Numerical Analysis
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Lecture -8, Part - 2
Properties of divided differences,
Introduction to Inverse Interpolation

So let us now work out some problems based on divided differences.

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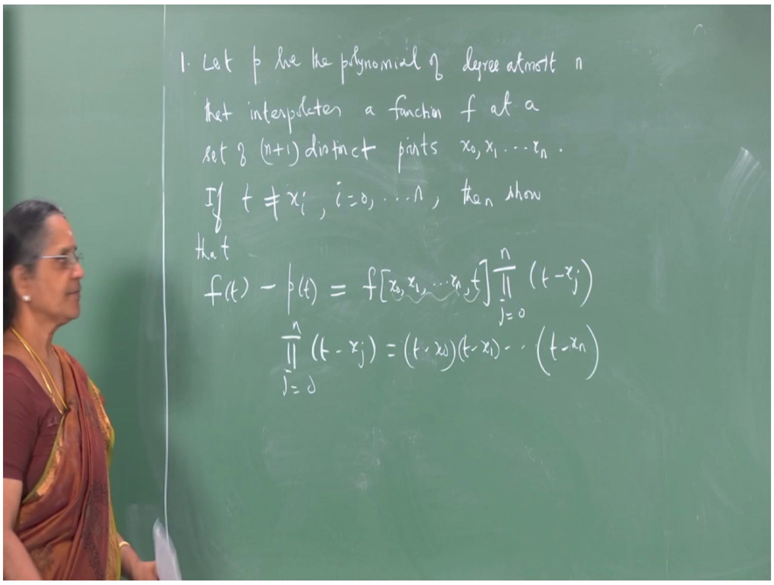

Let p be the polynomial of degree at most n that interpolates a function f at a set of n plus 1 distinct points say x 0, x 1 , etc x n. If t is different from these x i for i is equal to 0,1,2,3 etc n. Then show that f(t) minus p(t) so what is it? What is f(t) minus p(t), p(t) interpolates the function at a set of discrete points which are distinct namely x 0, x 1, etc x n.

And t is any point in that interval which is different from these x i. So f(t) minus p(t) gives you error in the approximation of f(t) by p(t) at the point t in that interval. The result says this error is f[x 0 x 1 etc x n, t] so there are n plus 1 points and there is an additional point n plus 2 points so it is the n plus 1 th order divided difference multiply by the product of the factors j is equal to 0 to n (t minus x j). Thereby j equal to 0 to n (t minus x j) we mean (t minus x 0) into (t minus x 1) etc upto (t minus x n).

So the problem essentially gives us an expression for error in interpolation in terms of the divided differences.

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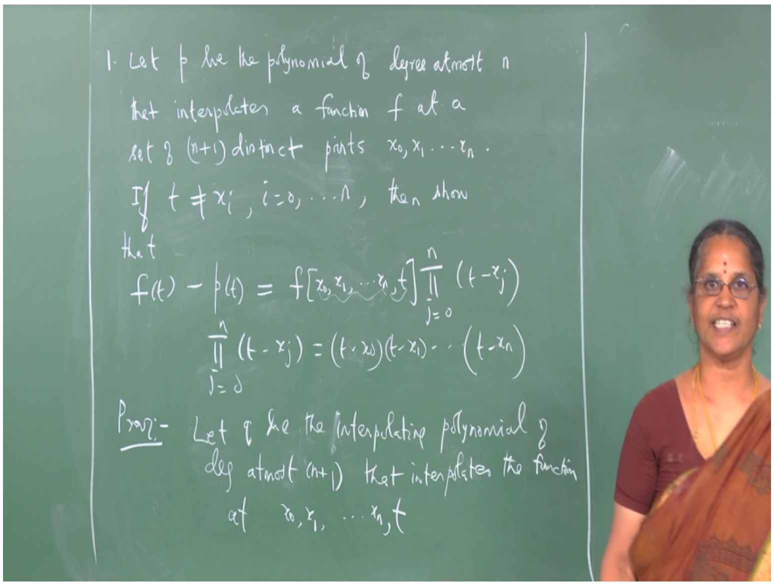

So let us try to work out the details so let hue be the interpolating polynomial of degree at most n plus 1 that interpolates the function at its a polynomial of degree at most n plus 1, so it interpolates the function at a set of n plus 2 discrete points.

So at say x 0 x 1 etc x n we have n plus 1 of them and t. So we have n plus 2 such points. And so can we write down what this polynomial q when we know the polynomial p yes that is the advantage of divided difference interpolation polynomial that we had seen earlier.

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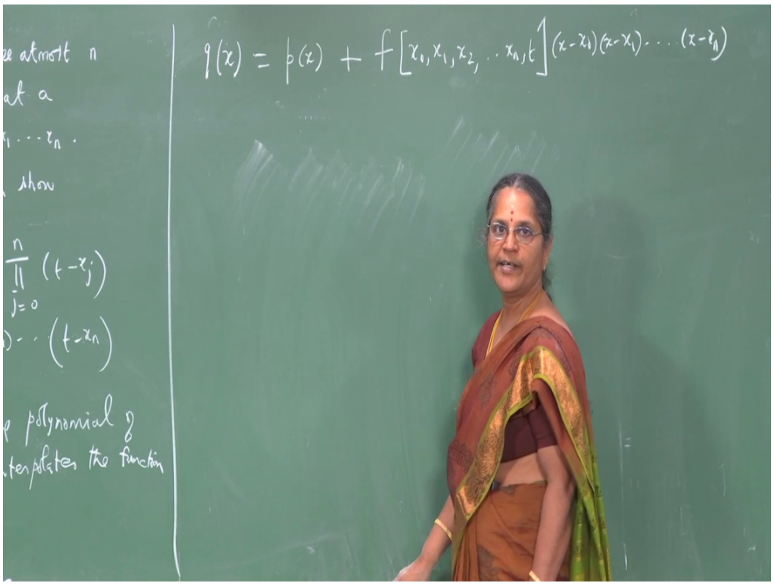

So I can write down q at any x will be equal to p (x) polynomial of degree n plus the additional term will be a n plus 1 into product of certain factors. That a n plus 1 will be the divided difference f[x 0 x 1 etc x n, t] multiplied by the factor (x minus x 0) (x minus x 1) etc upto (x minus x n).

So how many such factors are there starting from x minus x 0 to x minus x n there are n plus 1 factors so this is a polynomial of degree n plus 1 its coefficient is this n plus 1 th order divided difference. And it is added to a polynomial of degree at most n. So that will be the polynomial q(x).

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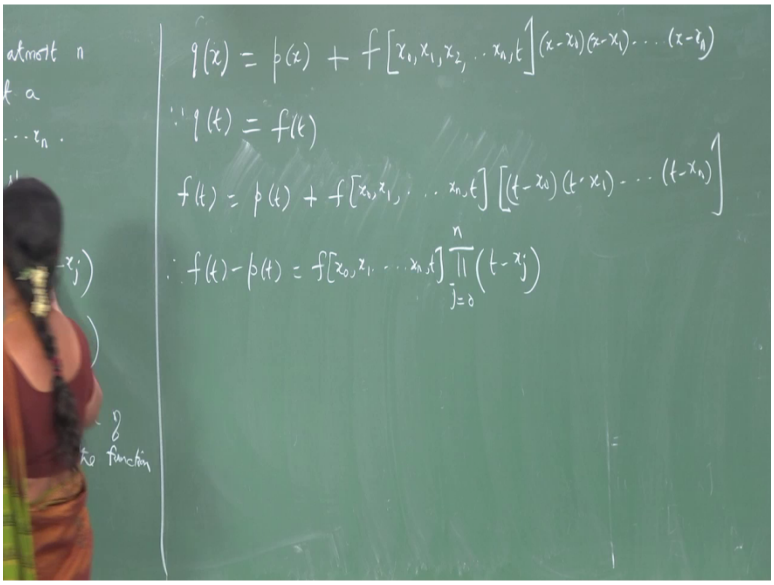

So therefore and what do we know about q (t). So q(t) is f(t) why? how have we constructed q? q is the interpolating polynomial of degree at most n plus 1 that interpolates the function at [x 0 x 1 etc x n, t] so q(t) will be f(t). So I evaluate this at t so I have f(t) will be p(t) plus f[x 0 x 1 etc x n, t] multiplied by [t minus x 0) into (t minus x 1) etc upto t minus x n)] therefore we have f(t) minus p(t) to be equal to f[x 0 x 1 etc x n, t] into product say j is equal to 0 to n (t minus x j).

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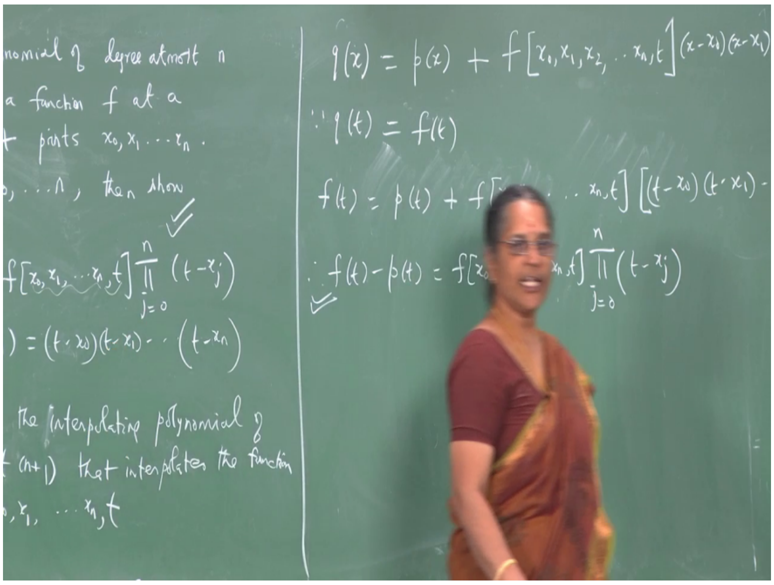

And that is what we have to show here.

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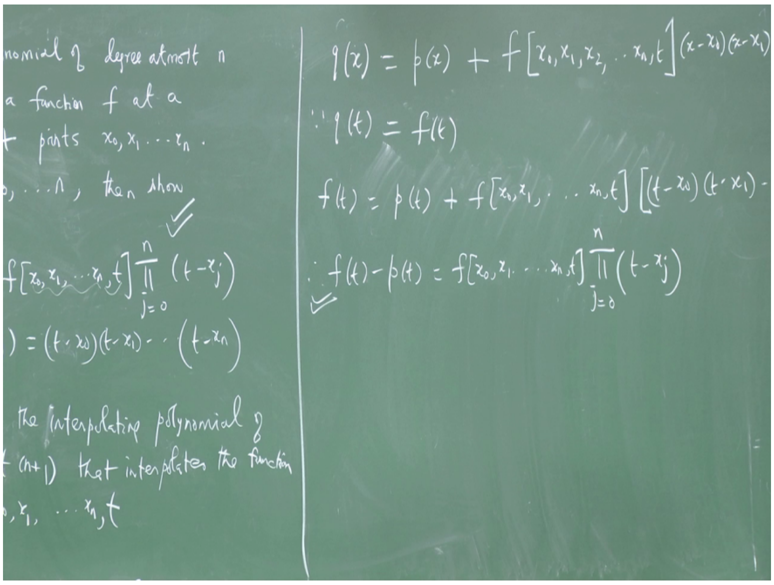

This gives you the error in interpolation in terms of the divided differences.

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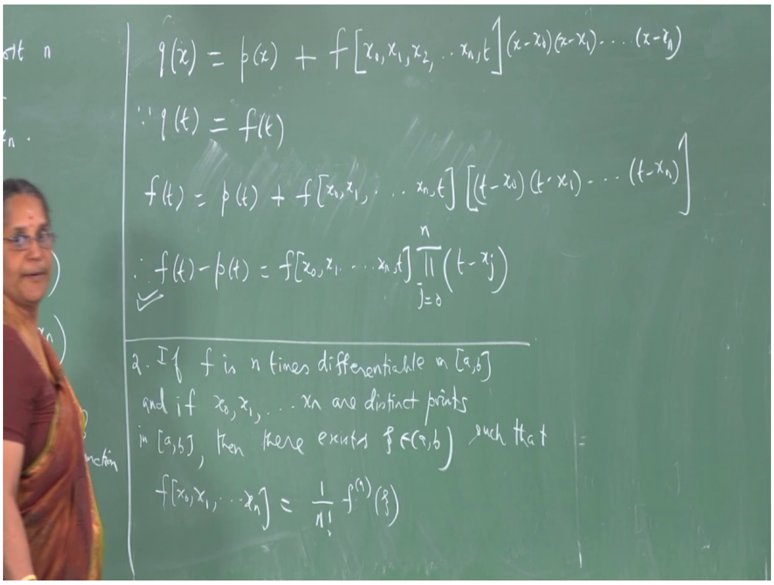

So let us now continue with another problem which says if f is n times differentiable say on the interval [a, b]. And if [x 0 x 1 etc x n] are distinct points in the interval [a,b] then there exists Psi in the open interval (a,b) such that f[x 0 x 1 etc x n] the n th order divided difference is 1 by n factorial into the n th derivative of f (psi). The n th order divided difference is the nth derivative at psi by n factorial.

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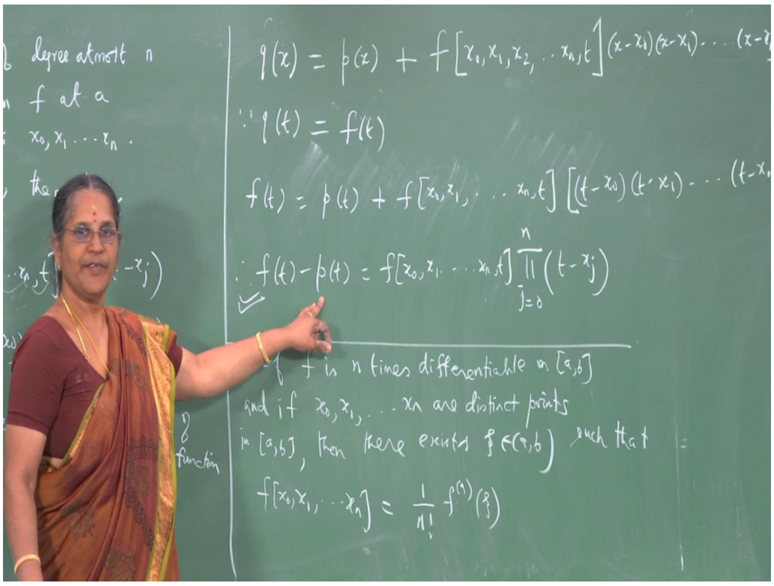

And you now look at the previous problem and this problem if you use this information in this result the n th order divided difference is 1 by n factorial into n th derivative at psi. So I have a n plus 1 th order divided difference. So if I prove this problem then I can express this as the n plus 1 th order derivative of f(psi) by n plus 1 factorial right? Provided f is n plus 1 times differentiable in that interval.

So if I substitute in terms of this result then I essentially get the result which we have already computed for error in interpolation namely f(t) minus p(t) is n plus 1 th derivative at psi by n plus 1 factorial into the product of certain factors which we had denoted earlier by pi n plus 1 (x).

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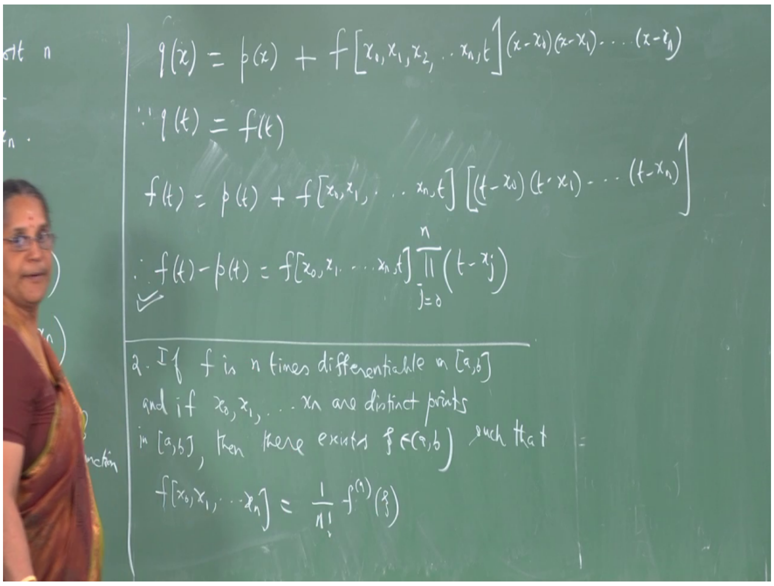

So let us work out the details of the proof of this problem. So what we have to show f is n times differentiable on (a,b) and there are n plus 1 distinct points in (a,b) then there exists a psi in (a,b) such that the n th order divided difference is this given in terms of the n th derivative of f at some psi which belongs to this interval (a,b) by n factorial.

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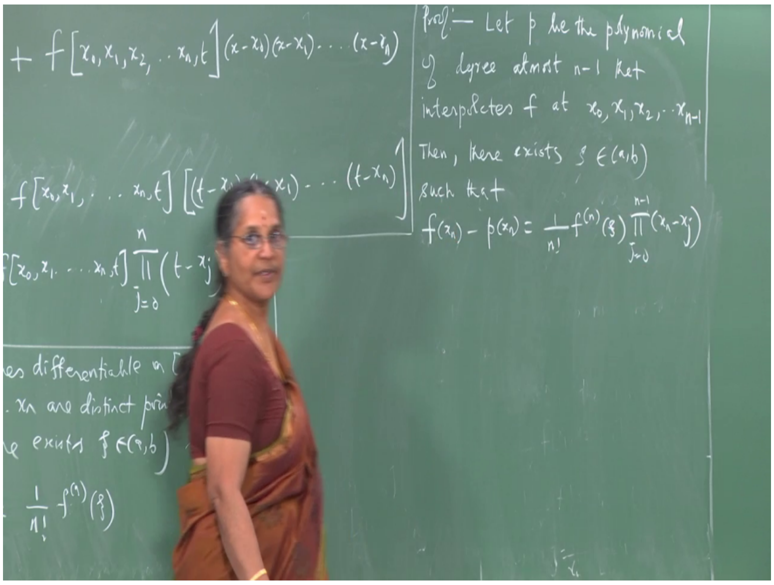

So let us take p be the polynomial of course the interpolating polynomial of degree at most say n minus 1 that interpolates the function f at how many points? P is the polynomial of degree at most n minus 1.

So there must be n points starting from x 0, it is x 0, x 1, x 2, etc at x n minus 1. So there are n points at which the polynomial interpolates this function. Then there exists so I am going to now use the previous result. If p is a polynomial of degree at most n that interpolates the function at n plus 1 points than at any t which is different from x i which I have used f(t) minus p(t) can be expressed this way. So now I am considering and denoting the polynomial that interpolates the function at a set of n points is a polynomial of degree n minus 1.

So using the previous problem I know that there exists psi belonging to the interval (a, b) such that f(x n) minus p(x n) is 1 by n factorial into n th derivative (psi) into product j is equal to 0 to n minus 1 x n minus x j. So this x n is one of the points in this list. So here t is not one of the points in that list recall the error in interpolation that we have already have determined.

There also we had the point denoted by x which is not any of the nodal points at which the polynomial interpolates the function. So I choose the point x n which is not any of these points and use the error in interpolation formula which we had computed earlier and write down the result for the error at the point x n which is this.

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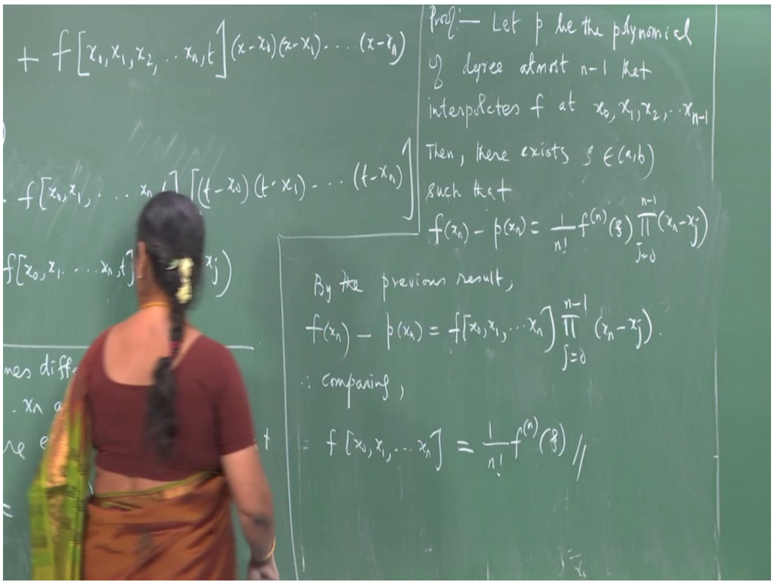

Now at this stage I use the previous result I know by the previous result that f(x n|) now x n plays the role of t in this problem. So f(x n) minus p(x n) will be equal to f[x 0 x 1 etc upto x n] into product j is equal to 0 to n minus 1 into x n minus x j.

So the error in interpolation formula which we have already done gives the error at x n to be this. The previous problem tells me the error in interpolation in terms of the divided differences of order n as this. The left hand sides of the same they give you the error at the point x n and therefore comparing the two results we show that f[x 0 x 1 etc x n] the n th order divided difference is 1 by n factorial into n th derivative (psi) and that is what we are asked to show.

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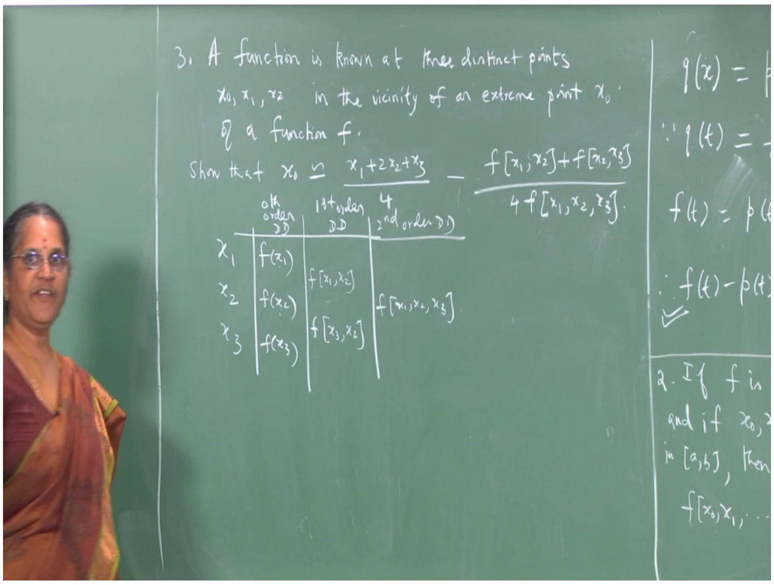

So let us work out the following the function is known at say three distinct points. So let us call these points as x 0, x 1 x 2 in the visility of an extreme point x 0 of a function f(x). Show that x 0 can be approximated by x 1 plus 2 x 2 plus x 3 by 4 minus f[x 1, x2] plus f[x 2,x 3] divided by 4 times f[x 1, x 2x 3]. So if you know information about a function say at a set of three distinct points which lie in a neighbourhood of an extreme point x 0 of that function. Then you can approximately get the extreme point x 0 at by computing the right hand side.

So what does it say you have information at x 1, x 2, x 3 about this function so the corresponding values are f(x 1) f(x 2) f|(x 3). Since the result involves divided differences we immediately form the divided difference table. So if I find the difference between the two that will be f(x 1, x 2) which will be f(x 2) minus f(x 1) by (x 2 minus x 1) and this will give me f(x 3 x 2) this is a 0 th order divided difference and this will be the first order divided difference. So I can form the second order divided difference which will be f(x 1 x 2x 3) and the higher order divided difference will all be 0.

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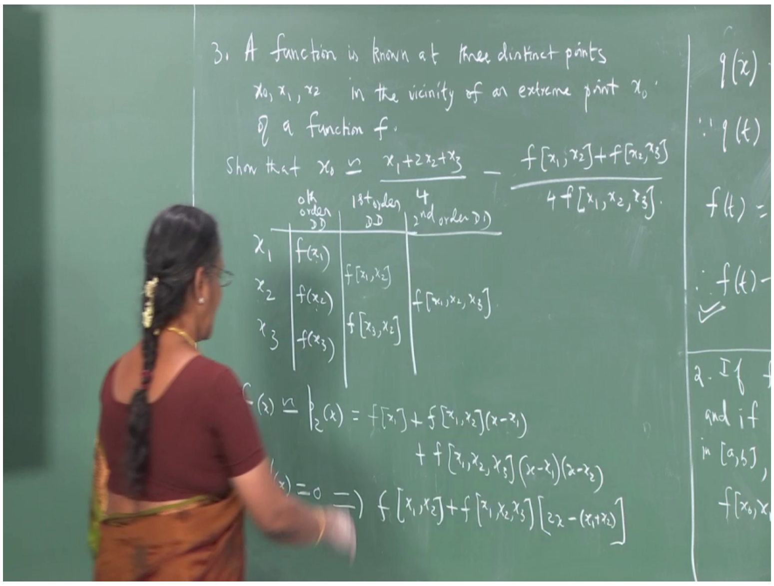

So now that you have the divided difference table you can find the interpolating polynomial interpolates this function at a set of these discrete points namely f(x) can be approximated by a polynomial p 2(x) of degree 2 and in terms of the divided difference interpolation polynomial this will be f(x 1) plus f(x 1 x 2) multiplied by x minus the first point which is x 1plus f(x 1 x 2 x 3) multiplied by the factor x minus x 1 into x minus x 2 that is the interpolating polynomial. Now f dash (x) is 0 so that will give you f(x 1 x 2) into derivative plus f(x 1 x 2 x 3) into derivative of this that will give you 2x minus x 1 plus x 2 and that must be equal to 0.

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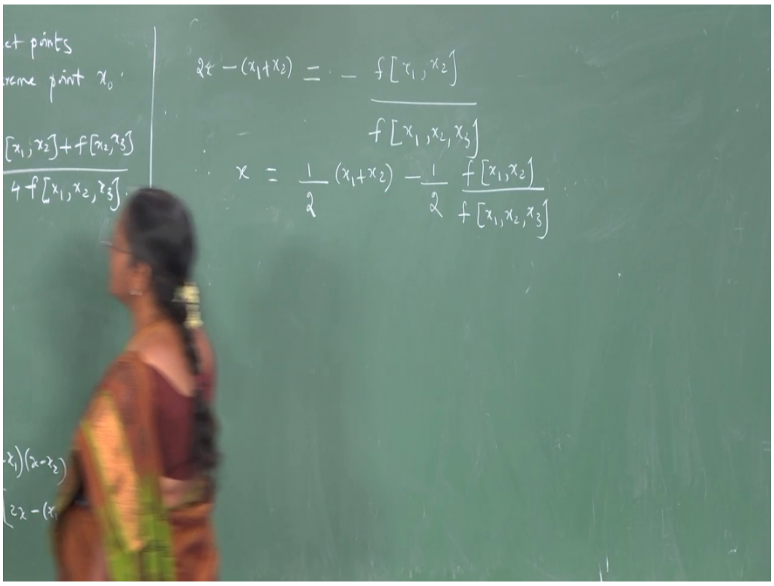

2x minus x 1 plus x 2 will be equal to minus f(x 1 x 2) divided by f(x 1x 2 x 3) and therefore x is equal to half of x 1 plus x 2 minus half of f(x 1 x 2) by f(x 1x 2 x 3).

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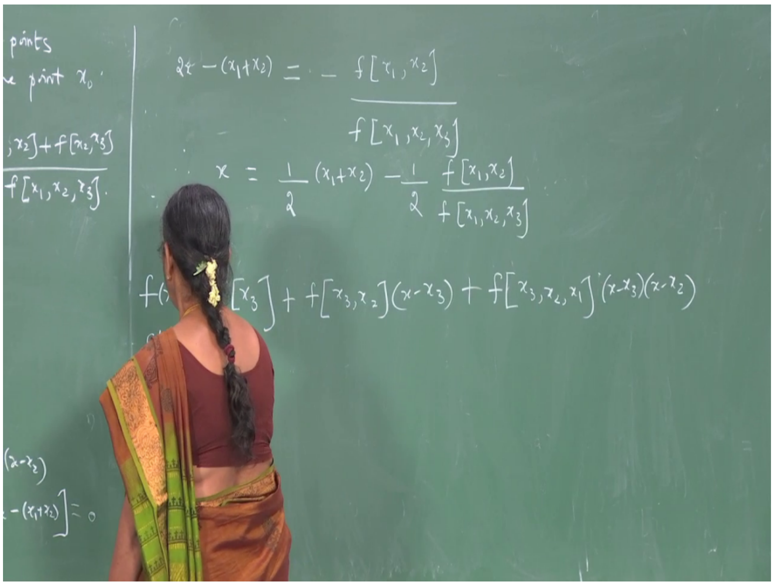

But when we look at the result it also involves information at x 3, so I write down the interpolating polynomial again by looking at the set of values x 2, x 3, x 1 and taking information along this diagonal. So by interpolating polynomial that interpolates the function f(x) can also be written as f(x 3) plus f(x 3 x 2) multiplied by so what should be the factor there then it is f(x 1 x 2) it comes out to be x minus x 1 so here f(x 3 x 2) so the factor must be x minus x 3.

Then the next term will be f(x 3 x 2 x 1) isn't it? Which is the same as f(x 1 x 2 x 3) you have already seen the divided differences is symmetric in its argument. So what will be the factors there? That will be x minus x 3) into (x minus x 2).

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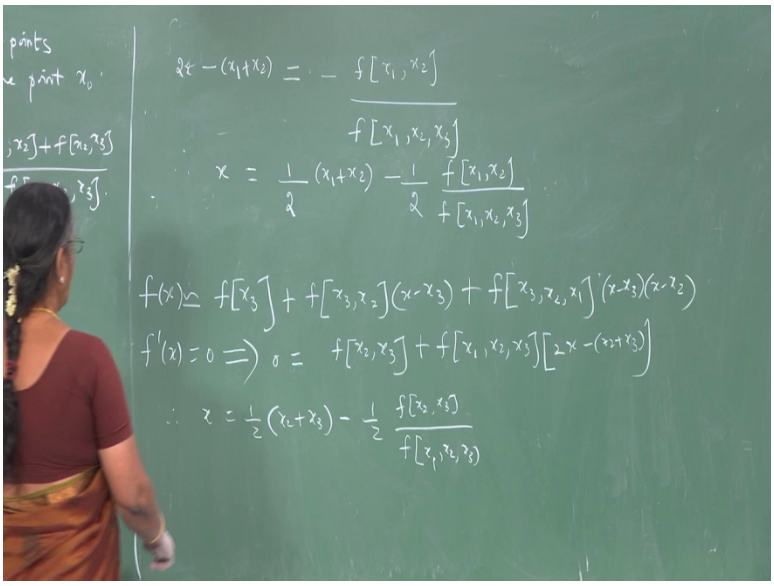

So let us find out x prime x is equal to 0 gives you 0 equal to f(x 3 x 2) as (x 2 x 3) then plus f(x 1x 2 x 3) into again 2x minus x 2 plus x 3 from which we can get x to be equal to half of x 2 plus x 3 minus half of f (x 2, x 3) divided by f(x 1x 2 x 3).

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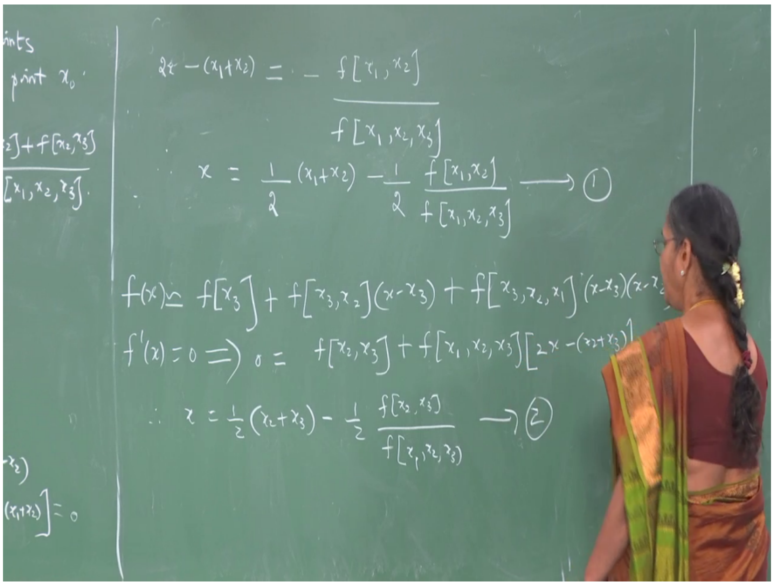

So f prime x is 0 when x is equal to this. So at an extreme point which I denote by x 0 we have x 0 to be given by this as well as this both are approximations to the extreme points x 0. So if I call this as say result 1, and this as result 2.

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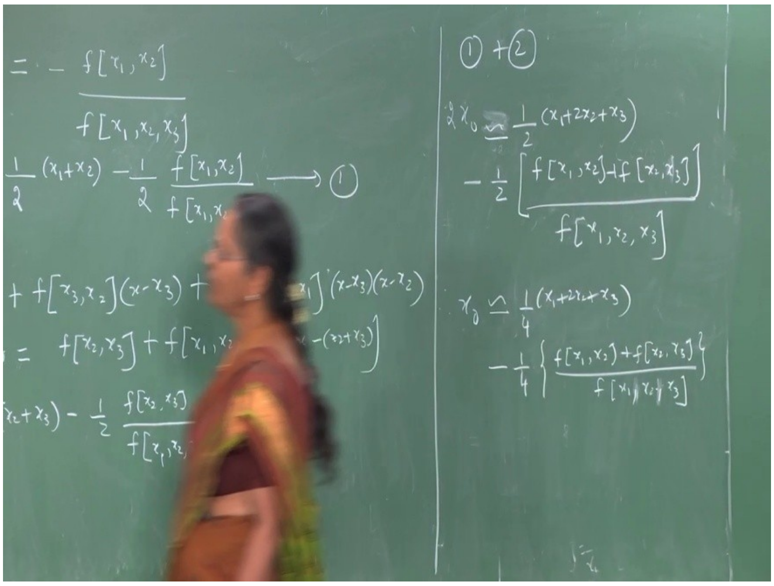

And use both these results 1 and 2 and that will give me two times x 0 equal to half of [x 1 plus 2 x 2 plus x 3) minus half of f(x 1 x 2) plus f (x 2 x 3)] divided by f[x 1 x 2 x 3]. And therefore x 0 will be given by of course this is an approximation so I shall denote it as this so x 0 is approximately 1 by 4 th (x 1 plus 2 x 2 plus x 3) minus 1 by 4 th [f(x 1 x 2) plus f (x 2 x 3)] divided by f[x 1 x 2 x 3].

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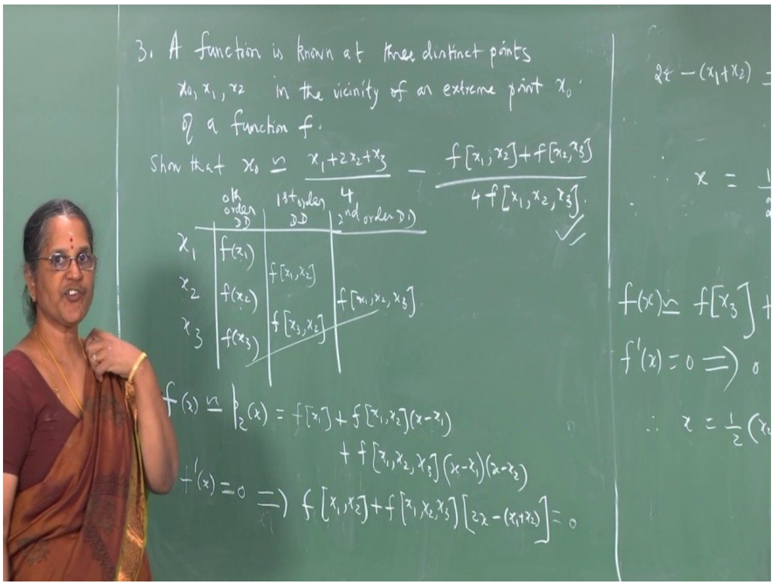

So and that is what we have to show that if a function is known at the three distinct points in the visility of the extreme point x 0 then x0 can be approximated by the value on the right hand side and that is what we have shown.

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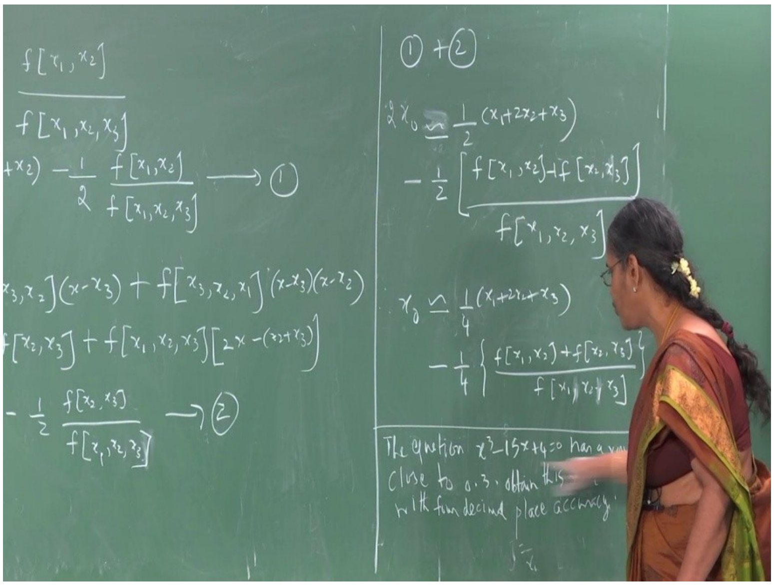

Let us now see how we can solve the following problem it says the equation x cube minus 15 x plus 4 equal to 0 has a root close to say 0.3 and you are asked to obtain this root with 4 decimal place accuracy we are given that this equation has a root close to point 3.

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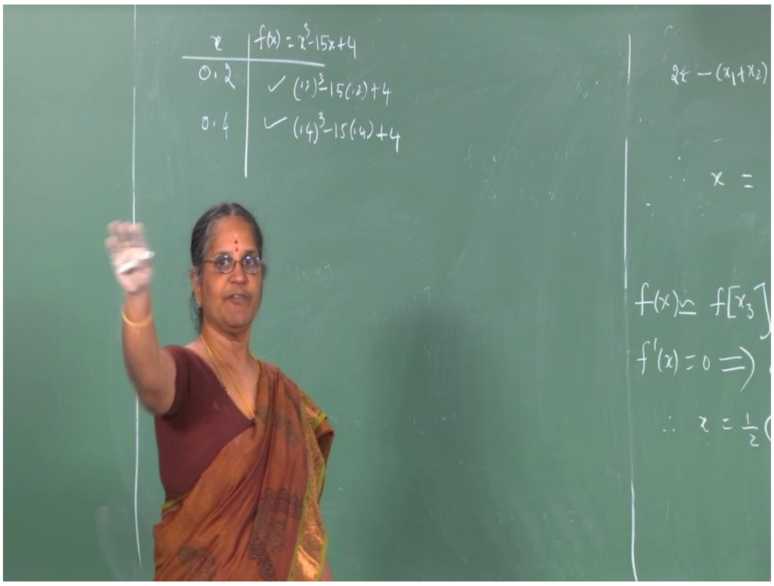

Suppose I get the information about this function x which is x cube minus 15 x plus 4 at some points in a neighbourhood of 0.3 say at 0.2 and 0.4 compute what is f(x) by substituting excess 0.2 and access 0. 4 these two points are close to 0.3 at a distance of 0.1 unit from 0.3. So I can write down what is the value of f(x) which is 0.2 cube minus 15 into 0.2 plus 4 this is 0.4 cube minus 15 into 0.4 plus 4. So since it is said that it is close to 0.3 I would like to find out the values at 0.2 and 0.4 and since it is a route at which the function must vanish it has to change its sign either from a positive value to a negative value or from a negative value to a positive value so that it will cross the point access equal to 0 on the x axis.

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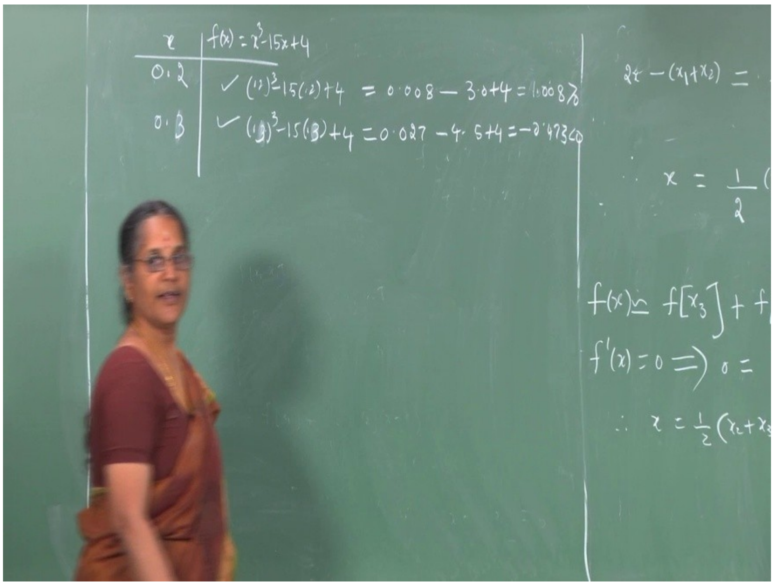

So I would like to look at the value of the function at 0.2 so this gives you 0.008 minus 3 plus 4 which is 1 .008 which is positive in if I compute the value at 0.4 may be since it is said that it is close to 0.3 I can even take this to be 0.3 and 05 so 15 into 0. 3 plus 4 and the values will be 0.027 minus 4.5 plus 4 and it turns out to be minus 0.473 which is negative.

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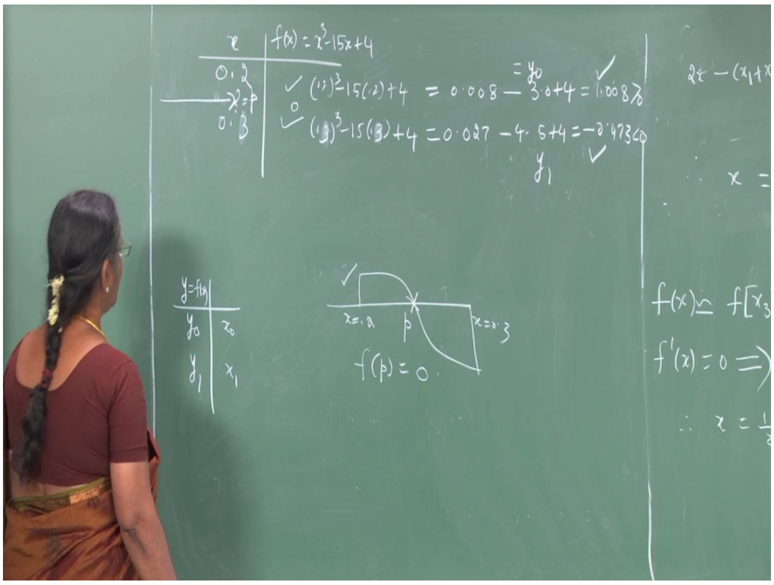

So I have at axis equal to 0.2 f(x) is positive and axis equal to say 0.3 is negative. So it has to cross the x axis at some point between 0.2 and 0.3 and it is this point which I call as p which I seek which will give me f(p) to be equal to 0 and that axis equal to p will be a route of that equation. So what is it that I require now, what is the x which I call as p such that at this p f(x) is equal to 0 that is what is asked. So I observe that I can compute this point p by taking the interpolating polynomial on these two nodes if I call them y 0 and y 1. So I have the information about the y which is f(x) at y 0 y 1 which are given at x 0 x 1.

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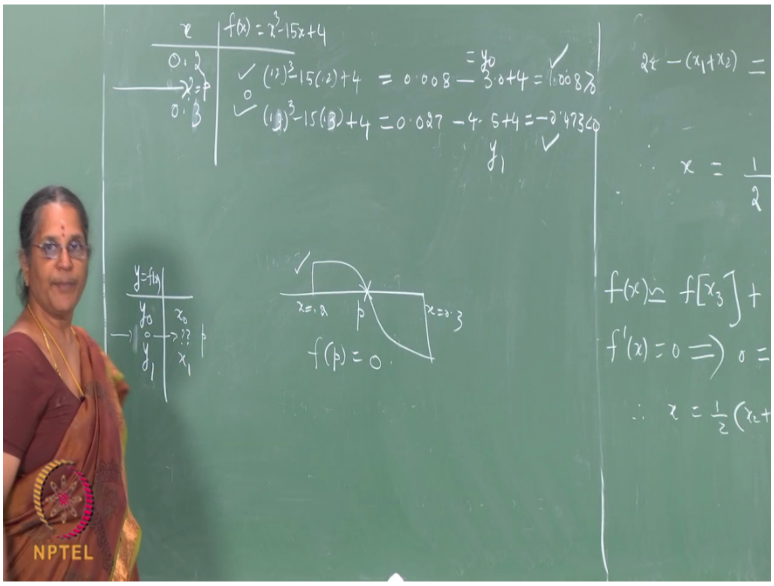

So there is a y namely 0 at which I do not have the information about the x which I call as p. So if I do the interpolating polynomial on the nodes y 0 y 1 and find out that x at which y becomes 0 then I will have the answer to this problem.

So when I do the interpolation on the nodes y 0, y 1 I essentially do what is known as inverse interpolation, so when I take these two points y 0l y 1 I will be getting a polynomial of degree at most 1 a linear inverse interpolation polynomial once I get the polynomial then I will get the information about the x at which this y becomes 0 and that would solve the problem I need to understand the ideas in inverse interpolation and we shall continue this in the next class.