Numerical Analysis
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Lecture - 9
Inverse Interpolation, Remarks on Polynomial Interpolation

In last class we started our discussion on inverse interpolation. In inverse interpolation you have to determine the value of the dependent variable x when the corresponding value of the independent variable f(x) is prescribed. So we wanted to give an example and understand this.

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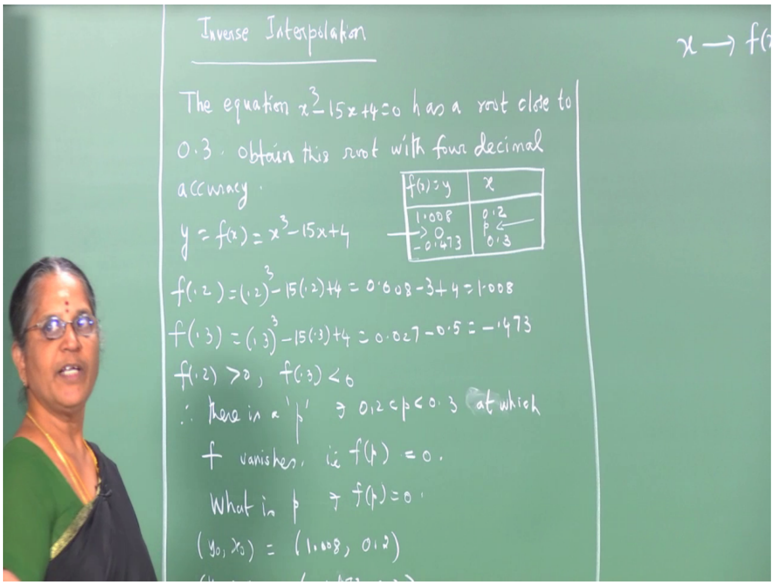

So we started with this problem namely equation has a route close to 0.3 and obtain this route with 4 decimal accuracy inverse interpolation is useful in solving non linear equation of this form f(x) is equal to 0. So let us see how inverse interpolation helps us to obtain this route which is near to 0.3 connect to 4 decimal accuracy.

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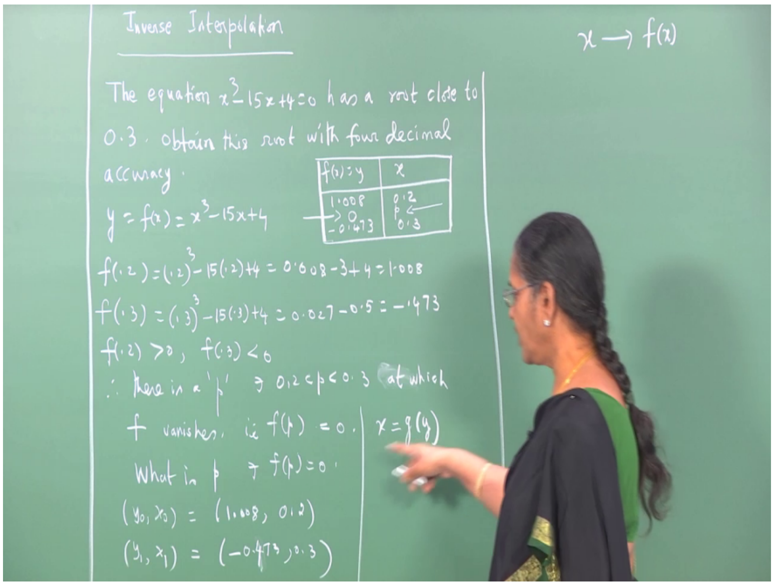

So we took two points x is equal 2.3 close to this 0.3 and then evaluated the function it assumes positive value at x is equal to 0.2 and it is negative at x is equal to 0.3, so there is a change in sign and so there is a route of this equation f(x) is equal to 0 between 0.2 and 0.3.

So our goal therefore is to find that p which lies between 0.2 and 0.3 at which f vanishes. Namely f(p) equal to 0 so we now interpolate on the nodes y 0 y 1 and determine a linear interpolating polynomial that passes through the points y 0 x 0 y 1 x 1. So we determine an interpolating polynomial x is equal to g(y) that interpolates this function at a set of these two points y 0 x 0 y 1 x 1.

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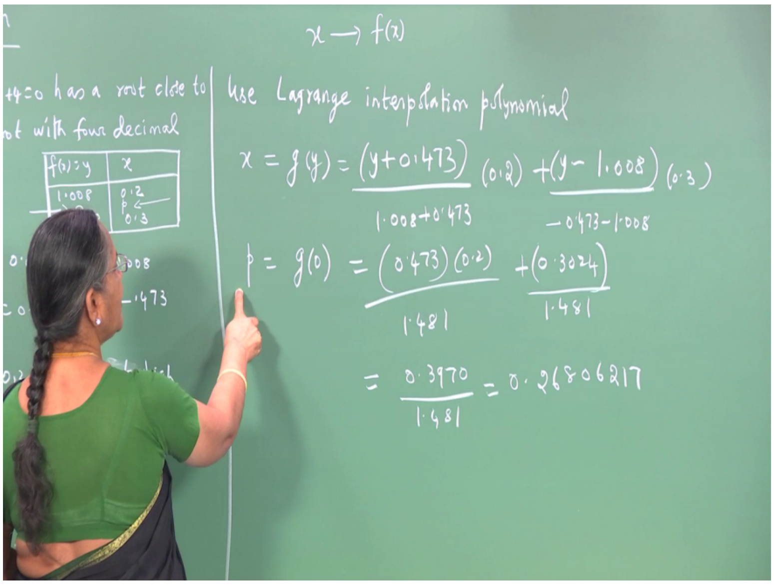

So we use lagrange linear interpolation polynomial and write down the interpolation polynomial. So use lagrange interpolation polynomial and write down this polynomial say x is equal to g(y) so that will be y plus 0.473 by 1.008 plus 0.473 multiplied by the value at y 0 namely x 0. And the next term will be y minus 1.008 by minus 0.473 minus 1.008 into x 1 which is 0.3.

So we have the linear interpolation polynomial g(y), so we want to find out that x which we call as p such that g(0) is going to be this p. So this gives you 0.473 into 0.2 by 1.481 plus 0.3024 by 1.481, so that gives you 0.3970 by 1.481 namely 0.26806217 after the decimal place. So we have been able to obtain an estimate p for root of the equation that lies between 0.2 and 0.3.

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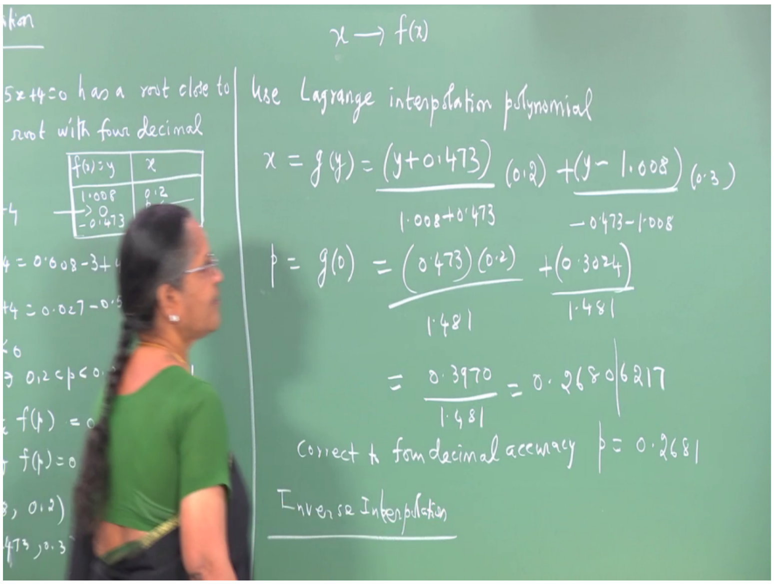

So correct to 4 decimal accuracy this result will be given by p equal to 0.2681 because in the fifth decimal place we have a digit which is more than 5.What we have done is known as inverse interpolation. So let us understand what essentially that we have done.

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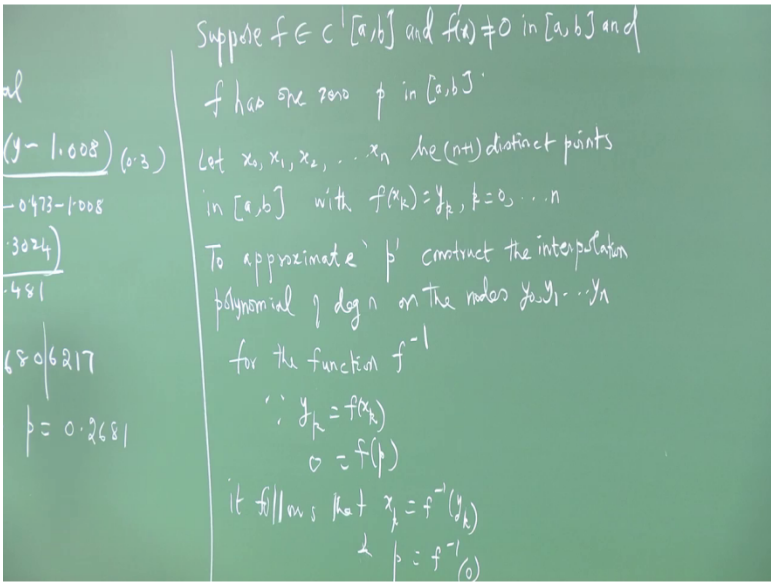

So suppose say our function f is in c 1 of a b and f prime (x) is different from 0 in that interval say on the interval [a,b]   and f has 10 say p in the interval [a,b]. So we take points x 0, x 1,etc ,x 2, x n namely n plus 1 distinct points. In the interval [a,b] with f(x k) as y k from k is equal to 0, 1,2,3 upto n.

So we need to approximate p so to approximate p we construct the interpolation polynomial of degree n on the nodes y 0 y 1 y 2 etc y n for the function which is f inverse. So since y k is f(x k) under 0 is f(p) it follows that x k will be f inverse of y k and p is f inverse of 0.

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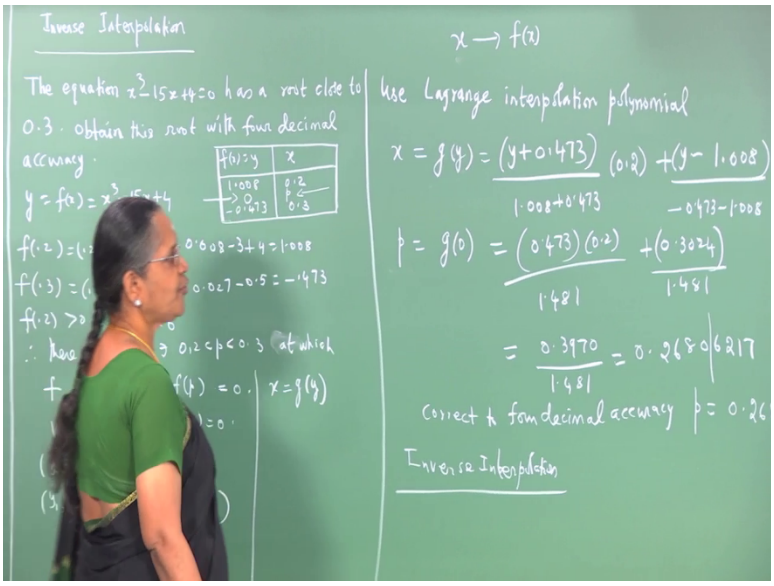

So this is what we have done while computing an estimate of route of this equation which lies between 0.2 and 0.3.

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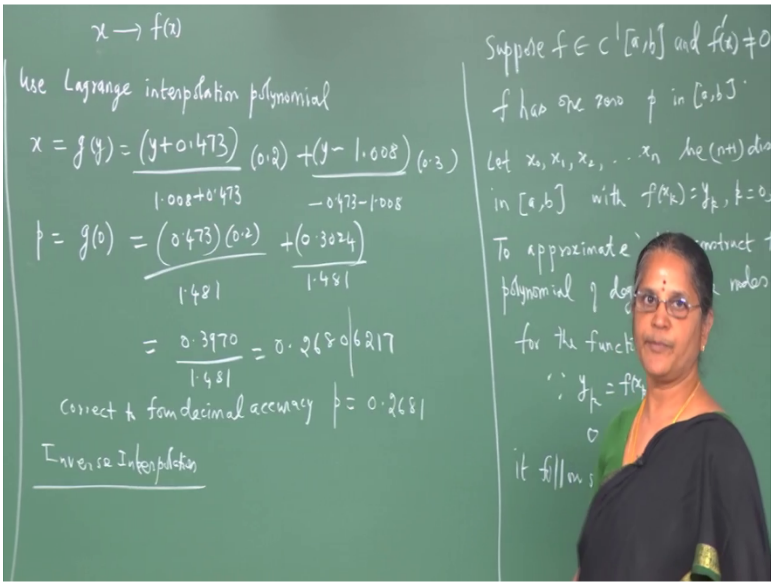

So we compute the interpolation polynomial on the nodes y 0 y 1 etc y n such that f (x k) is y k and determine that x namely p at which y turned out to be 0. So inverse interpolation ideas are very useful in solving non linear equations numerically. We would like to see how much of error is incurred in linear inverse interpolation.

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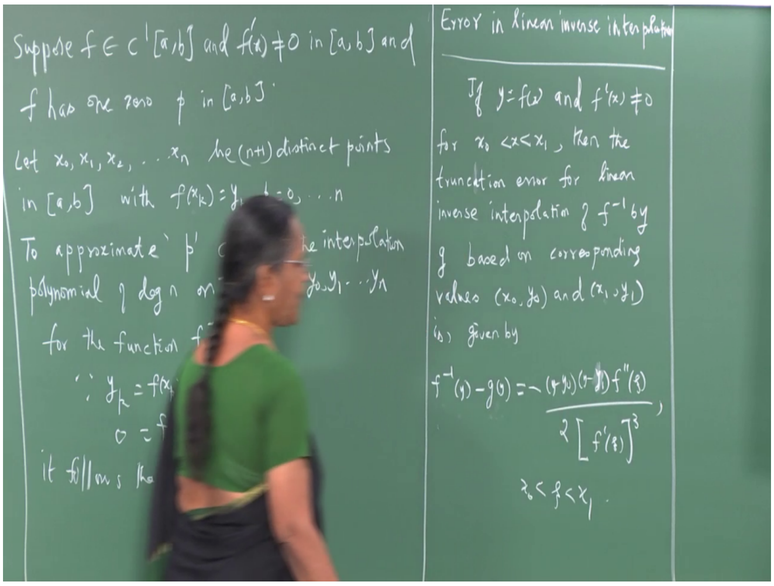

So let us present the result first and then obtain the expression for error in linear inverse interpolation. So the result says that if y equal to f(x) and f prime (x) is different from 0 for x lying between x 0 and x 1, then the truncation error for linear inverse interpolation of f inverse y g based on corresponding values x 0 y 0 x 1 y 1 is given by f inverse of y minus g (y) is y minus y 0 into y minus y 1 into f double dash (psi) by twice [f prime(psi)] to the whole cube where psi lies between x 0 and x 1.

So the left hand side gives the error in interpolation namely inverse linear interpolation where you reconstruct the function f inverse by an interpolation polynomial g(y) of degree 1 when the information is given to you at the points x 0 y 0 x 1 y 1 you obtain the linear interpolation polynomial on the nodes y 0 y 1 and that gives you g(y), and that reconstructs the function f inverse in this interval x 0 to x 1. The error that is incurred at any y in that interval is given by the expression on the right hand side.

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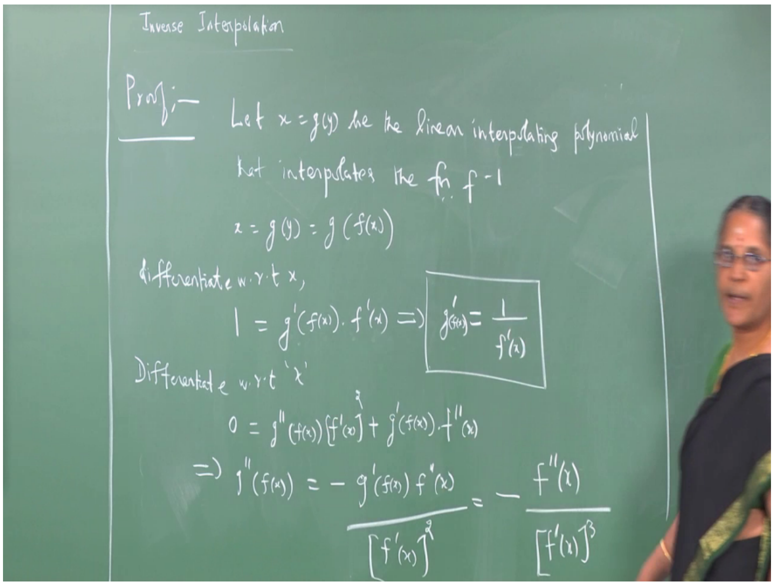

So let us give the proof of this result. So let x is equal to g(y) be the linear interpolating polynomial that interpolates the function f inverse so s is g(y) f(x) because y is equal to f(x) with the condition that f prime x is different from 0 in the interval [x 0 to x1]. So let us differentiate with respect to x. Then I have 1 as g prime (f(x)) into f prime (x) this gives me g prime to be equal to 1 by f prime(x) so g prime (f(x)) is 1 by f prime (x).

So let us now differentiate this with respect to x again then derivative of 1 with respect to x will give you g double prime (f(x)) into f prime (x) I already have a f prime (x) so f prime(x) the whole square plus the next term g prime (f(x)) into derivative of f prime(x) that is f double prime(x). So this gives me g double prime (f(x)) to be equal to minus g prime (f(x)) into f double prime (x) divided by f prime (x) the whole square.

But I know that g prime(f(x)) is 1 by f prime(x). So g double prime will be minus f double prime (x) divided by [f prime (x) the whole cube]. So I have an expression for g double prime now at this stage I write down the error in interpolation of a function which we have already obtained.

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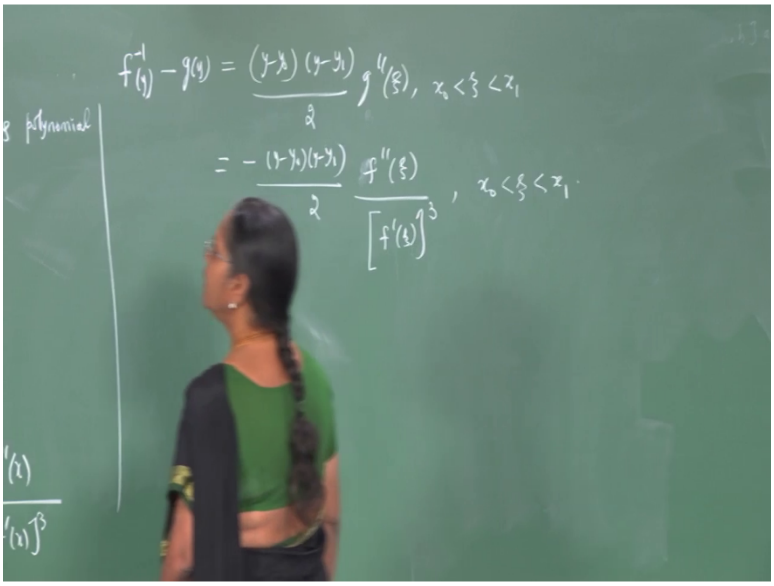

So if g(y) interpolates the function f inverse in some interval then the error in interpolation is given by y minus y 0 into y minus y 1 by factorial 2 into g double prime (psi) where psi lies between the interval x 0 to x 1. So here I know what is g double prime of f(x) so that will be minus y minus y 0 into y minus y 1 by factorial 2 into g double prime is the f double prime (psi) by [f prime (psi)] the whole cube.

So this is the error in interpolation. Remember while doing inverse interpolation we must ensure that the inverse function exists and it is single valued otherwise our computations are absolutely useless.

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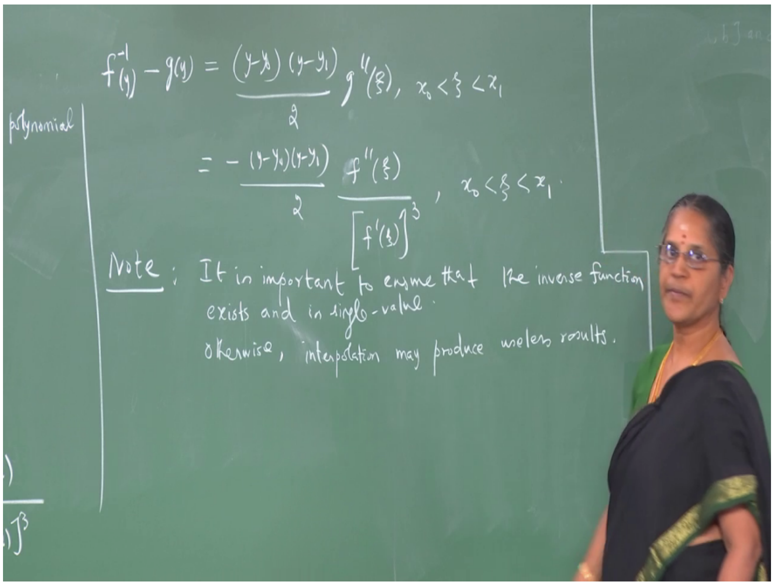

So we just give that as a note that it is important to ensure that the inverse function exists and it is single valued otherwise our we will have meaningless results. So interpolation will produce useless results.

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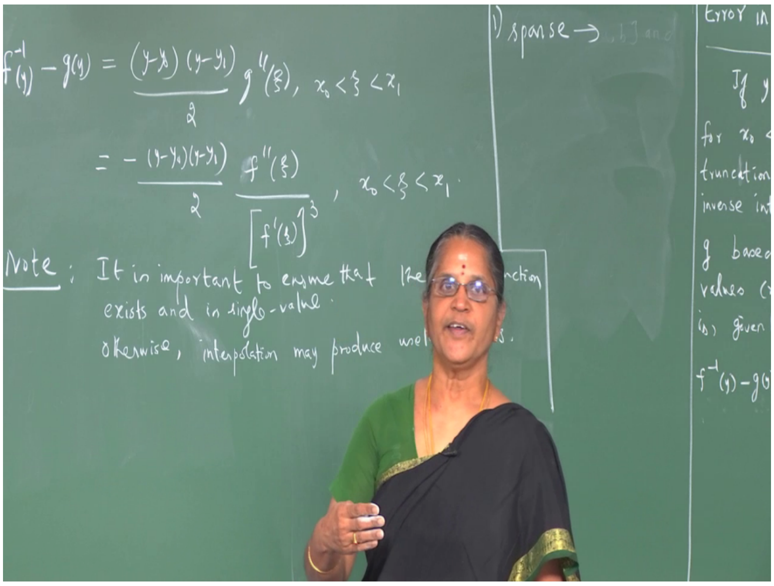

Now I would like to conclude this section on interpolation by polynomial by making the following remarks and observations. the first remark that I would like to tell you is that if a table of values of a function are prescribed to you and that the table is sparse then in that case the polynomial interpolation that you perform may not help you in obtaining the result to the desired degree of accuracy no matter how many terms you include in your polynomial.

This happens when the table of values given to you is sparse. Then the question arises is there any indication that my results are unreliable.

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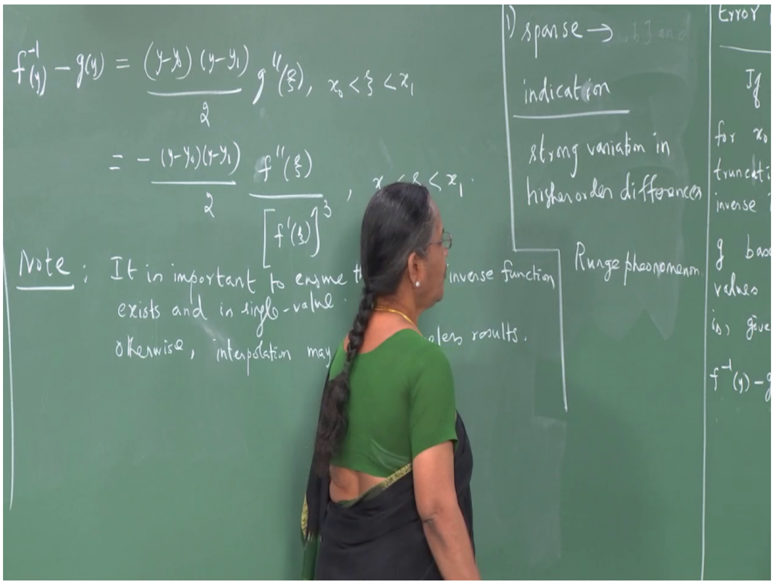

If there are some indications one such indication is that when you found the difference table you will see that there is a strong variation in higher order differences is that happens in your computation then you can immediately say that oh!

This indicates that my result is not going to be very reliable that is one indication. Another indication is that the terms that you add to interpolation polynomial by making the degree of the interpolation polynomial higher and higher in order to get more accuracy it may be such that the terms added may not decrease very fast as the terms are included that is another indication.

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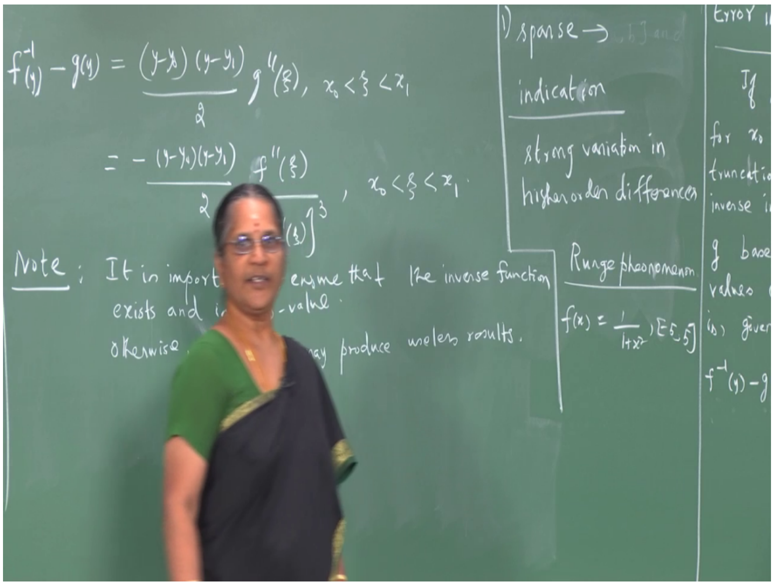

Then there is third indication which you can get by seeing whether the problem that you are dealing with displays what is known as Runge Phenomena. So what is this Runge Phenomena? Runge observed this behaviour when he tried to interpolate the function f(x) is equal to 1 by 1 plus x square in the interval minus [5 to 5]. So let us see what Runge observed when he tried to interpolate this function in this interval..

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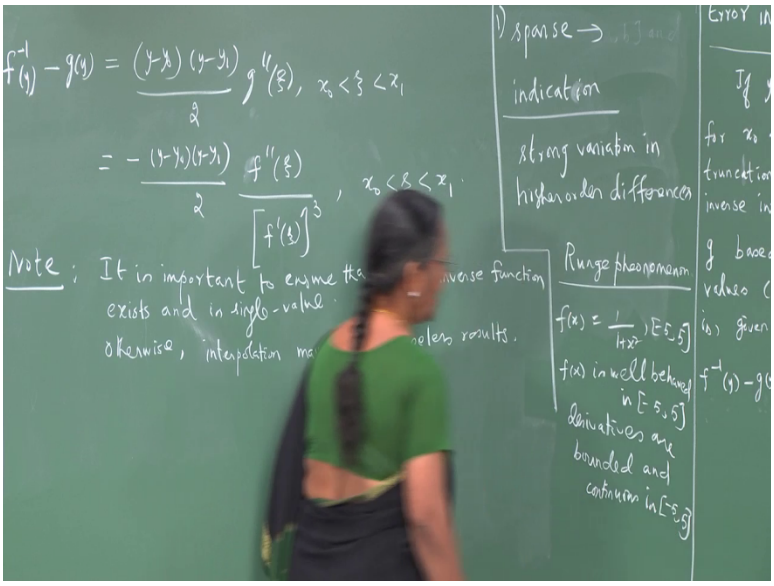

You observed that this function f(x) is well behaved in this interval [minus 5 to 5] and its derivatives write its derivatives they are all bounded and continuous in this interval [minus 5 to 5] so you expect that if you approximate this function by an appropriate polynomial in the interval [minus 5 to 5] depending upon the accuracy that is required by you. You have reliable results.

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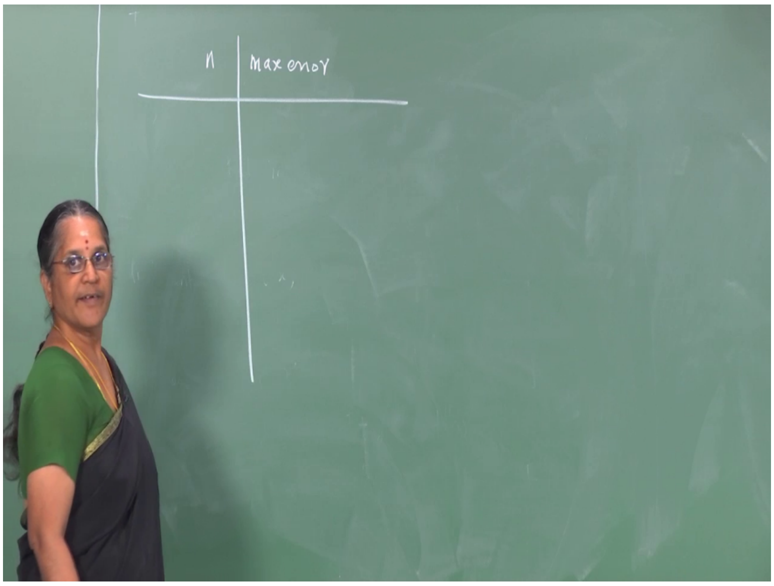

Let us see this did not happen. So let us consider the details that are provided by Runge. So he constructed interpolation polynomials that interpolate this function in this interval [minus 5 to 5] by taking various degree polynomials. So I shall write down the degree of the polynomial here and give the maximum error that was incurred in interpolating this function by an nth degree polynomial.

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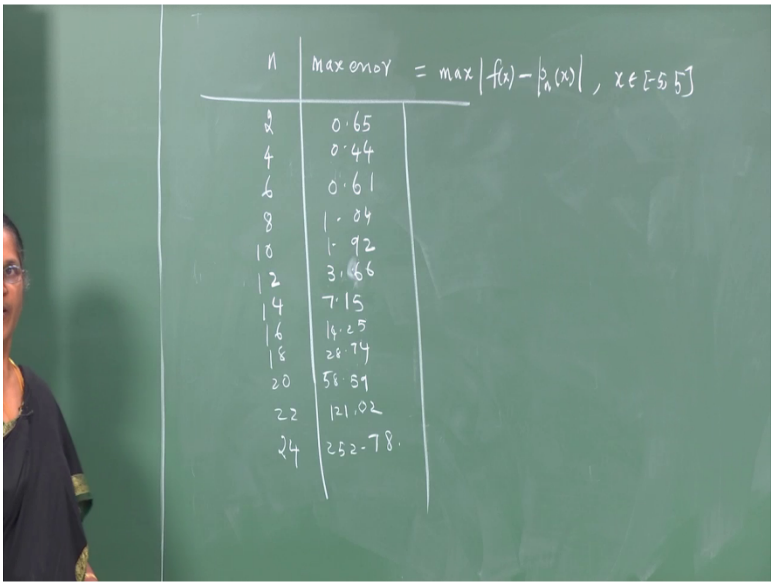

What is the maximum error it is the maximum of the difference between the function f(x) and p n(x) for x in this interval [minus 5 to 5]. We know the expression for it and we can always evaluate the bound on the error so when he approximated it by second degree polynomial then he obtained the maximum error to be 0.65.

When he approximated by a 4 th degree polynomial it turned out to be 0.44. Right? This is what we wanted the errors maximum errors should decrease as n increases. Namely as the degree of the polynomial increases when n as 6 the error turned out to be 0.61 it started increasing. When n was 8 it was 1.04.

So I shall give the details for some more n values where n denotes the degree of the interpolating polynomial that interpolates the function f(x) which is 1 by 1 plus x square. So this was 1.92 when n is 10, then at 12 it 3.66, 14 7.15, 16 14.25 and 18 28.74, 20 58.59 22 121.02 and at degree 24 it was 252.78.

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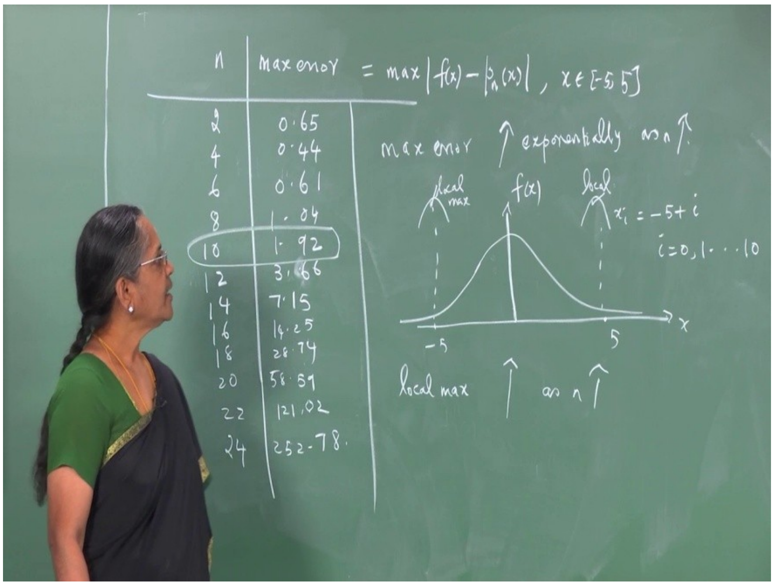

So Runge performed the computations for various degree polynomial starting from degree 2 upto say degree 24 and he observed and computed the maximum error to have these values and we see that the maximum error increases exponentially as n increases exponentially as n increases in addition he also observed that if we plot this function f(x) as a function of x in the interval [minus 5 to 5] we see that our x is equal to 0 it takes a value 1 and where x is [minus 5 to 5] it takes a value which is 1 by 1 plus 25 that is 1 by 26, the graph of the function is something like this.

When he approximated this function by a polynomial of degree 10 namely here by taking the 11 points. So x i which is equal to [minus 5 plus i for i is equal to 0 1 2 3 upto 10 these were the points that he chose. So x i runs between [minus 5 and 5]. So there are 11 points using these 11 points approximated the function by a polynomial of degree 10 and he observed that at x is equal to plus 5 and minus 5 the polynomial has a local maximum.

While the graph of the function is such that f(minus 5 and plus 5) where 1 by 26 at minus 5 and plus 5 interpolating polynomial had the values which were the maximum values in that interval for the interpolating polynomial. And this local maxima which was obtained at these n point at the points minus 5 and plus 5 increased exponentially again as n increased.

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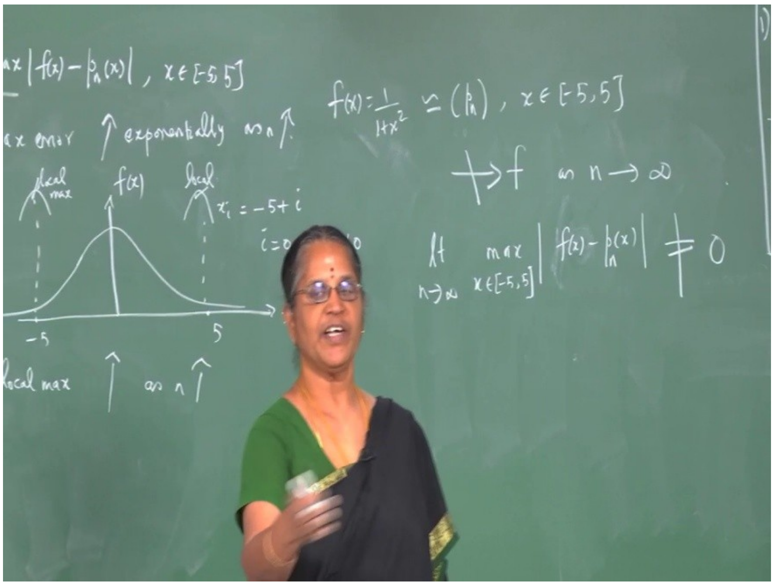

And as we remarked earlier the function is well behaved and the derivatives are grounded and continuous but the sequence of polynomials which are interpolation polynomials that interpolates this function f(x) which is equal to 1 by 1 plus x square in the interval minus 5 to 5 is such that this sequence did not converge to the function f as n tends to infinity Or in other words Runge showed limit as n tending to infinity of maximum of the error which is f(x) minus p(x) for appropriate degree of the polynomial.

For x lying between minus 5 and 5 was never 0. So the sequence of interpolation polynomials as n increases namely the degree of the polynomial increases. We want it to converge to the function which we are approximating but in this example although the function has nice behaviour this did not happen because the maximum error the sequence of maximum error as n increases did not converge in turn the sequence of polynomials p n as n tends to infinity did not converge to f which is 1 by 1 plus x square. This phenomenon is referred to Runge phenomenon.

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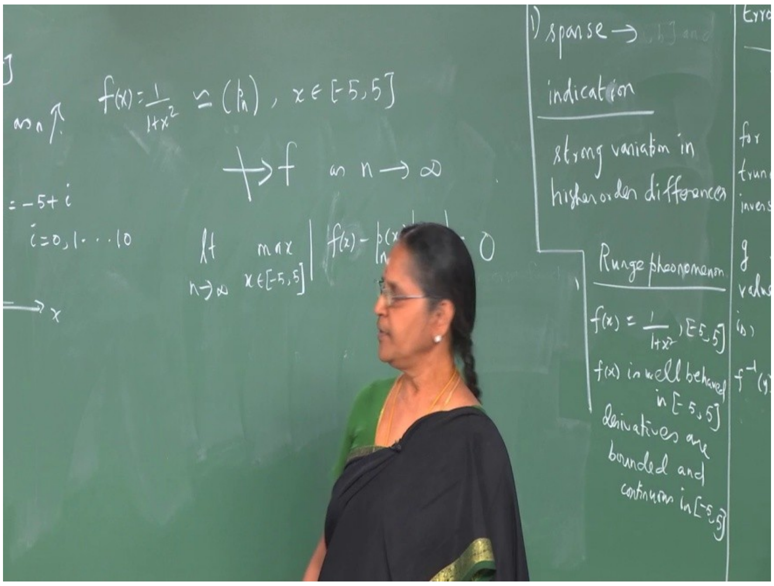

So that is an indication of the fact that when you perform interpolation by polynomials if you observe such a phenomenal referred to as Runge phenomenal then surely expects that your results are going to be totally unreliable. So while doing the problem you should take care to see that you do not come across any of these indications.

So you will think that yes in all when we perform the computations we can look for such indications and then obtain results and then we can be confident about what the results are and how reliable our results are. But that may not happen all the time. So let us give some examples where we may get totally unreliable results without any indications so let us consider the following examples.

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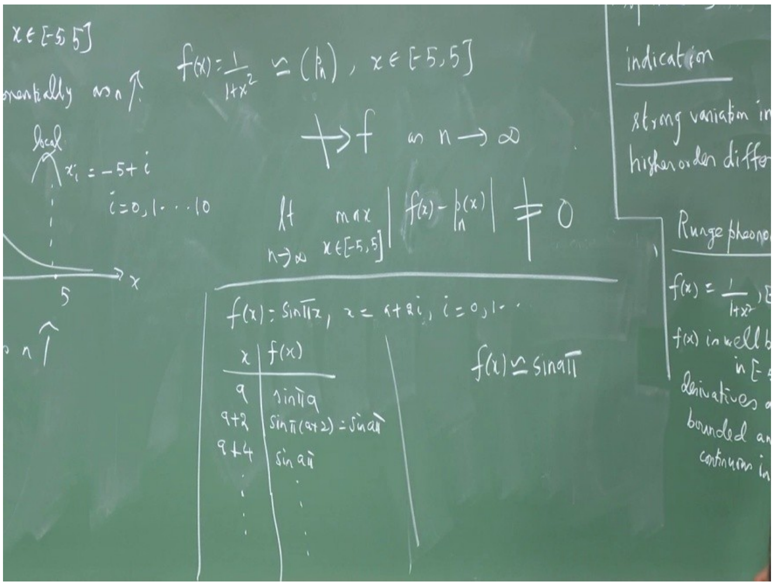

So suppose I consider the function f(x) which is equal to sin pi x and then take the interpolation points nodes as say x is equal to a plus 2 i for i is equal to 0 1 2 3 and so on. Then I see that when I write down at these points x the corresponding values f(x) when x is equal to a when i is 0 I get this to be sin pi a, when x is a plus 2 into 1 then it is sin pi a plus 2 which is again sin a pi, at x is equal to a plus 2 into 2 so a plus 4 it is again sin a pi, so and I observe that all these nodes x i the function assumes the same value and therefore all its differences are 0.

And hence a polynomial interpolation of this function f(x) gives you sin a pi as the interpolation polynomial which is a constant polynomial which is totally unreliable. There are absolutely no indications which we had listed namely we did not observe any strong variation in the higher order differences and so we thought that our results would be reliable but this does not happen let us see another example.

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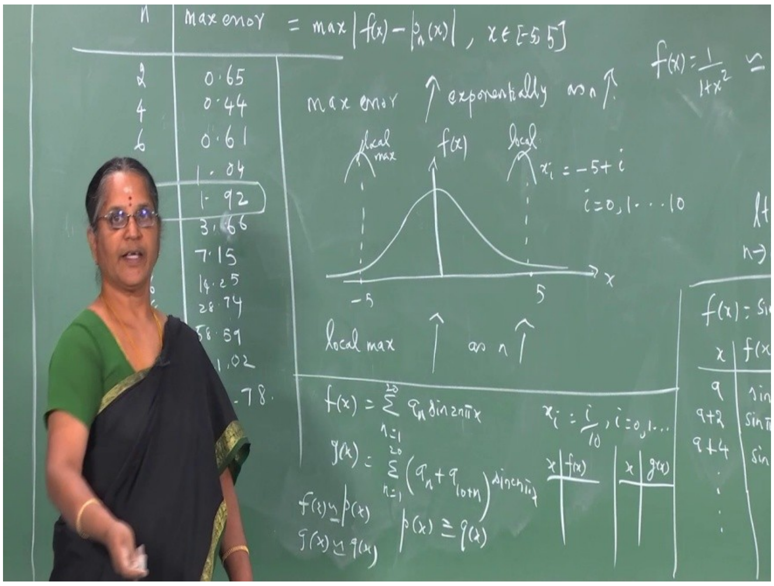

Suppose I consider the functions f(x) which is n is equal to 1 to 20 a n into sin 2 n pi x an another function say g(x) which is n is equal to 1 to 20 a n plus a n plus n) into sin 2 n pi x. And I shall consider the nodes x i which is equal to i by 10 for i is equal to 0 1 2 3 etc. And write down the table of values for the function f(x) and the table of values for the function g(x).

You will observe that the table of value show f(x) and g(x) have identical values at the set of point x i that you have considered namely i by 10. And therefore when you write down the interpolating polynomial say p(x) for the function f(x) and say q(x) is the interpolating polynomial that interpolates g(x) at these nodes then since they have identical function values you will get your interpolating polynomials to be identically the same.

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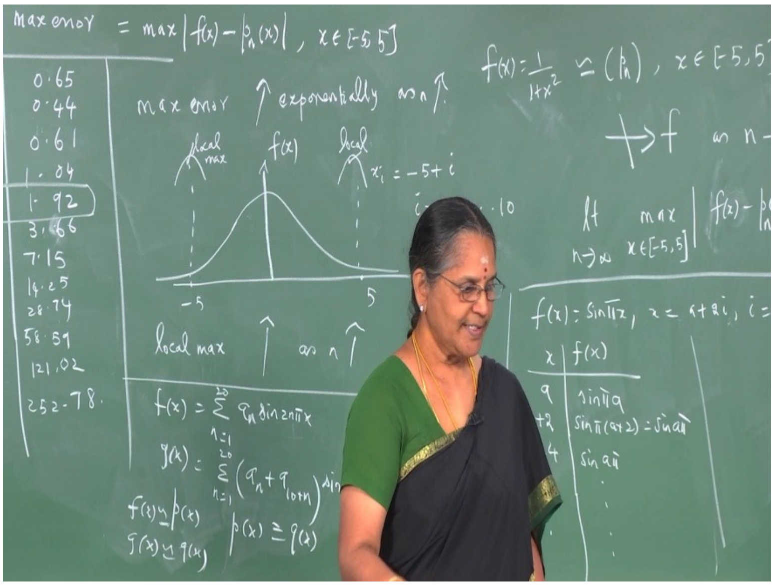

Suppose a prior you did not know f(x) g(x) you are only given a set of table values namely either this or this they are one and the same, either this or this. Then you would obtain an interpolating polynomial called that is p(x) and the set of say n plus 1points. Then what is the function that you have reconstructed by this polynomial you do not know you have reconstructed functions f(x) as well as g(x) with the information that has been provided to you.

And the two functions of totally different and therefore you do not have a reliable information that you have reconstructed only the function for which the function values are even it may be f(x) or g(x) and they are totally different. So these examples illustrate that your interpolation computation can be totally unreliable there are absolutely no indications of th fact that it is an unreliable result.

So the examples the remarks and the comments that we have given so far clearly tell us that we must have a clear idea about the typical scales at which the function changes significantly. Unless we have this information our results based on polynomial interpolation will be totally useless.

Again you may be given a table of values which is smooth without the function itself without not being smooth then thirdly suppose say the table of values which are given to you are entirely from a few experimentally measurements then it is not possible to give an estimate of the total truncation error that is incurred in approximation. So in all these situations we observe that polynomial interpolation is not reliable is not useful to us. Then what do we do?

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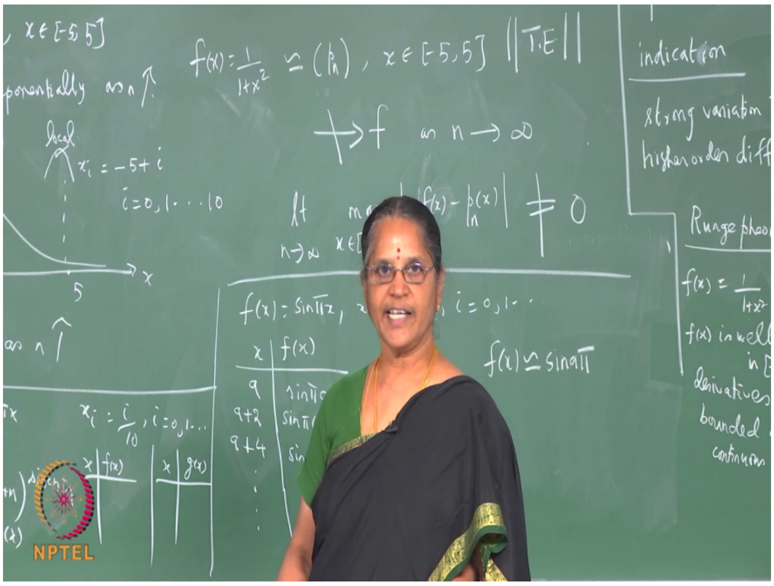

We have to take records to a more general class of approximation. Which is based on minimizing the norm of an appropriate truncation error in approximation? And this topic is beyond this course basic numerical analysis which we are studying now and so I leave at this stage our discussion on the topic on interpolation by polynomials. We move on to the next topic that is of interest to us in this course.