Numerical Analysis
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Lecture 13
Numerical Integration 2
Error in Trapezoidal Rule
Simpson's Rule

In todays lecture we shall see how we can develop some numerical integration methods

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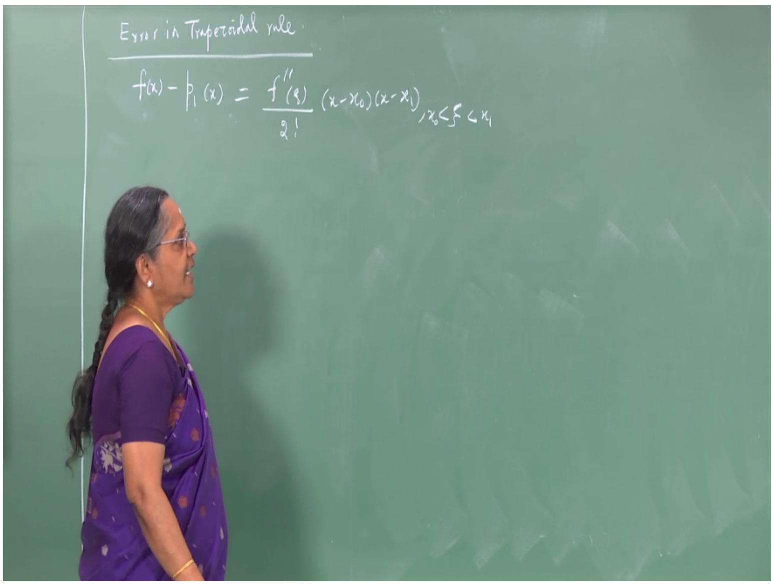

So we work out the details for error in Trapezoidal rule. So what is the error in f(x) minus p 1(x) and we have already obtained the expression for error in interpolation at any x which belongs to the interval x 0 to x 1. So what is the error in the interpolation it is f(x) minus p 1(x) and that is equal to the second derivative evaluated at Psi divided by 2 factorial multiplied by (x minus x 0) into (x minus x 1) for Psi lying between x 0 and x 1.

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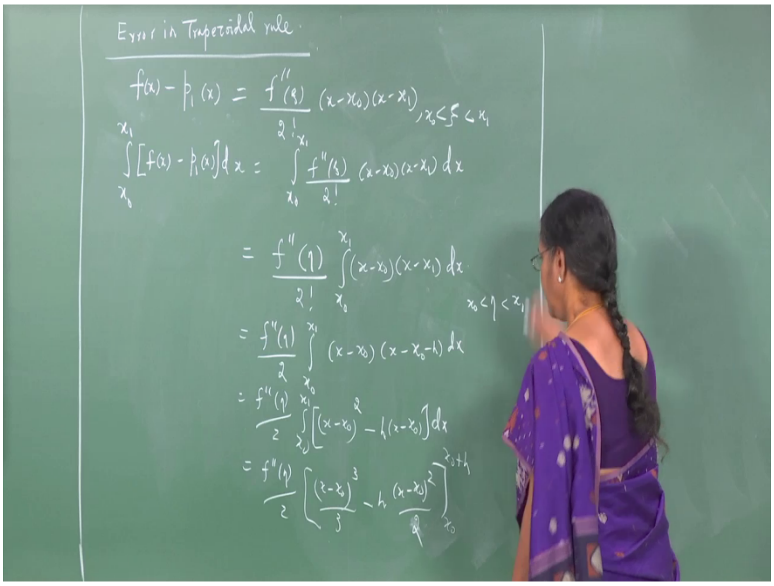

And therefore error in this evaluation of the definite integral will be obtained by integrating the error in interpolation between x 0 and x 1. So p 1(x) dx will be equal to integral x 0 to x 1 f double dash (Psi) by 2 factorial into (x minus x 0) into)(x minus x 1) into dx. So by using mean value theorem for integral calculus we have f double dash at some(Eta) by 2 factorial integral x 0 to x 1 x minus x 0 into x minus x 1 into dx.

So eta is now in x 0 to x 1. So this will give you f double dash (eta) by factorial 2 into let us evaluate this so x 0 to x 1 (x minus x 0) (x minus I use the fact that x 1 is x 0 plus h) dx. So that gives you f double dash (eta) by 2 into x minus x 0 the whole square minus h (x minus x 0) and that must be integrated with respect to x.

So by 2 into x minus x 0 the whole cube by 3 minus h into (x minus x 0) the whole square by 2] between the limits x 0 and x 0 plus h.

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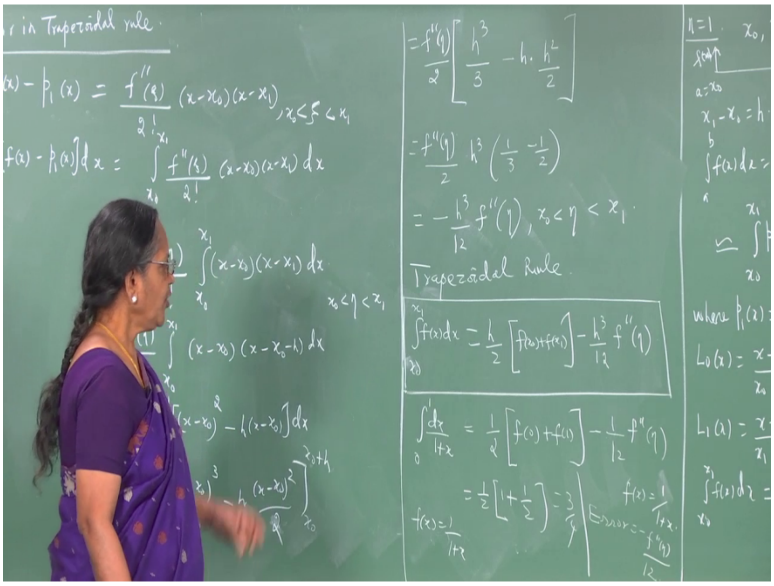

So this is f double prime (eta) by 2 into at the upper limit x 0 plus h this will give you x 0 plus h minus x 0. So h cube by 3 and at the lower limit this will vanish and the next term minus h into at the upper limit x 0 plus h this will give you h square by 2] and at the lower limit this will vanish. So this will give f double prime (eta) by 2 into h cube into (1 by 3 minus 1 by 2) so that will give you minus h cube by 12 into f double dash (eta) eta lying between x 0 and x 1.

So error in Trapezoidal role is this where h is the step size namely the width of the interval.. So you now can write down the Trapezoidal role as if you want integral x 0 to x 1 f(x) dx then it is h by 2 into f(x 0) plus f(x 1) plus the error namely minus h cube by 12 into the second derivative (eta).

So if I ask you to evaluate integral say 0 to 1 dx by 1 plus x by Trapezoidal rule, then it is simply h is 1 because b minus a into [f(0) plus f(1)] and by doing this you have made an error of h is 1 so 1 by 12 into f double prime (eta) where f(x) is 1 by 1 plus x.

So this essentially will give you half of f(0), f(x) is 1 by 1 plus x so f(0) is 1 plus f(1) is half so the result is 3 by 4 then the error committed is minus f double prime (eta) divided by 12. So if you want the bound on the error then it will be the absolute value of f double prime (eta) by 12 knowing f(x) which is 1 by 1 plus x you can compute the second derivative.

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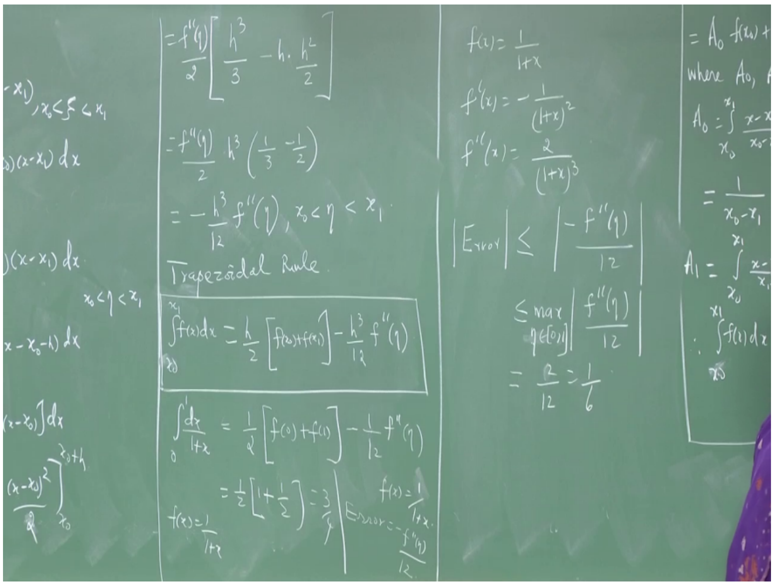

So the second derivative will be f(x) is 1 by 1 plus x f dash (x) will be minus 1 by (1 plus x) the whole square and f double dash(x) will be 2 by (1 plus x) the whole cube. So if I want the bound on this error then this will be minus f double prime(eta) by 12 and so I observe that this will be less than or equal to the maximum of f double prime (eta) by 12 and so for maximum I take this eta belonging to the interval 0 to 1. Having computed the second derivative I observe that the maximum will occur at x is equal to 0 and therefore it is 2, so the result is 2 by 12 and so 1 by 6. So the error is always bounded by 1 by 6 in this approximation of the interval which is given by this.

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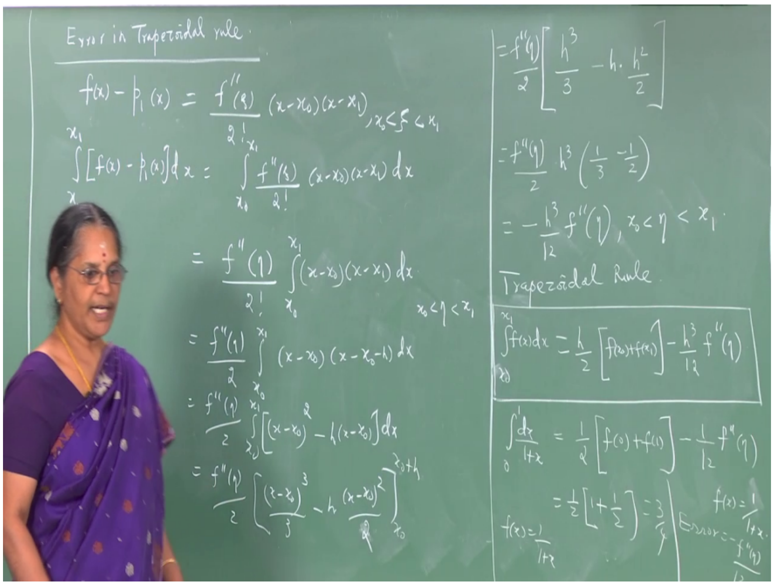

So this method was obtained by approximating f(x) by a linear polynomial, So we move ahead to see what happens if we approximate f(x) by a second degree polynomial namely a quadratic interpolation polynomial the resulting method that we get is called Sampson's rule. So we shall derive now Simpson's rule.

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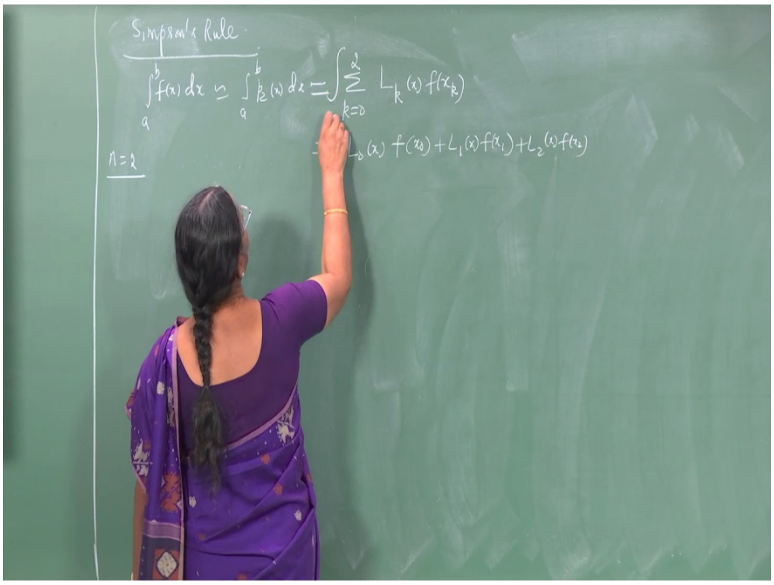

The method is obtained by approximating the function f(x) which has to be evaluated in this interval by a polynomial of degree 2. So I take my n to be 2 because I am interested in deriving Newton codes method. So I would like to get a method which is of the form sigma k is equal to 0 to 2 L k (x) into f(x k). So this will be L 0 (x) into f(x 0) plus L 1(x) into f(x 1) plus L 2(x) into f(x 2).

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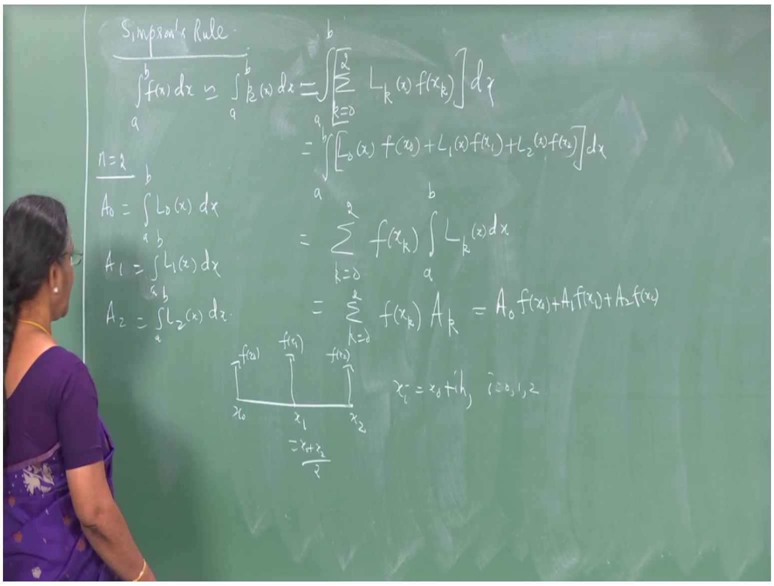

So I have to integrate this with respect to x, so integral of this with respect to x, so which will give me sigma f(x k) into integral a to b L k (x) dx for k is equal to 0 to 2. So this will be k equal to 0 to 2 f(x k), I shall call this integral L k(x) g(x) to be A k. So I am interested n getting the constants A 0, A 1, and A 2 in this formula where A 0 will be equal to integral a to b L 0(x) dx, A 1 is integral a to b L 1(x)dx, and A 2 is integral a to b L 2(x)dx.

So I require three points x 0, x 1 x 2 and we are deriving Newton codes method. So the points must be equally spaced so I shall take the points x 0 x 1 and x 2 to be such that x i is x 0 plus ih for i is equal to 0,1,2. So my x 1 is nothing but x 0 plus x 2 by 2 the mid point of the interval. So I evaluate the ordinates at these points. So I have the information about the ordinates because I know what f(x) is? If I evaluate A 0 A 1 A 2 then I will have the formula where f(x) is approximated by a quadratic interpolation polynomial.

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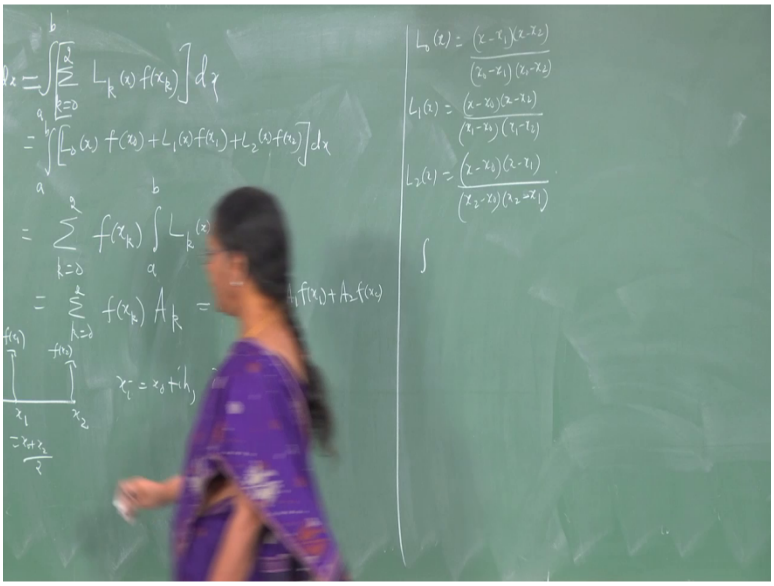

So we write down what these L i(x) are so L 0(x) is equal to (x minus x 1) into x minus x 2) by (x 0 minus x 1) into (x 0 minus x 2). L 1(x) will be (x minus x 1) into (x minus x 2) by (x 1 minus x 0) into (x 1 minus x 2) and L 2(x) is (x minus x 0) into (x minus x 1) by (x 2 minus x 0) into (x 2 minus x 1). So I have integral a to b what is my a?

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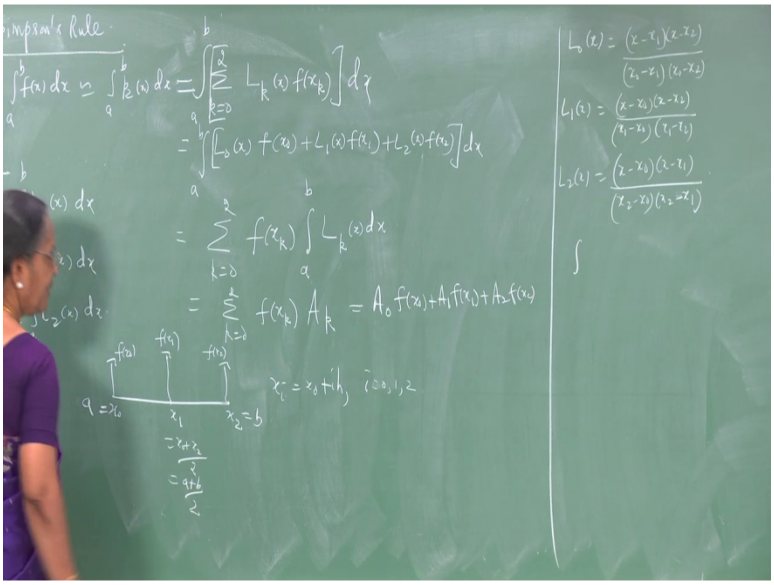

a is x 0 and b is x 2 and the midpoint is a plus b by 2.

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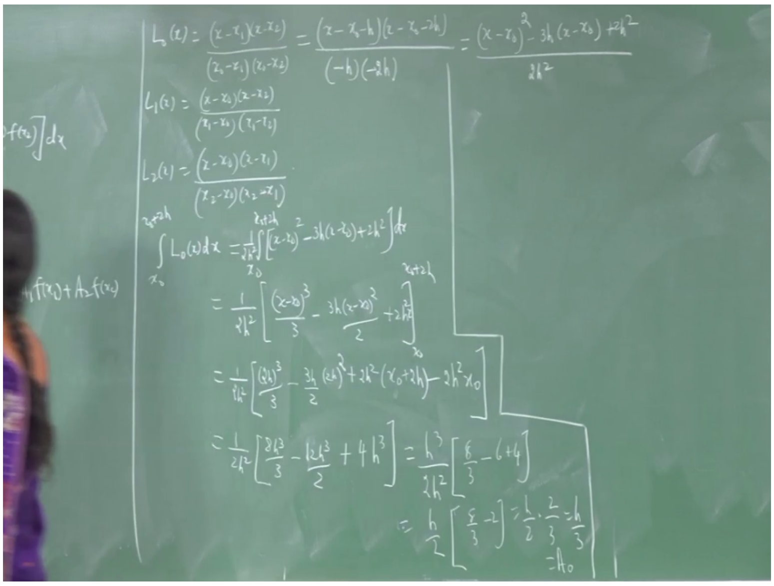

So I require integral x 0 to x 0 plus 2h of L 0 (x) dx. So I can rewrite this as x minus x 0 minus h)into (x minus x 0 minus 2h) by (x 0 minus x 1) is (minus h), (x 0 minus x 2) is (minus 2h), so I have (x minus x 0) the whole square then minus 3h into (x minus x 0) plus 2h square divided by 2h square.

So this will give me the integral x 0 to x 0 plus 2h (1 by 2h) square into (x minus x 0) the whole square minus 3h into (x minus x 0) plus 2h square integrated with respect to x. So you have h cube by 2h square into [8 by 3 minus 6 plus 4] so h by 2 into[8 by 3 minus 2] so h by 2 into [2 by 3] so you end up with h by 3 that is integral L 0 (x) dx So similarly we must evaluate integral L 1 (x) dx and L 2(x) dx. And what is it that we have got we have obtained a value of the constant A 0 which is integral L0(x) dx.

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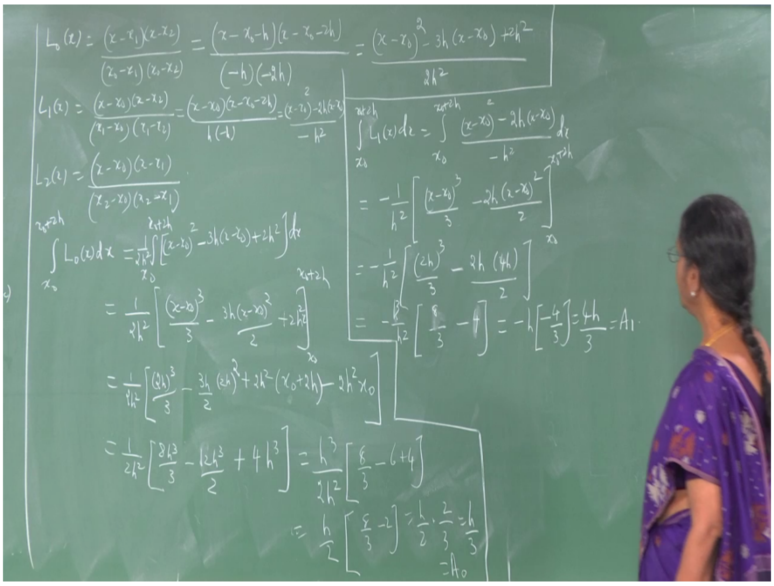

So let us now compute A 1, so let us simplify L 1(x) so x is s minus x 0) into (x minus x 0 minus 2h) by (x 1 minus x 0) is h (x 1 minus x 2) is (minus 2h) so this gives you (x minus x 0) the whole square (minus 2h) into (x minus x 0) by (minus h) square. So integral x 0 to x 0 plus 2h of L 1(x) dx will be integral x 0 to x 0 plus 2h x minus x 0 the whole square minus 2h into x minus x 0 by minus h square into dx.

So minus 1 by h square into let us evaluate x minus x 0 the whole cube by 3 minus 2h into x minus x 0 the whole square by 2 between x 0 and x 0 plus 2h. So that gives you minus 1 by h square at the upper limit it is 2h the whole cube by 3 minus 2h into 2h the whole square by 2. So that gives you minus h into minus 4 by 3 and so the constant A 1 is 4h by 3. Similarly we evaluate integral L 2(x) dx.

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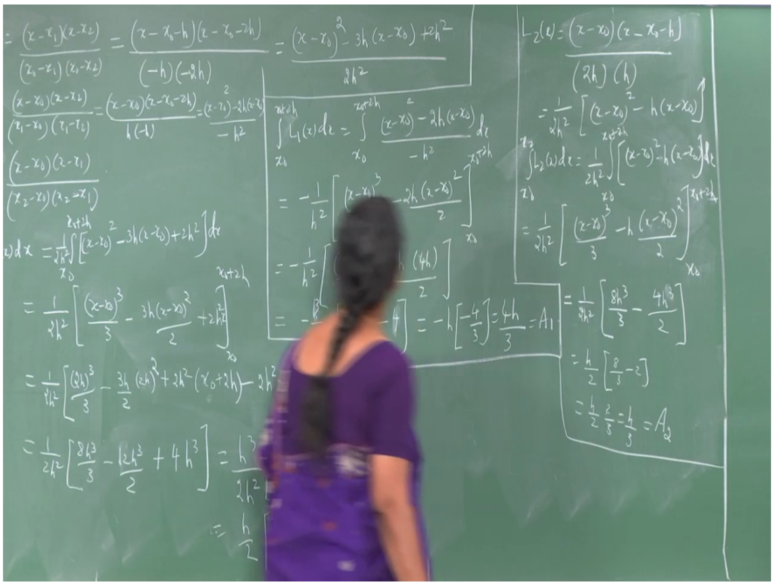

So let us simplify L 2(x) dx which is x minus x 0 into x minus x 1 so x 0 minus h x 2 minus x 0 which 2h x 2 minus x 1 is 2h square into x minus x 0) the whole square minus h into x minus x 0. So integral x 0 to x 2 L 2(x) dx will be 1 by 2h square integral x 0 to x 0 plus 2h (x minus x 0) the whole square minus h into x minus x 0 integrated with respect to x. So this will give you 1 by 2h square into first term integrated will give you x minus x 0 the whole cube by 3 minus h into x minus x 0 whole square by 2 between x 0 and x 0 plus 2h.

So applying the limits of integration so 1 by 2h square (x 0) this will give you 2h the whole cube so 8h cube by 3 this will give you x minus x 0 will be x 0 plus 2h minus x 0 so 2h the whole square divided by 2 into h so this will be h cube at the lower limit they vanish so this gives you h cube 2 h square h by 2 into 8 by 3 minus 2 so h y 2 into 2 by 3 and the constant is h by 3 so what we have evaluated is A 2. So we have the constants A 0 A 1 and A 2 determined.

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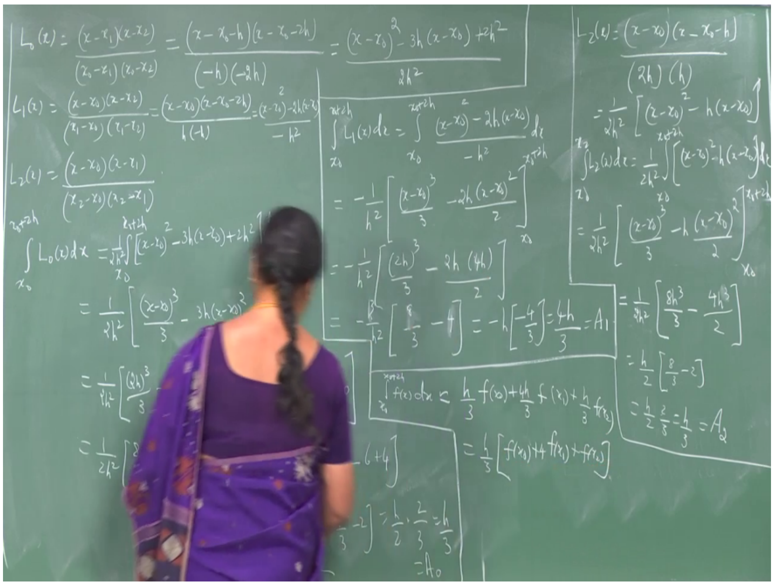

So we write down the method which says integral x 0 to x 0 plus 2h f(x) dx is A 0 which is h by 3 into f(x 0) plus A 1 which is 4h by 3 into f(x 1) plus A 2 which is h by 3 into f(x 2) and so this gives you h by 3 into f(x 0) plus 4 f(x 1) plus f(x 2). So let us see what this formula tells us.

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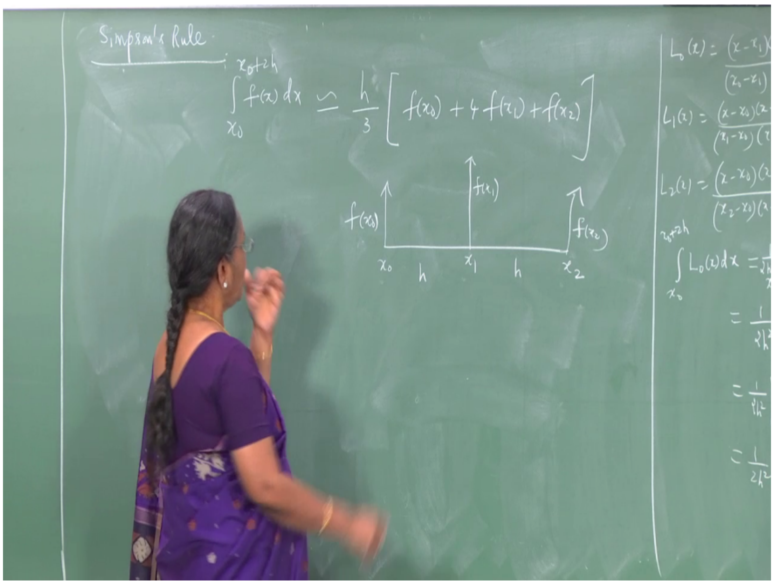

If we want to evaluate integral f(x) dx between x 0 to x 0 plus 2h then it is h by 3 times f(x 0) plus 4 times f(x 1) plus f(x 2). So namely if x 0 x 1 x 2 are equally spaced such that the step size the distance between any two points is h then the formula tells this integral has a value which is approximate and is given by 1 third of this h multiplied by the sum of the n ordinates namely f(x 0) plus f(x 2) plus 4 times the ordinate at (x 1) so 4 f(x 1) it is easy to remember applying Simpson's rule it is equivalent to getting the sum of the n ordinate values plus 4 times the intermediate ordinate namely the ordinate the midpoint of the two end points and multiplying it by h by 3. What does it actually mean let us see geometrically?

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So we have in the xy plane a curve y is equal to f(x) and we are asked to integrate between x 0 to x 0 plus 2h namely we are asked to get the value of I mean we are asked to find the area bounded by y is equal to f(x) the ordinate at (x 0) and x 2 and the x axis between x 0 and x 2. So what did we do? We approximated this function f(x) by a parabola passing through the points x 0 x 1 and x 2.

So we have a quadratic interpolation polynomial namely a parabola passing through these three points so that this parabola approximates this curve which is equal to f(x). Alright and we have evaluated the area which is now this bounded by the parabola which is this the axis at that point clear? Our parabola is here this is our parabola passing through the three points right?

So the shaded region is area that we have computed so in doing that we have committed an error namely this part of the area has been omitted and this part of the area has been included and so some amount of error has incurred in obtaining this integration method namely in Simpson's rule.

So our goal now is to see how much of error is incurred in obtaining Simpson's rule and ask before we can get this error by integrating the corresponding expression for error in interpolation.

So we observe that we have derived Newton codes formulas by approximating a function in an interval by a linear polynomial and we obtained Trapezoidal rule and by a quadratic interpolation polynomial and we obtained Simpson's rule and these rules are valid for the case when the nodes are equally spaced and geometrically they represent the corresponding areas bounded by a Trapezium in the case of Trapezoidal rule and the area bounded by a parabola passing through the three points x 0, f(x 0) x 1 f(x 1) and x 2 f(x 2) and the ordinates of the end points and the segment along the x axis between the end points.

That is the geometric way of looking at this Newton codes formulas where n is equal to 1 and n is equal to 2. So one thing that remains to be done is computation of the error in Simpson's rule method for evaluating and integral.