one is ready to sacrifice the requirement that the nodes must be equally spaced. So if you sacrifice the requirement there are nodes within that interval must be equally spaced it is possible for you to construct an integration method such that the method is exact for polynomials of degree greater than n.

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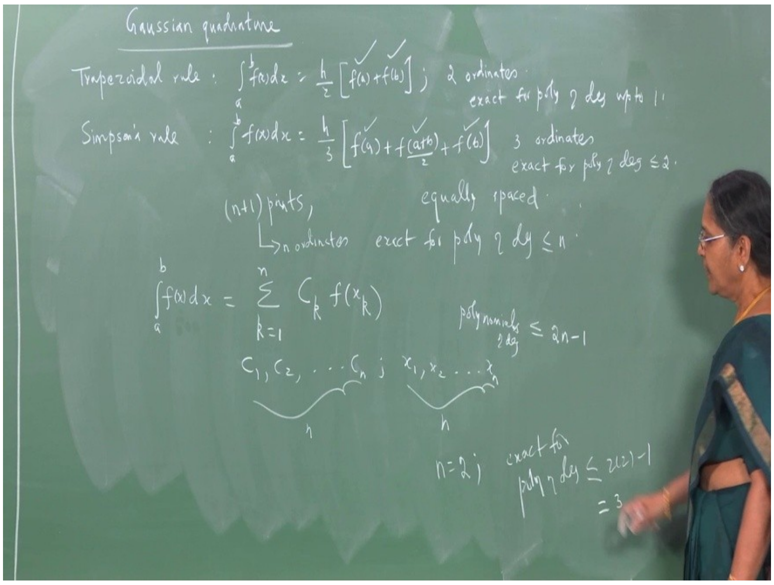

So he said that one can obtain integration methods of the form integral a to b f(x)dx which is of the form sigma say k is equal to 1 to n C k f (x k) so k runs from 1 to n. So there are n unknown constants C k which are C 1, C 2 etc C n. In addition we have x k which are also not known to us. So x 1 x 2 etc x k that is k running from 1 to n they are the nodes that we have to select in that interval a to b which need not be equally spaced so that we have another n unknowns.

So totally we have 2 n unknowns to be determined such that sigma k equal to 1 to n C k f(x k) will evaluate this integral and the method will be exact for polynomials of degree less than or equal to 2n minus 1. So how many ordinates that you use in this method there are only n ordinates. The ordinates are at x 1 x 2 etc x n. So you involve only information about the function at n nodes evaluation involves only n ordinates.

But the method that you derive is such that it is exact for polynomials of degree less than or equal to 2 n minus 1. Say for example if n is equal to 2 then you will derive a method which is exact for polynomials of degree less than or equal to 2 into 2 minus 1 namely less than or equal to 3. You will see that the x i which are located here need not be equally spaced. So let us work out the details proposed by Gauss and arrive at a Gaussian Quadrature.

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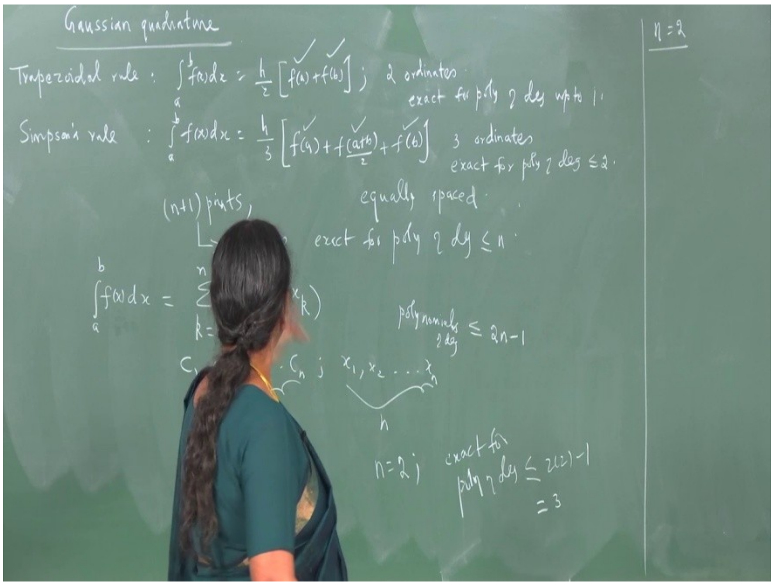

Say for the case when n is equal to 2. So we require an integral of the form a to b f(x) dx is sigam C 1 f(x 1) plus C 2(f x 2) you do not know C 1 C 2 you also do not know x 1 x 2 but the method must be exact for polynomials of degree less than or equal to 2 into 2 minus 1 namely less than or equal to 3. So this requires the knowledge of the properties of Legendre polynomials. So we briefly present those results and use them in the derivation of Gaussian Quadrature methods.

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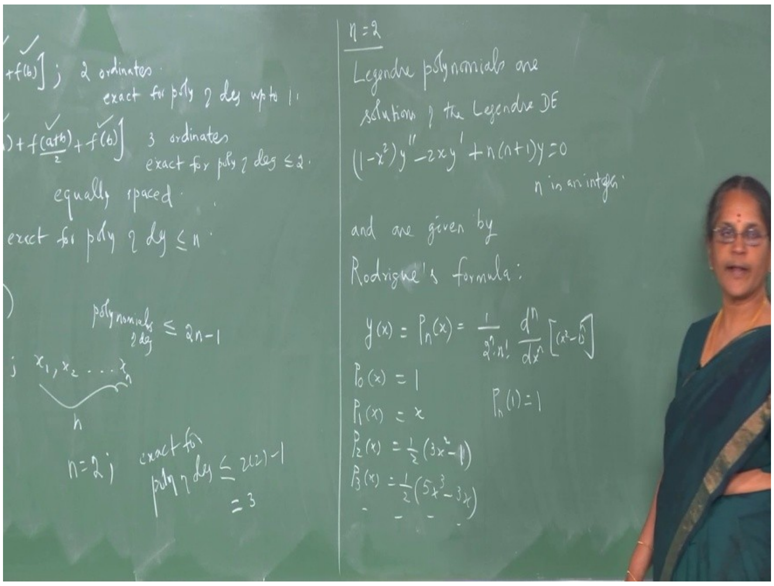

So what are Legendre polynomials? The Legendre polynomials are solutions of the Legendre equation Legendre differential equation which is 1 minus x square into y double prime minus 2x into y prime plus n(n plus 1) into y equal to 0 where n is an integer. And solutions are given by Rodrigues formula you must have derived these in your course on differential equation when you studied about special functions.

So Rodrigues formula tells you that the solution of this equation y(x) which is a Legendre polynomial of degree n when appears here in the differential equation in the form n into n plus 1 then the solution is given by 1 by 2 power n into n factorial into n th derivative of [x square minus 1 to the power of n]. So when n is 0 you get the Legendre polynomial of degree 0 which is 1 and then when n is 1 from here you will obtain that it is x P 2(x) will be half of (3xsquare) minus 1) and P 3(x) will be half of (5x cube minus 3x) and so on.

You can get higher degree polynomials and you observe that polynomials with suffix or odd polynomials odd degree polynomials and those denoted by even suffixes are even degree polynomials. In addition the polynomials have the property that P n(1) is equal to 1. You can check P 2(1) is 2 by 2 P 3(1) is again 1 and so on. So we satisfy the property that P n(1) is 1.

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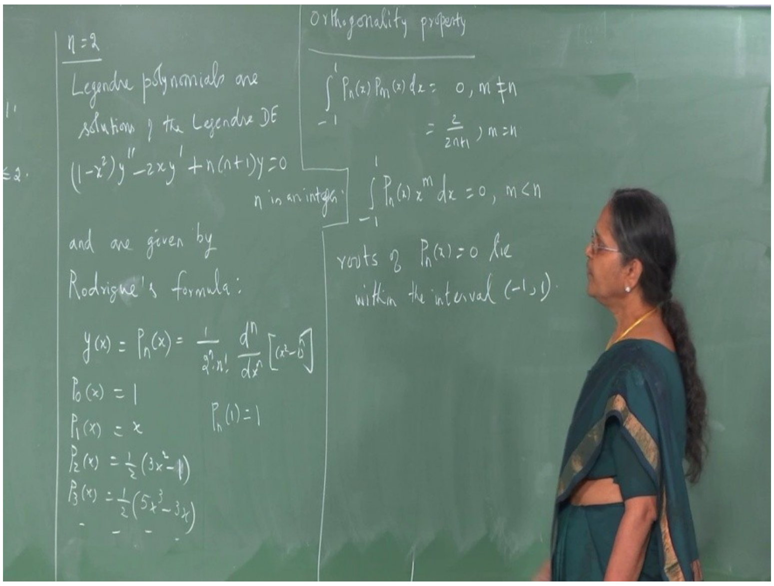

Further we satisfy an important property which is called Orthogonality Property which says integral minus 1 to 1 P n(x) into P n (x) dx is 0 whenever the polynomials are of different degree and it is 2 by 2 n plus 1 when we have the same degree.

And these polynomials are such that if you consider (minus 1 to 1) P n(x) into x to the power of m dx that will be 0 where m is less than n. Further the roots of these polynomials or zeroes of these polynomials are the roots of P n(x) is equal to 0 like within the interval (minus 1 to 1).

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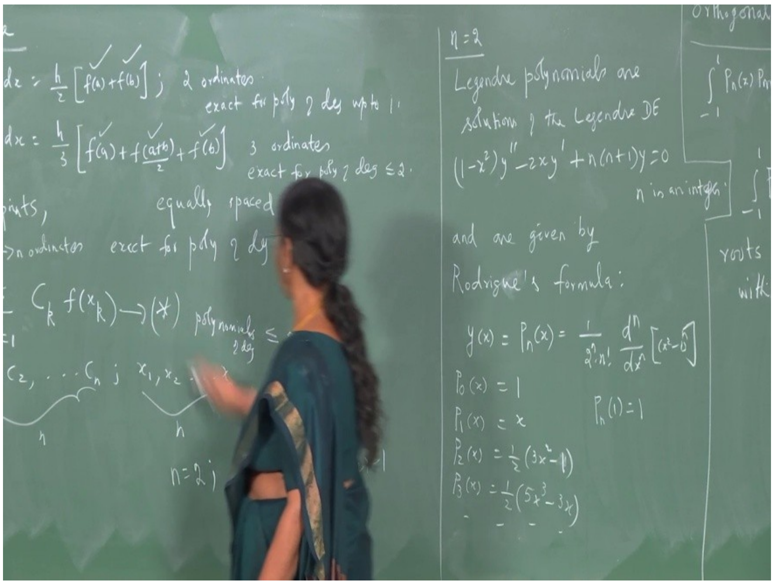

So these are some properties which we require in deriving Gaussian Quadrature formulas. So we have recalled these results. So let us now work out the details of Gauss Quadrature method says when n is equal to 2 in this formula.

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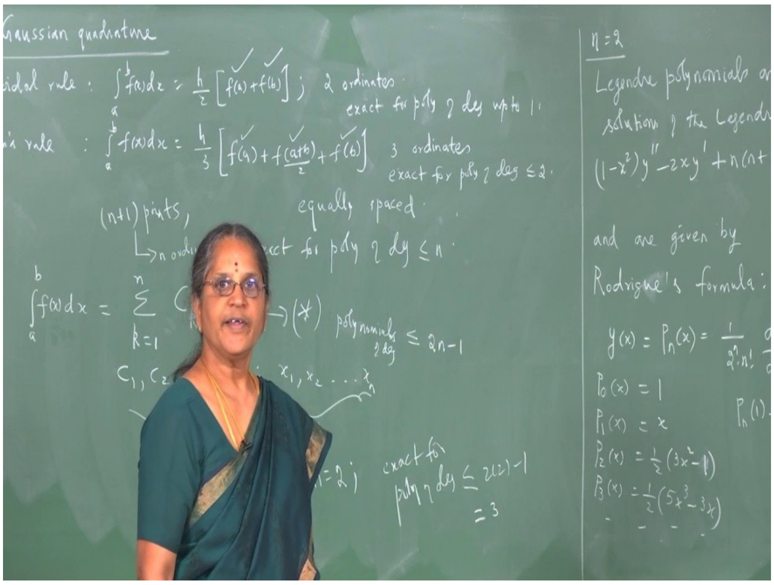

So let n be equal to 2 in star. So since I know that the Legendre polynomials are such that the roots of the Legendre polynomials P n (x) is equal to 0 lie within the interval (minus 1 to 1) Gaussian integration methods have been derived for integral minus 1 to 1 f(x) dx so we seek an integration method such that integral minus 1 to 1 f(x) dx is sigma k is equal to 1 to n C k f(x k).

So you may ask me then how do you get if you are asked to evaluate an integral of the form integral a to b f(x) dx. We will illustrate it we simply have to use a transformation for change of variable such that the integral between a to b goes to integral between minus 1 to 1). So we shall discuss that later, let us now derive the integration method where n is equal to 2.

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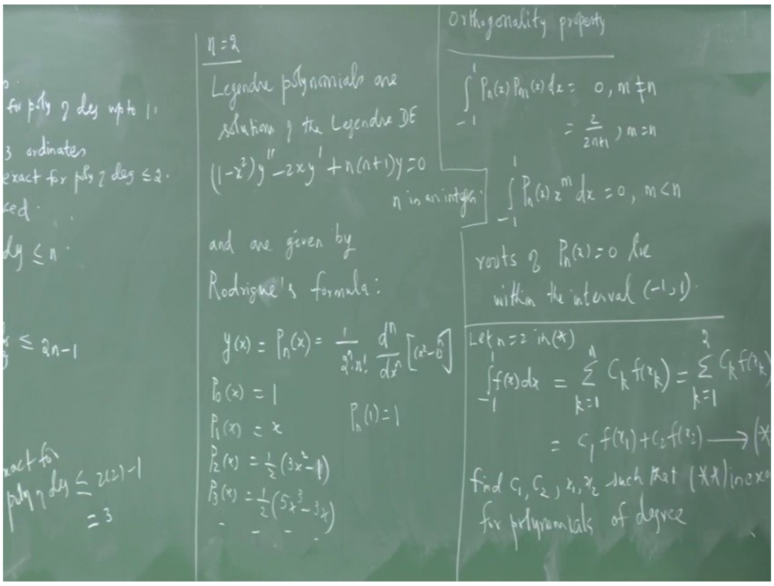

So we require a method such that integral minus 1 to 1 dx is sigma k equal to 1 to 2 C k f(x k) so mainly C 1 into f(x 1) plus C 2 into f(x 2). So find C 1 C 2 x 1 x 2 such that this integration method let us call this as double star such that double star is exact for polynomials of degree n is 2 the method says for polynomials of degree less than or equal to 2n minus 1.

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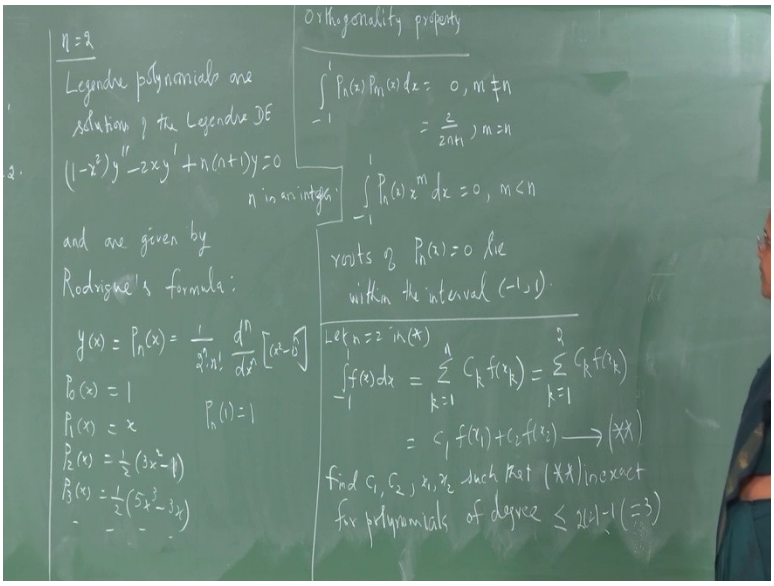

So here we determine these unknowns in such a way that the method is exact for polynomials of degree less than or equal to 2 into 2 minus 1 namely 3. So we know we have to workout the details.

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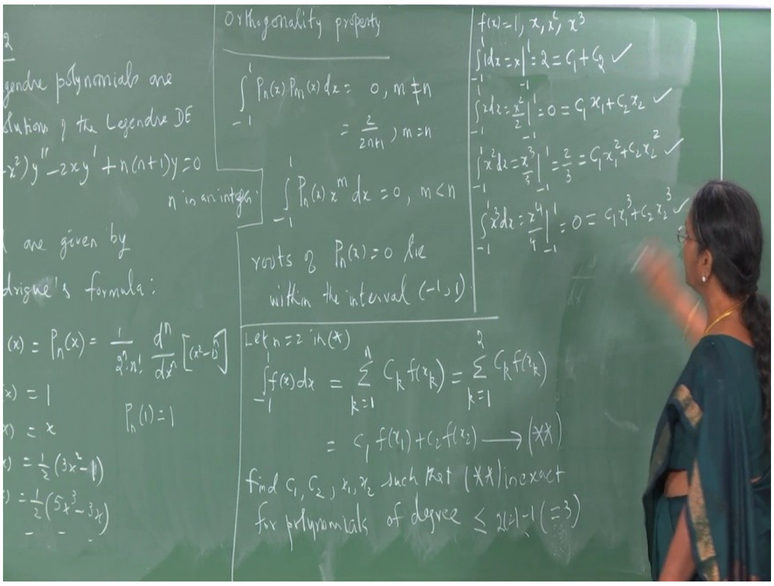

So we take f(x) to be equal to therefore 1 ,x,x square and x cube and substitute in double star and then see how you can make use of the properties of the Legendre polynomials and evaluate this integral so when I take f(x) to be 1 then I get it is x between minus 1 and 1 so that is 2 and that will be equal to C 1 into f(x 1) which is 1 plus C 2 which is f(x 2).

Then integral minus 1 to 1 x dx is x square by 2 between minus 1 and 1 and that will be 0 and that will give you C 1 into f (x 1) is x 1 C 2 into f(x 2) is x 2. Then integral minus 1 to 1 x square dx will be x cube by 3 between minus 1 and 1 so that will give you 2 by 3 into C 1 into x 1 square plus C 2 into x 2 square. Then minus 1 to 1 x cube dx will be x to the power of 4 by 4 between minus 1 and 1 and that again will be 0 and that is equal to C 1 into x 1 cube plus C 2 into x 2 cube. So we observe that we have 4 equations to determine the four unknowns C 1 C 2 x 1 and X 2.

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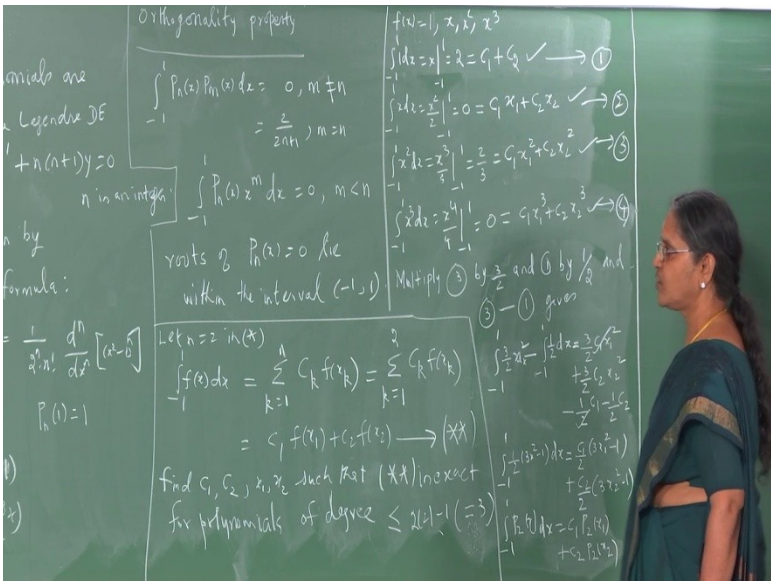

So at this stage we would like to bring in the knowledge of Legendre polynomials that we have and use thier properties. So what I do is suppose I call these equations as 1 2 3 4 I shall multiply equation 3 by 3 by 2. So you will understand why we do is so multiply equation 3 by 3 by 2 and equation 1 by half and resulting equation 3 minus 1 gives the following.

So let us write minus 1 to 1 3 by 2 x square that is the left hand side then I have to subtract half times the first equation so integral minus 1 to 1 half so integration with respect to x. And what does that give that gives you from here I multiplied the equation 3 by 3 by 2.

So I have 3 by 2 into C 1 x 1 square plus 3 by 2 C 2 x 2 square and I subtract it half times the first equation, so half C 1 minus half C 2. So left hand side is minus 1 to 1 half of 3x square minus 1 dx and the right hand side gives me C 1 by 2 into 3 x 1 square minus so have taken these two terms, plus c 2 by 2 into 3 x 2 square minus 1.

So now it must be clear to you what is it that we have here? we have integral minus 1 to 1 P 2(x) dx is equal to C 1 into what is half of 3 x 1 square minus 1 it is P 2(x 1) plus C two times again P 2(x 2).

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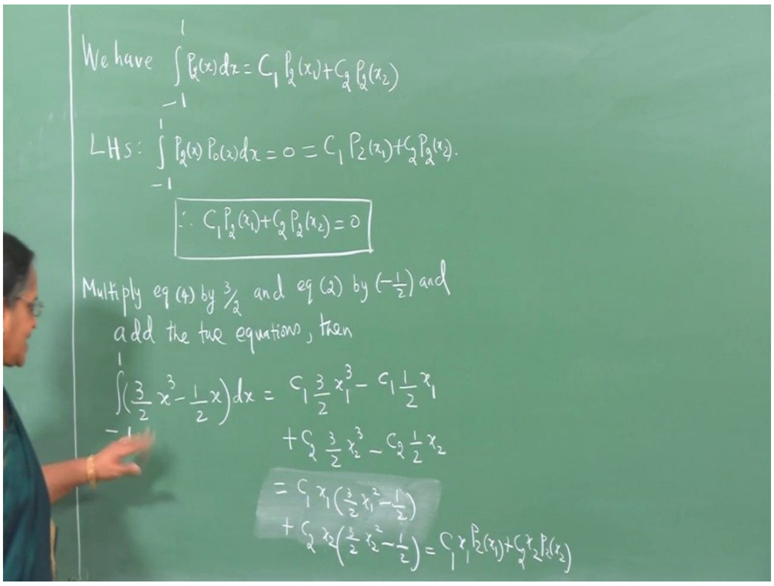

We have integral minus 1 to 1 P 2(x) dx as C 1 P 2(x 1) plus C 2 P 2(x 2). The left hand side is integral minus 1 to 1 P 2(x) dx which is P 2(x) into P 0 (x) dx and that is 0 by Orthogonality property of Legendre polynomials. So this will be C 1 P 2(x 1) plus C 2 P 2 (x 2) this gives therefore C 1 P 2(x 1) plus C 2 P 2(x 2) is 0. It is now derived another equation.

Suppose I multiply equation 4 by 3 by 2 and equation 2 by minus half and add the two equations then we get integral minus 1 to 1 3 by 2 x cube minus half x into dx is C 1 into 3 by 2 into x 1 cube minus C 1 into half x 1 plus C 2 into 3 by 2 x 2 cube minus C 2 into half x 2 this gives C 1 into x 1 into 3 by 2 x 1 square minus half) plus C 2 into x 2 into 3 by x 2 square minus half which is C 1 x 1 into half of 3 x 1 square minus 1 that is P 2(x 1) plus C 2 into x 2 into half of 3x 2 square minus 1 which is P 2(x 2).

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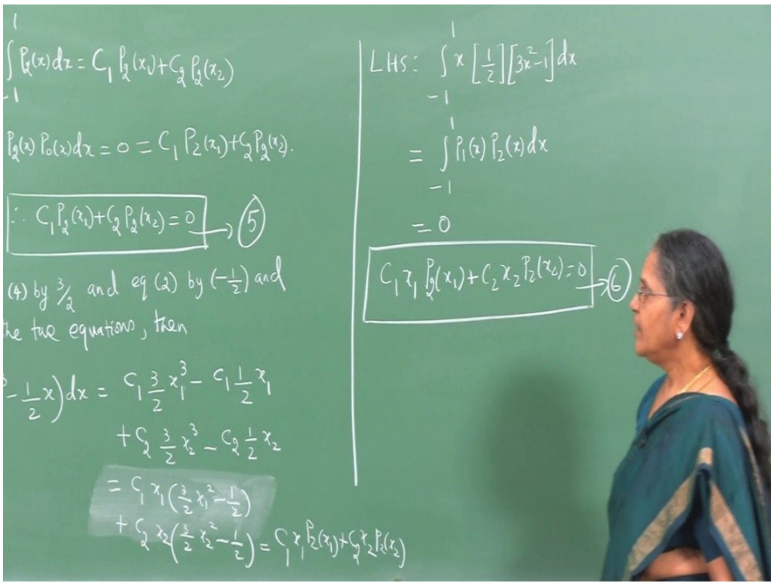

Now let us look at the left hand side so we have left hand side as integral minus 1 to 1 x into half of 3 x square minus 1 integrated with respect to x and that is integral minus 1 to 1 P 1(x) into P 2(x) dx which is again by the Orthogonality property of theLegendre polynomials 0. Next we have the second equation is to be given by C 1 into x 1 into P 2(x 1) plus C 2 into x 2 into P2(x 2) is 0. So let us call these equations as 5 and 6.

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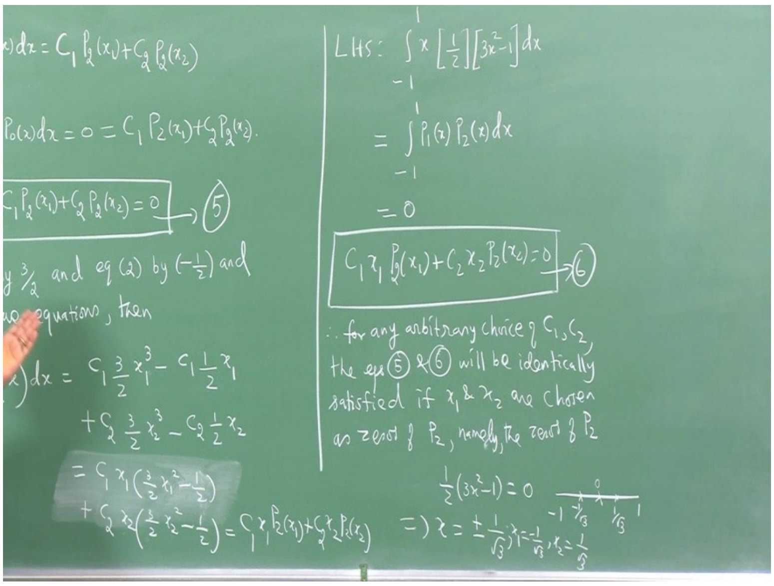

Now looking at equations 5 and 6 we observe that these two equations are identically satisfied for any choice of the constants C 1 and C 2. If we choose x 1 and x 2 to be the zeroes of P 2 namely the legendar polynomials of degree 2 in which case P 2(x 1) is 0 P 2(x 2) is 0 and therefore this equation will be identically satisfied and this will also be identically satisfied and for any arbitrary choice of C 1 C 2 equations 5 and 6 will be satisfied.

So therefore for any arbitrary choice of C 1 and C 2 the equations 5 and 6 will be identically satisfied if x 1 and x 2 are chosen as zeroes of P 2 namely the zeroes of half of 3x square minus 1 equal to 0. This is P 2(x) so if I consider the roots of this equation which are zeroes of P 2. Then they identically satisfy equation 5 and 6. So what are the zeroes they are given by x is equal to plus or minus 1 by root 3.

So we integrate between the limits minus 1 and 1 and the roots of P 2(x) is equal to 0 are given by x 1 is equal to minus 1 by root 3 and x 2 is equal to plus 1 by root 3. So these two nodes lie in the interval minus 1 to 1 and they are located symmetrically about the origin and both these nodes are interior points they lie within the interval minus 1 to 1. So we have determinded x 1 and x 2 the only problem that remains to be solved is determine the constants C 1 and C 2.

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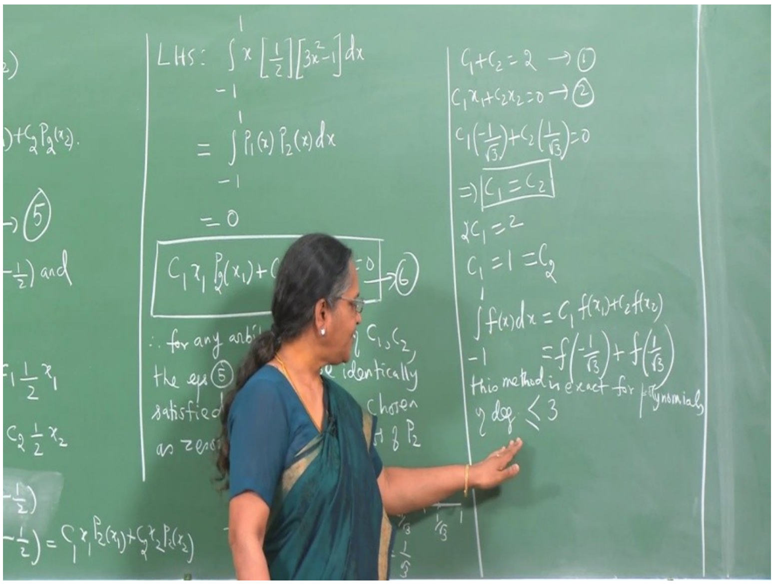

So we consider the first two equations mainly C 1 plus C 2 is equal to 2 C 1 x 1 plus C 2 x 2 equal to 0 these are the first two equations which we had written down. Now that I have determined x 1 and x 2 it is C 1 into (minus 1 by root 3) plus C 2 into (plus 1 by root 3) and that is 0 which shows that C 1 is equal to C 2. So substitute here that will give you C 1 is C 2 so C 1 is 1 but C 1 is C 2 so C 1 equal to C 2 equal to 1.

And therefore we have the integration method to be given by integral minus 1 to 1 f(x) dx is equal to C 1 f(x 1) plus C 2 f(x 2) C 1 is 1 x 1 is minus (1 by root 3) C 2 is 1 and x 2 is plus 1 by root 3 so knowing the function f(x) which is the integrant in the given integral. You evaluate f(minus 1 by root 3) and at(plus 1 by root 3).

Then the sum of these values will give an approximation to this integral and the method is exact for polynomials of degree less than or equal to 2n minus 1 where n is 2 here mainly for polynomials of degree less than or equal to 3, so this method is exact for polynomials of degree less than or equal to 3. There are only two nodes at which the function is evaluated. The method is exact for polynomials of degree less than or equal to 3.

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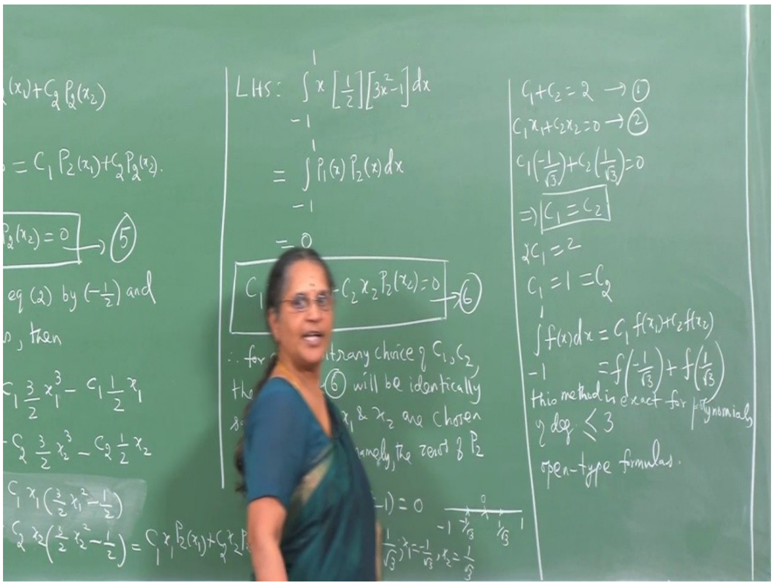

As we remarked earlier the two nodes lie within the interval and they are interior nodes and the method does not involve the function evaluation at the end points of the interval and therefore these methods are referred to as open type formulas which give you integration methods for evaluating a definite integral of the form minus 1 to 1 f(x)dx.

In Newton codes methods as we have discussed earlier for example the Trapezoidal rule Simpson's rule they both involved the ordinates of the function value at the end points of the interval they are referred to as closed type formulas whereas Gaussian Quadrature methods are open type formulas, since they do not involve function evaluation at the end points of the interval.

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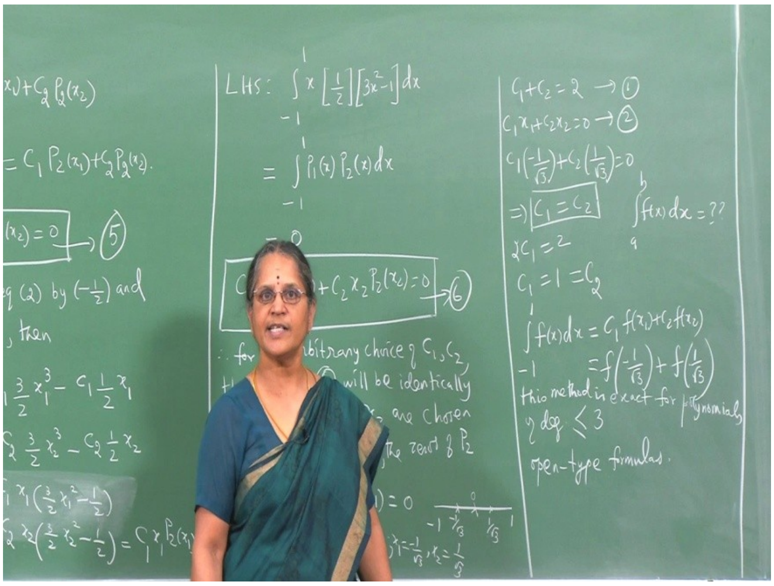

So we have derived two point Gaussian integration method given by this which requires the knowledge of roots of Legendre second degree polynomial so that once we know the function which has to be integrated between minus 1 and 1 evaluate this function at these two zeroes of P 2 or roots of equation P 2(x) is equal to 0.

And that will immediately give you the value of this definite integral. So usually we will be given an integral of the form integral a to b f(x) dx and we will be asked to evaluate this integral. Suppose that we are asked to evaluate such an integral using Gaussian quadrature method namely use two point Gaussian Quadrature technique to evaluate this integral we observe that we have derived this method where an integral of the form integral minus 1 to1 f(x)dx is expressed as f(minus 1 by root 3)plus f(1 by root 3).

But the given problem involves limit as a and b where a and b need not be minus 1 and 1respectively so we have to convert this integral a to b f(x)dx to an integral of the form integral minus 1 to 1 f(x)dx. So we consider the transformation which converts integral between a and b 2 and integral between minus 1 and 1. So let us see how we can do this.

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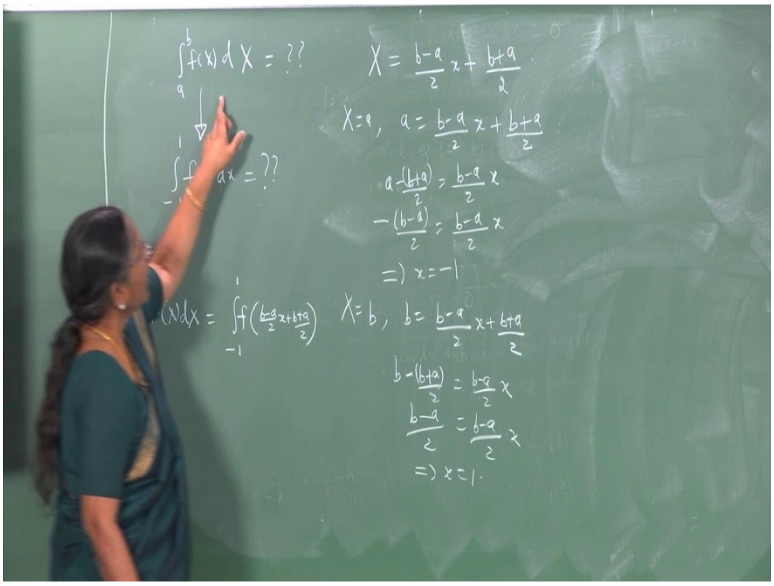

So let us consider integral a to b f(x) dx, so I would like to convert this integral to the form integral minus 1 to 1 say f(x)dx and then evaluate it using Gaussian Quadrature so we make use of this transformation x is b minus a by 2 into x plus b plus a by 2. Let us see whether it works. So when x takes the lower limit value a then we have a equal to b minus a by 2 into x plus b plus a by 2. So we have a minus b plus a by 2 is b minus a by 2 into x.

So we have a minus a by 2 minus b by 2 which is minus of b minus a by 2 and that is minus b minus a by 2 into x which tells that x takes the value minus 1. So when capital x is a small x takes the value minus 1. What happens when x takes the value b that gives you b minus b plus a by 2 is b minus a by 2 into x. So b minus b by 2 is b by 2 minus a by 2 that is the left hand side and the right hand side is b minus a by 2 into x. So that shows that x takes the value 1. So therefore when captial x varies between a and b small x varies between minus 1 and 1.

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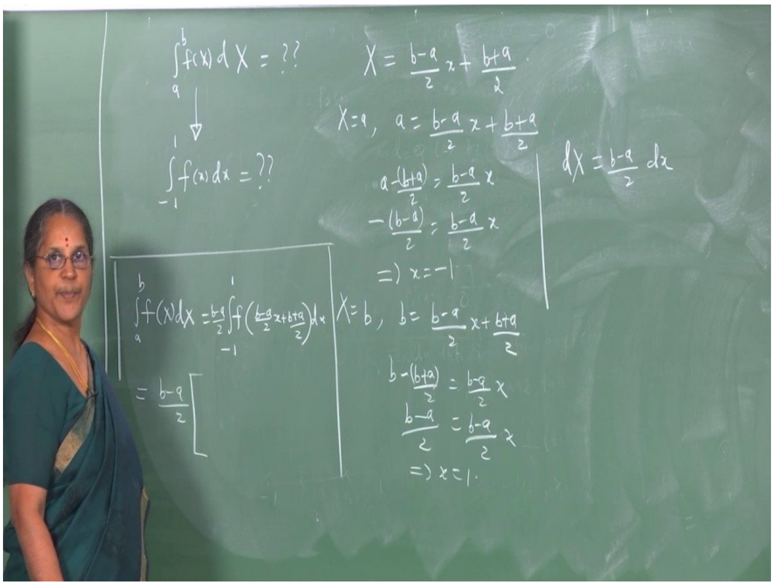

And therefore this integral, integral a to b f(x) dx will be equal to under this transformation minus 1 to 1 f of what is capital X, capital X is (b minus a by 2 into small x plus b plus a by 2 and what is b capital X, b capital X from here is b minus a by 2 into dx. So b minus a by 2 into dx. So if I have integral a to b f(X) dx b minus a by 2 integral minus 1 to 1 f evaluated at this node,into dx. So suppose I apply Gaussian Quadrature with n is equal to 2 and evaluate this integral what is it b minus a by 2 into minus 1 to 1 f of this integration with respect to x.

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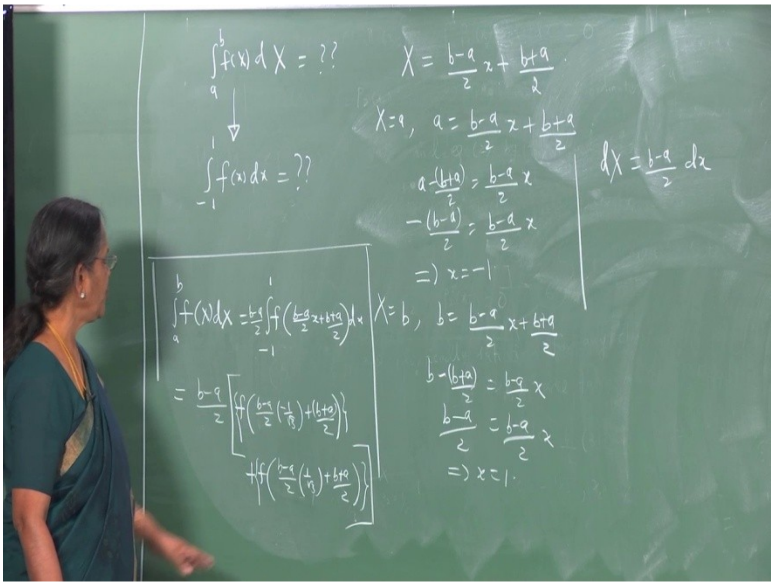

So that gives you, you should take the node at minus 1 by root 3. So at x is equal to minus 1 by root 3 the function must be evaluated. So that gives you this function evaluated at b minus a by 2 into minus [(1 by root 3) plus b plus a by 2 that is the first term and then the next term will be f(evaluated at 1 by root 3 so b minus a by 2 into (1 by root 3) plus b plus a by 2 that is going to be the second term. So this integral has been evaluated using Gaussian Quadrature you know a b etc because the upper and the lower limits are known and therefore substitute those values and this will give you the value of this integral between a and b.

So let us take example and illustrate this.

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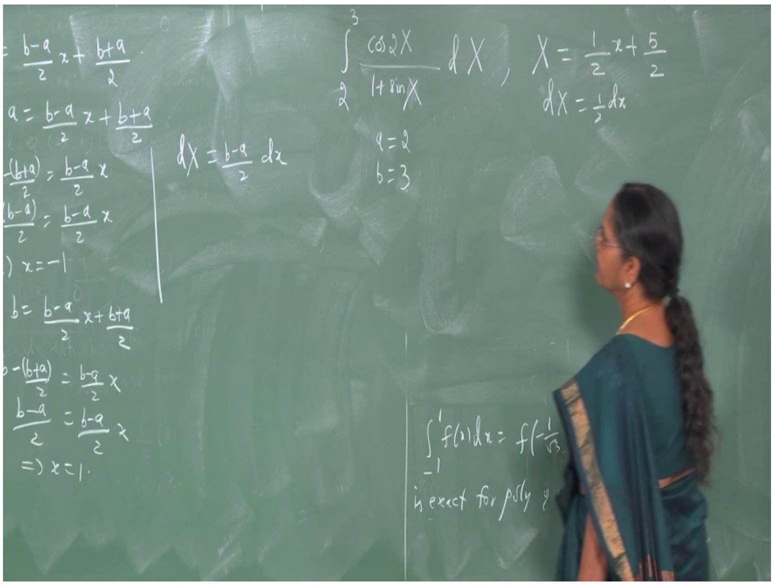

Evaluate integral minus 1 to 1 or let us do for a general integral of the form a to b so integral 2 to 3 cos 2X by 1 plus sinX integration with respect to x using Gauss quadrature method then is equal to 2. So we call this as Gauss two point Quadrature formula. So evaluate this integral using Gauss two point integration method. So we write down the transformation X is b minus a so a is 2 b is 3 so b minus a so 1 by 2 into x plus b plus a so 2 plus 3 so 5 by 2. So this is the transformation and dX is half dx.

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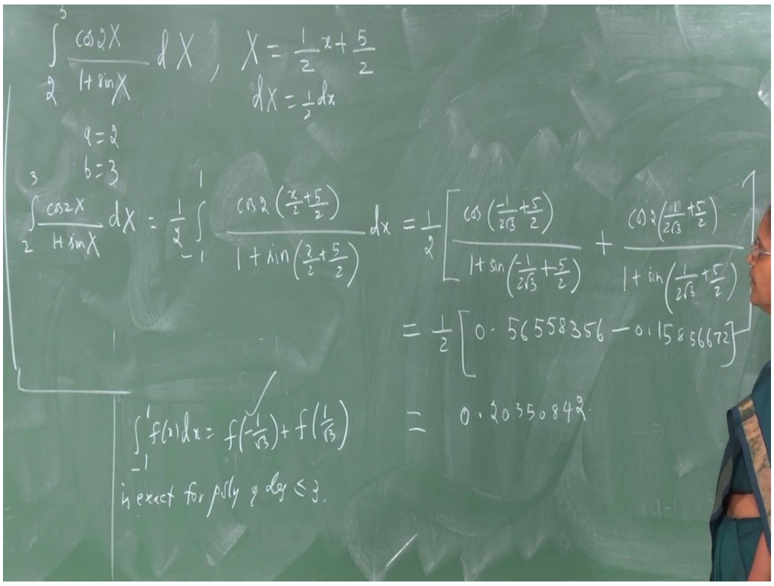

So by that formula integral 2 to 3 cos 2X by 1 plus sin X dx will be integral minus 1 to 1 f (cos 2 what is x (x by 2 plus 5 by 2) divided by 1 plus sin (x by 2 plus 5 by 2) into dX that is half of dx. So now we would like to evaluate this using two point integration method so what does it say? It simply says get the function value at my x is equal to minus 1 by 3.

So the integral give you half of cos (x which is minus 1 by root 3) divided by 1 plus sin (x is equal to minus 1 by root 3). So (minus 1 y 2 root 3) plus 5 by 2. So we have evaluated f(minus 1 by root 3) then the second term at x is equal to 1 by root 3 this should be evaluated so cos (x 1 by root 3 plus 5 by 2) divided by 1 plus sin (1 by root 3 plus 5 by 2).

So both the ordinates have been computed and this gives you the value of the integral. So you can use your calculator and then show that it is equal to half of 0.56558356 minus 0.15856672 and when you simplify you get the result 0.2350842. So using 2 point integration method we have been able to evaluate this integral.

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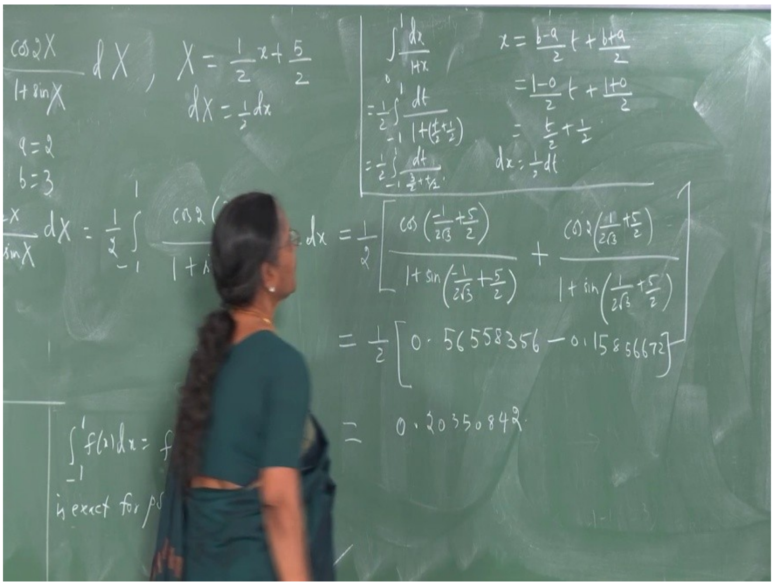

Let us take another example so evaluate integral 0 to 1 f(x)dx where f(x) is 1 by 1 plus x. So the interval a to b is now 0 to 1. So I have to make a transformation so that this integral is converted to an integral of the form minus 1 to 1 in order that I am able to apply Gauss 2 point integration method.

So I make the transformation x is equal to b minus a by 2 into t plus b plus a by 2 that is 1 minus 0 by 2 is 0 plus 1 plus 0 by 2 so it is t by 2 plus half. So that dx is half dt. So what is the value of this integral it is half integral minus 1 to 1 right? dx is half dt divided by 1 plus what is x t by 2 plus half. So it is half of integral minus 1 to 1 dt by 3 by 2 plus t by 2.

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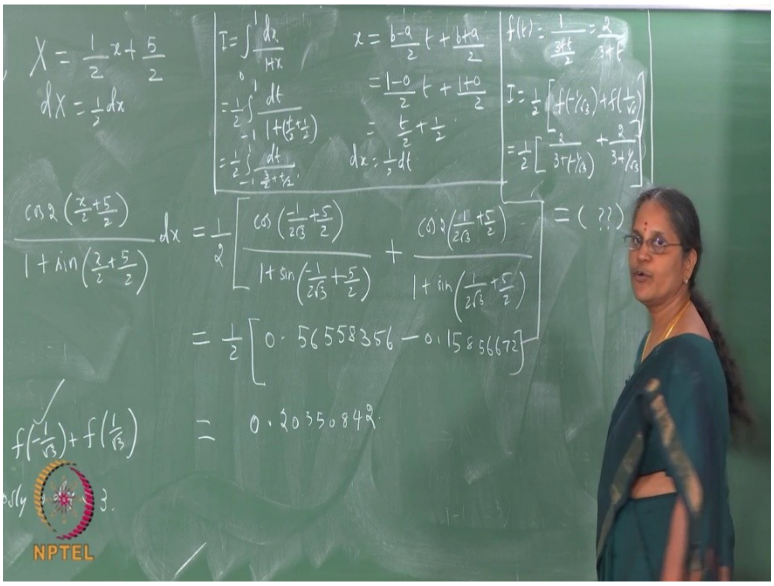

So what is my function , my function here is f(t) and that is 1 by 3 by 2 or 3 plus t by 2 or 2 by 3 plus t that is my f(t). So this integral if I call this say as i then integral i will be half of f evaluated at minus (1 by root 3) plus f evaluated at plus (1 by root 3) that is Gauss 2 point method so that will give you half of we know what the function is 2 by 3 plus t is minus 1 by root 3 plus 2 by 3 plus t is plus 1 by root 3].

So simplify this and write down what is the value of the integrals. So the natural interest is to see what happens if you increase this n namely immediate curiosity is to see what is going to be Gauss 3point integration Quadrature method so we shall derive Gauss 3 point integration method in the next lecture. Thank You!