# Numerical Analysis

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**Lecture -8 Part - 1**

## Properties of divided differences, Introduction to Inverse Interpolation

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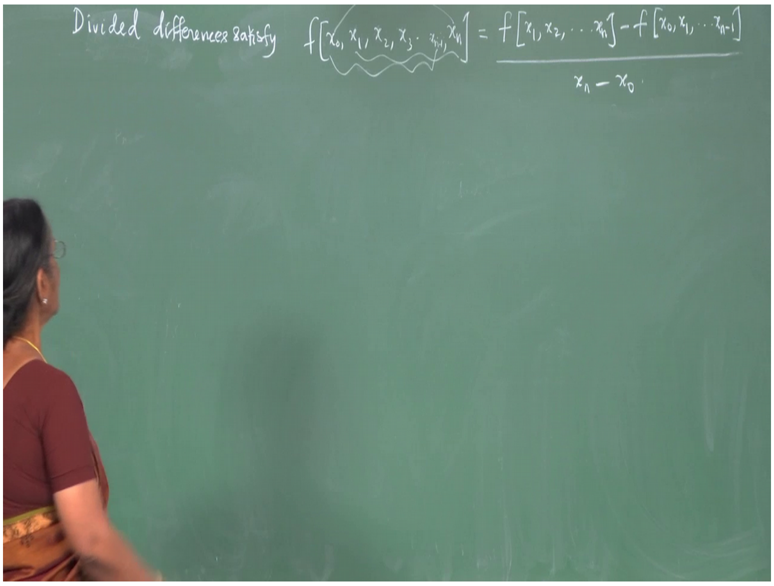


Ok so we said that we will look at the properties of a divided differences so I just recall while constructing the divided difference interpolating polynomial we had constants a 0 a 1 etc a n. We explicitly showed that  a 0 is the 0 th order difference a1st first order difference a 2 is the second order difference and So on a n is the n th order difference.

And we denoted the n th order difference by f [x 0, x 1, x 2, etc, x 3, x n]. And we said that a n is this. So how are we going to obtain this a n from the difference divided difference table or what is it actually. So the set of the first order difference is expressed in terms of the zeroth order difference the second order difference is expressed in terms of the first order Differences and so on.

The n th order divided difference is expressed in terms of the n minus 1 th order divided differences. And we show this property Namely the divided differences satisfy the property that the n th order difference f[x 0, x 1, x 2, etc, x n] is f [x 1, x 2, etc, x n] starting from here minus f [x 0, x 1, x 2, etc] going upto x n minus 1 divided by x n minus x 0.

So let us show this property so this will tell us how the higher order differences divided differences can be computed when we know the previous order divided differences. So to prove this result so this is property 1 where we need to show that the divided differences satisfy this property.

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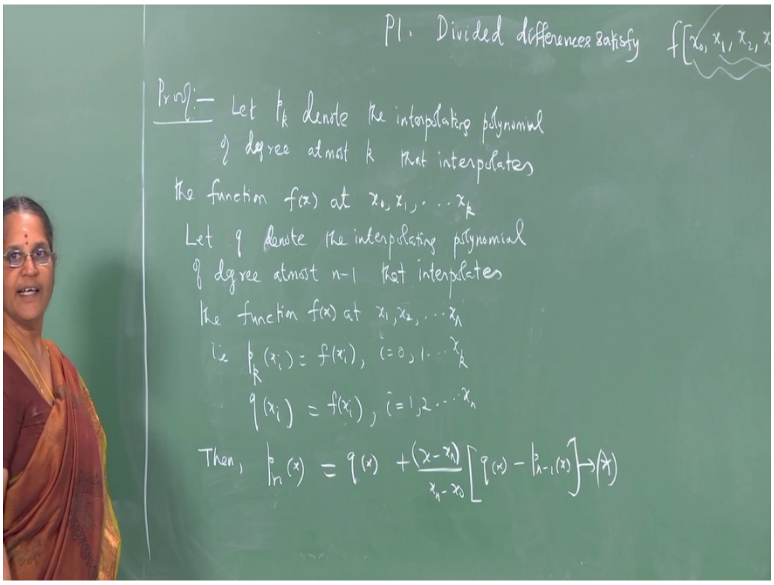


So to prove this let us denote by p k (x) or let p k denote the interpolating polynomial of degree at most k that interpolates the function f(x) at points [x 0, x 1, etc, upto x k] So we are given information at k plus 1 points. So we can seek a polynomial of degree at most k that interpolates the function at these points.

And let q denote the interpolating polynomial of degree at most n minus 1 that interpolates the function f(x) say upto points x 0, x 1, x 2, etc, x n.  How many points are there? There are n points. So we can seek an interpolating polynomial of degree at most n -1. And that we denote by q.

So let's write down the properties explicitly that is what does p k do ? p k at x i is f(x i) for what? For i is equal to 0, 1, 2, 3 up to x k. And what about q, q at x i is f(x i) for i is equal to 1, 2, 3 upto n. That is how we have denoted the polynomials interpolating polynomials. If this happens then we will now show that p n(x) polynomial of degree at most n that interpolates the function at x 0, x 1, x 2, etc, x n is q(x) plus x minus x n by x n minus x 0 into q x minus p n minus 1(x).

So we first show this result and from here we deduce the property of the divided differences. So what is it that we want to show we want to show that a polynomial of degree at most n that interpolates the function at n plus 1 points can be given in terms of this polynomial q and a polynomial of degree n minus 1 and then the term that is added to it has as a factor X minus x n in it.

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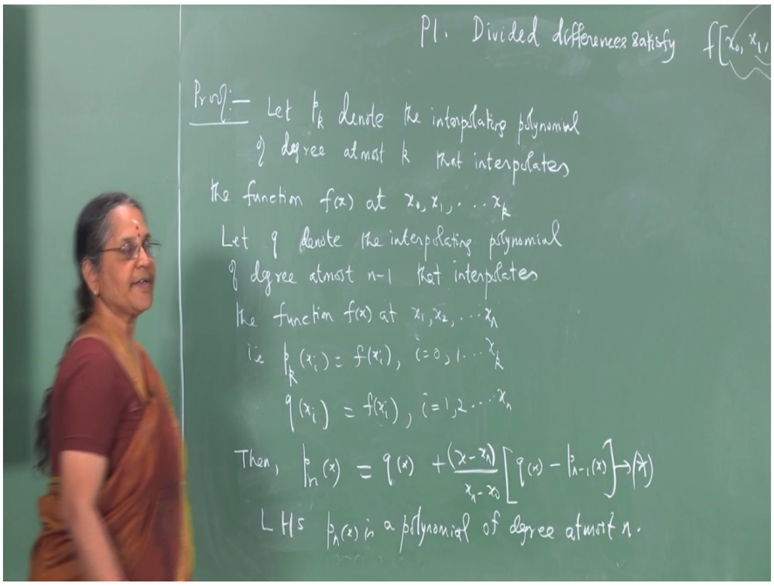


So let's look at the expression star and see what about the left hand side. Left hand side is a polynomial of degree n. How do I say that I have given already the notation? By p k I mean the interpolating polynomial that interpolates the function at x 0, x 1, etc x k and it is of degree at most K. So that notation tells me that p n(x) is a polynomial of degree at most n that interpolates the function at what points x 0, x 1, etc upto x n. So I know that the left hand side p n(x) is a polynomial of degree at most n.

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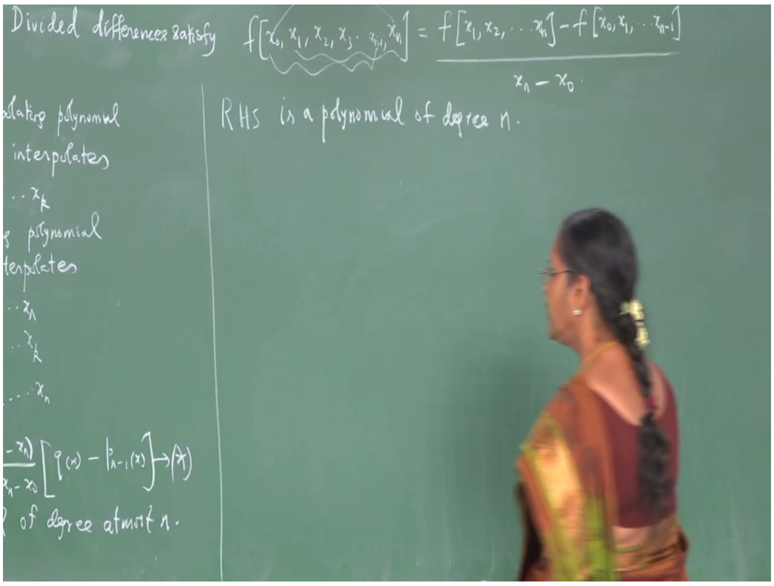


So now let us look at the right hand side and see what it represents. The right hand side if you see I have q(x) I know its a polynomial of degree at most n minus 1. So that polynomial of degree n minus 1. What about p n minus 1? p n minus 1 is a polynomial of degree at most n minus1 that interpolates the function f(x) such that p n minus 1of x i is f( x i) for i is equal to 0,1,2,3 up to x n minus 1.

So this is a polynomial of degree n minus 1, this is a polynomial of degree n minus 1. So the difference is a polynomial of degree at most n minus 1 and that is multiplied by a factor x minus x n. So the second term is a polynomial of degree at most n. So that is added to a polynomial of degree n minus 1 the whole the right hand side is also a polynomial of degree at most n. So the left hand side and the right hand side are both polynomials of degree at most n.

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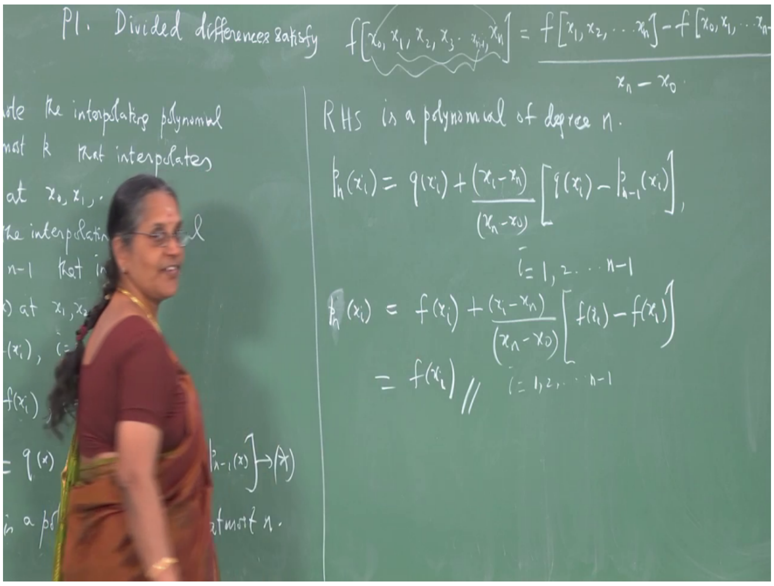


So let us now see what are the properties of the polynomials. Let us look at what is p n (x i)? Sp p n(x i) is q (x i) plus x i minus x n by x n minus x 0 into q (x i) minus p n minus 1 of (x i). So I would like to look at this for a i running from say 1,2,3 up to n minus 1. Let us first focus on these points what are they x 1, x 2, etc, x n minus 1.

What is p n(x i) for i is equal to 1to n minus 1. Just see p n(x i for i is equal to 1 to n minus 1 will be f(x i). So the left hand side is f(x i). What about q(x i)? What does q do q(x i) is f(x i) for i is equal to 1,2,3, upto x n minus 1 and x n and therefore upto n minus 1 q(x i) is f(x i) because it interpolates at these points. So plus x i minus x n divided by x n minus x 0 into what is q(x i) at these points it is f(x i).

What about p n minus 1 (x i)? p n minus 1 (x i) when k is n minus 1 will be f(x i) points 0,1,2,3 upto n minus 1 x n minus 1. And therefore this again will be f(x i), and so this results in f(x i) is equal to f(x i). For what values i is equal to 1,2,3 upto n minus 1. Or I can write this as p n(x i) itself. So p n(x i) is f(x i). So the polynomial that we have written has the property that at x 1, x 2, etc, x n minus 1 it interpolates the function.

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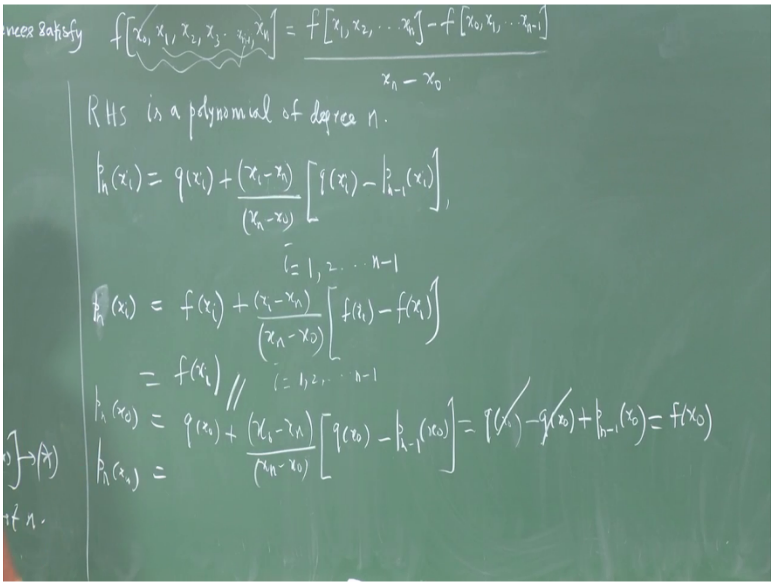


So let us see what it does at x 0 and what its value at x n these are the two points which we have not checked so far. So p n(x 0) will be q (x 0) plus just look at this  x 0 minus x n so substitute for x as x 0 divided by x n minus x 0 multiplied by q (x 0) p n minus 1 (x 0). So that will be equal to q (x 0)  first term (x 0 minus x n) by (x n minus x 0).

So the first term will give you q (x 0) and the next term will give you plus p n minus 1 at x 0. So this says p n (x 0) is the same as p n minus 1 (x 0) and let us see what p n minus 1 does? p n minus 1(x 0)  because p k(x i)is f(x 1) for i is equal to 0 that is it interpolates the function at x 0 and p k is a polynomial of degree at most k. So p n minus 1 is a polynomial of degree at most n minus 1 which interpolates the function at x 0. And so p n minus x 0 is nothing  but f(x 0). So what does it mean yield p n(x 0) is f(x 0). So p n interpolates the function at x 0.

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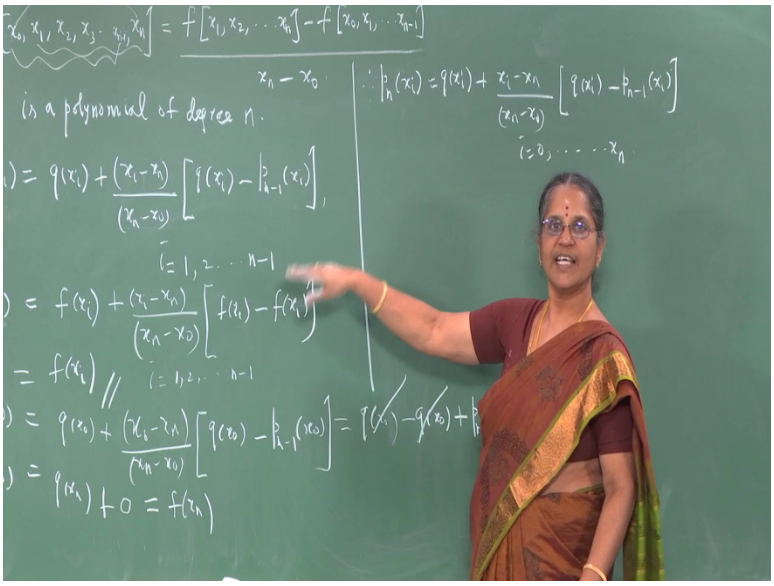


Let us now take p n(x n). What is p n(x n)? So let us use this star and then find out so p n(x n) is q(x n) then the next term x n minus x n. So the next doesn't contribute. Now we look at q(x n). What does q do? q interpolates the function at a set of points x 1, x 2, etc x n. So at x n q(x n)is f(x n)so this is equal to f(x n).

So what is it that we have proved in these steps we have shown p n is an interpolating polynomial of degree at most n that Interpolates the function at x 0 , that Interpolates the function at x 1, x 2, etc, upto x n minus 1, that Interpolates the function at x n that is p n is a polynomial of degree at most n that interpolates the function f (x) at a set of how many points at a set of n plus 1 points, namely it satisfies the property that p n(x i) is f(x i) for i is equal to 0 to n and the left hand side polynomial is equal to the right hand side polynomial at all these points namely we have shown that p n(x i) is equal to this not only at i is equal to 1to n minus 1, but also at x 0 and x n.

So we can write p n(x i) is equal to q(x i) plus x i minus x n by x n minus x 0 multiplied by q(x i) minus p n minus1 (x i) for points is equal to x 0, x 1 etc upto x n. A left hand side p n(x) is a polynomial of degree b the right hand side is also a polynomial of degree n and their values match and give you f(x i) effects and side is also a polynomial of degree n there values match and give you f(x i) for all these points x 0, x 1 etc x n.

So left hand side is an interpolating polynomial interpolating the function f(x) at the point  0, 1, 2, 3 upto n. Right hand side is also a polynomial of degree n that interpolates the function f(x) at these discrete points x 0, x 1 etc x n.  And therefore these two polynomials must be identical because there is a unique interpolating polynomial that interpolates the function at a set of discrete points. So these two polynomials be identical.

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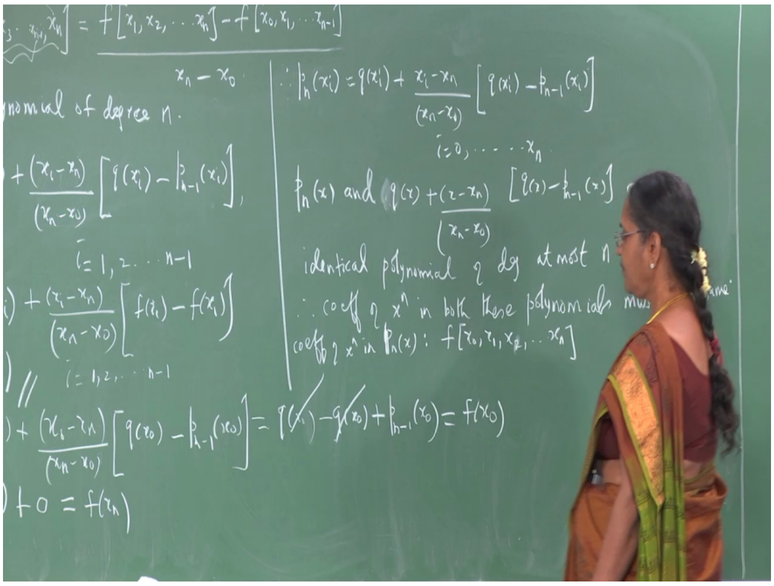

So p n(x) and the right hand side namely q(x) plus x minus x n by x n minus x 0 into  q(x) minus pn minus 1 (x) are identical polynomials of degree at most n that interpolates the function f at a set of discrete points x 0 to x n. And therefore coefficient of x to the power of n in both these polynomials must be the same.

So let us compute the coefficient of x to the power of n in p n(x). Do you know what is the coefficient of x to the power of n in p n(x)? Just recall the divided difference formula last term had an n th degree polynomial and the factors were x minus x 0 , x minus x 1, etc x minus x n minus 1. So the coefficient of x to the power of n is nothing but the n th order divided difference. So the coefficient of x power n in p n(x) is the n th order divided difference f(x 0 x 1 x 2 etc upto x n).

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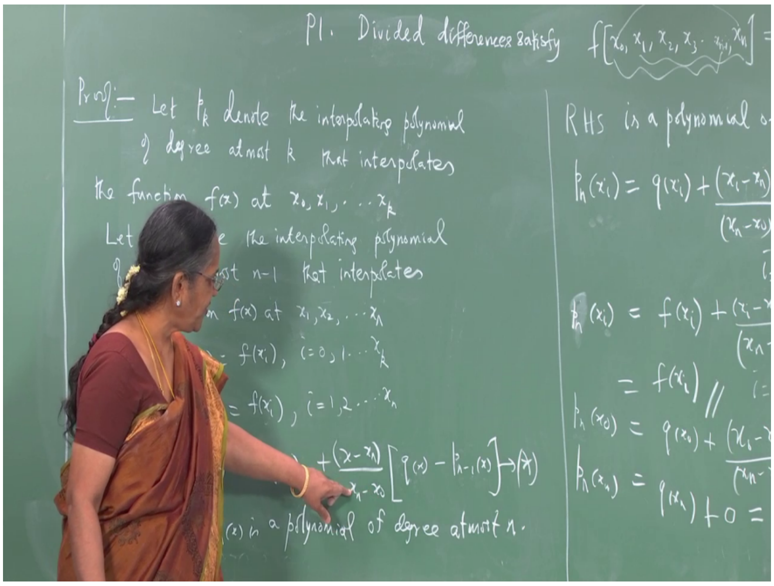


So we now write down the coefficient of x to the power of n in the right hand side of star. So we look at the first term the first term is q(x) which is a polynomial of degree at most n minus 1. So this doesn't contribute. So let's look at this, this is a polynomial of degree at most n, so the coefficient of x to the power of n coming from this polynomial is going to be see x minus x n multiplies a polynomial of degree n minus 1, right?

So this higher x power n minus 1 Coefficient multiplies the x term with 1 by (x n minus x 0) as the coefficient will give you the coefficient of x to the power of n. So what is the coefficient that is 1 by (x n minus x 0) into the coefficient of x to the power of n minus 1 in q(x) and we know q(x) is a polynomial of degree at most n minus 1 again we recall the divided difference interpolating polynomial and what is the coefficient of x to the power of n minus 1, it is nothing but an n minus 1 order divided difference but what are the points at which q interpolates the function at i which is 1,2,3 upto x n.

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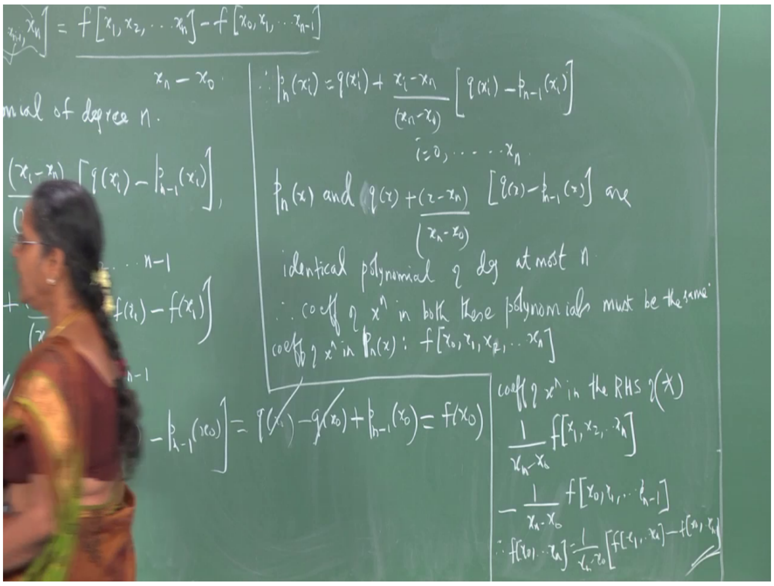

And therefore the coefficient will b f [ X 1 X 2 etc upto x n] that comes from the first term. What about the second term? There again p n minus 1 is a polynomial of degree n minus 1. That multiplies the factor x minus x n giving you a polynomial of degree n whose coefficient is going to be 1 by x n minus x 0 which is the coefficient of X there in to the coefficient of X n minus 1 in p n minus 1 at p n minus 1 is a polynomial of degree at most n minus 1 that interpolates the function where at x 0 x 1 x 2 etc upto x n minus 1 and therefore what is the coefficient there it's going to be f [x 0 x 1 x 2 etc upto x n  minus 1].

So these two coefficients must be the same, that is what we have concluded. So what does it give that gives you the left hand side is f [x 0 x 1 x 2 etc upto x n minus 1] and the right hand side is 1 by x n minus x 0 into first term f [x 0 x 1 x 2 etc x n] minus the second term f [x 0 x 1 x 2 etc upto x n  minus 1]. And therefore that proves the property of the divided difference that we want to establish. So therefore f [x 0 x 1 x 2 etc upto x n] must be 1 by (x n minus x 0)into f [x 0 x 1 etc  x n] f [x 0 x 1 etc x n  minus 1] proving the property of the divided differences.

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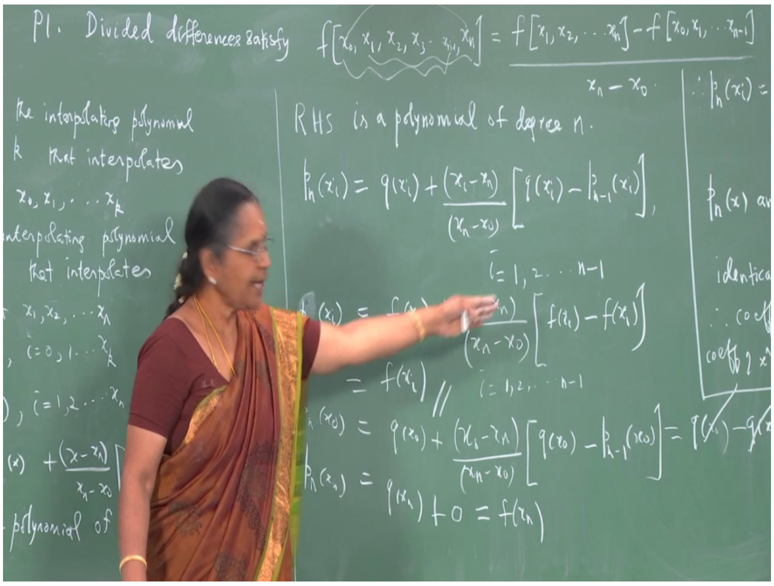


And therefore this property helps you compute a higher order divided difference in terms of the lower order divided differences. So when you form the table divided difference table then the entries in each of those columns can be computed in terms of so the higher order difference computed in terms of the lower order divided differences.

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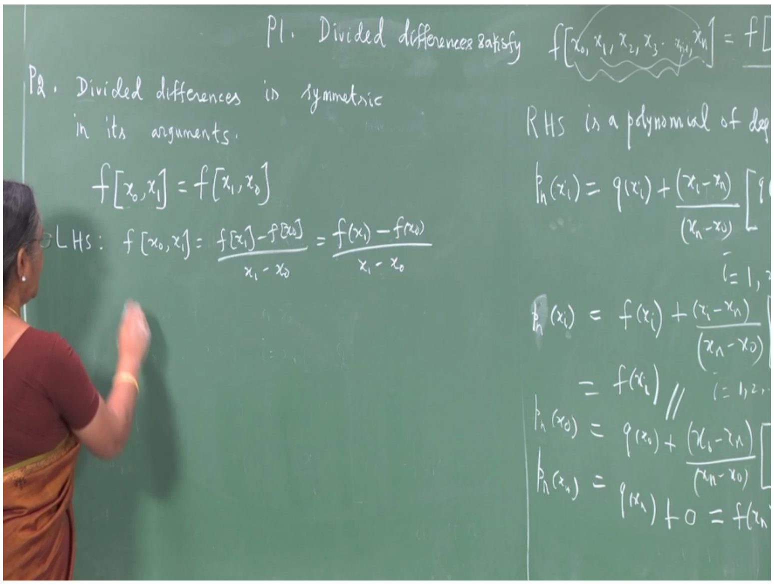


Let us now prove the second property namely divided differences is symmetric in its arguments. If you take divided difference of a particular order then it is symmetric in its arguments what does it mean? Let us just understand. If I take the first order divided difference f(x 0 x 1) the arguments are x 0 x 1 the result says it is the same as f(x 1 x 0) .

And that is obvious because what is the left hand side? The left hand side is f(x 0 x 1) by definition it is f(x 1) minus f(x 0) divided by x 1 minus x 0. I have used the previous property where I take n to be 1. I have f(x 0 x 1) and that is equal to n is 1, so f(x 1) minus n is 1 so f(x 0) by x 1 minus x 0. And what about the what is f(x 1) it is the 0 th order difference so it is the function value itself. So this by x 1 minus x 0.

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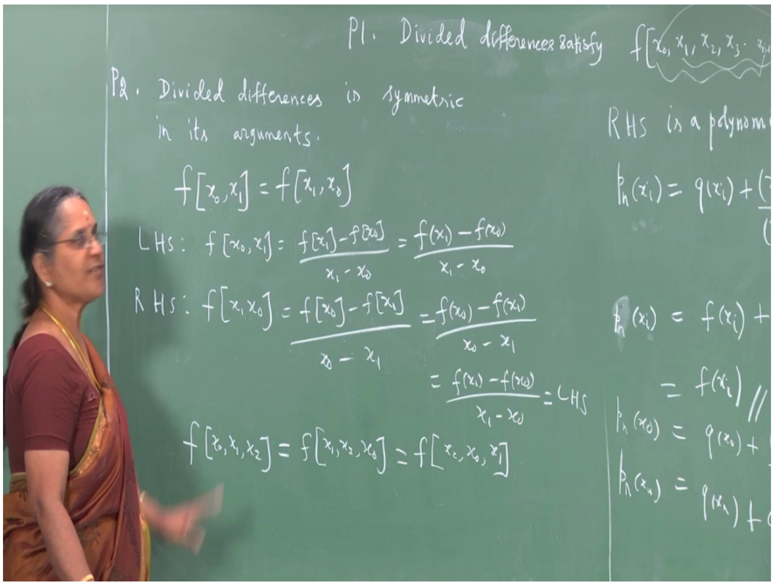


So let us take the right hand side which is f(x 1 x 0). So by definition what is it? It is x 0 minus f(x 1) divided by x 0 minus x 1. So I again use what the definition of 0 th order difference it is `the function value at x 0 minus x 1 divided by x 1 minus x 0. Which is f(x 1) minus f(x 0) divided by x 1 minus x 0. And that is the left hand side.

So it is symmetric in its arguments so what will be the result in the case of second order differences f(x 0 x 1 x 2 ) must be equal to f(x 1 x 2 x 0) and that must be the same as f( x 2 x 0 x 1) namely when I sightlically change these arguments.

So x 0 x 1 x 2 then x 1 x 2 x 0 then x 2x 0x 1 can form the second order difference then the result is the same is what this property tells you so let us prove it in general namely we will show that if I take the n th order divided difference f(x 0 x 1 etc x n ) and then arrange these slightlically in any manner then all these divided difference values will be the same that is what we would like to show it is symmetric in its arguments. That is what will have to be proved.

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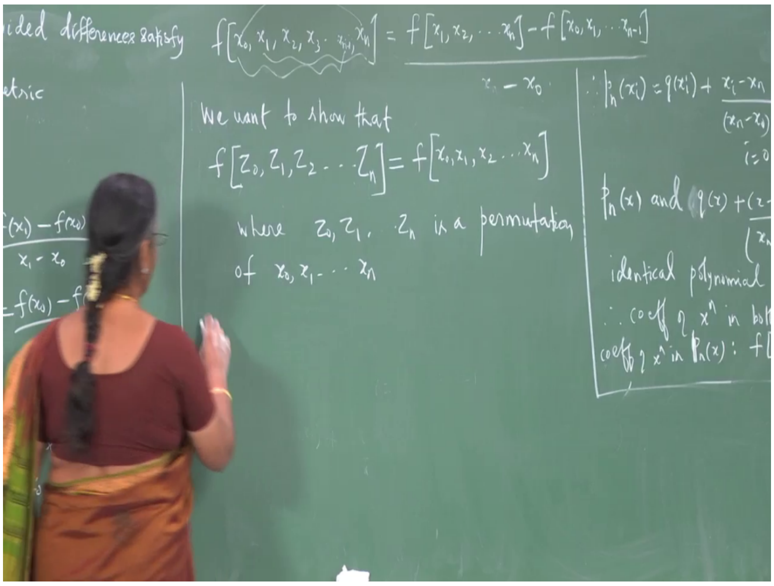


So can you think of how we should show this? So we want to show the following. Show that the n th order divided difference say f[ z 0, z 1, z 2 etc z n] is the same as the n th order divided difference x 0, x1 x 2 etc x n where Z 0 Z 1 etc Z n is a permutation of x 0, x1 x 2 etc x n. They are essentially x 0, x1 x 2 etc x n but some cyclic permutation has been done just as one can show f(x 0 x 1 x 2 ) is so this is my x 2 x 0 x 1 which I have called as z 0 z 1 z 2. So these are essentially x 0 x 1 x 2 arranged in some order. So now the proof must be umm easy what do we do?

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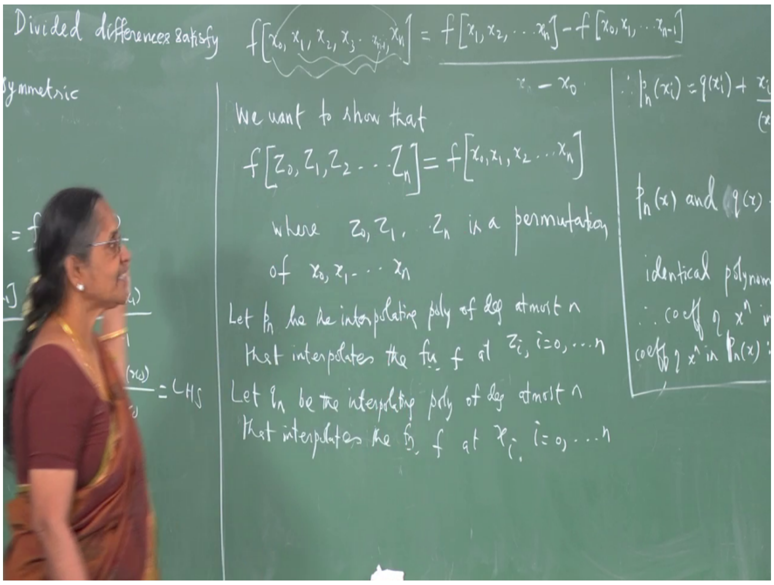

Let us construct a polynomial of degree at most n that interpolates the function at the set of points z 0 z 1 etc z n. And denote by say p n. So let p n be the interpolating polynomial of degree at most n that interpolates the function f(Z i) for i is equal to 0 1 2 3 upto n. And me let me denote by q n the interpolating polynomial of degree at most n that interpolates the function f ( x i) i is equal to 0 1 2 3 etc x n.

Remember z i are not different from x i they are just cyclic permutations o0f the x i. So they are essentially the same set of points at which the polynomial p n interpolates the function f. So p n must be identically same as q n they are polynomials of degree at most n each one interpolates the same function f at the set of same points x i and z i where z i is a cyclic permutation of x i. So the two polynomials must be identical because there exist a unique polynomial that interpolates the function at a set of discrete points.

And therefore the coefficient of x to the power of n in p n must be the same as coefficient of x to the power of n in q n. What is the coefficient of x to the power of n in p n recall the divided difference interpolation formula it is the n th order divided difference. What is it? It is f [z0 z 1 etc z n] that must be the same as the coefficient of x to the power of n in q n.

So recall the divided difference interpolation polynomial what is the coefficient of x to the power of n it is the n th order divided difference f ( x 0 x 1 etc x n). So the coefficient of x to the power of n in both the polynomials must be the same because they are identical polynomials and therefore these must be the same and therefore the divided differences is symmetric in its argument and that is how we prove the second property. So let us write down the details.

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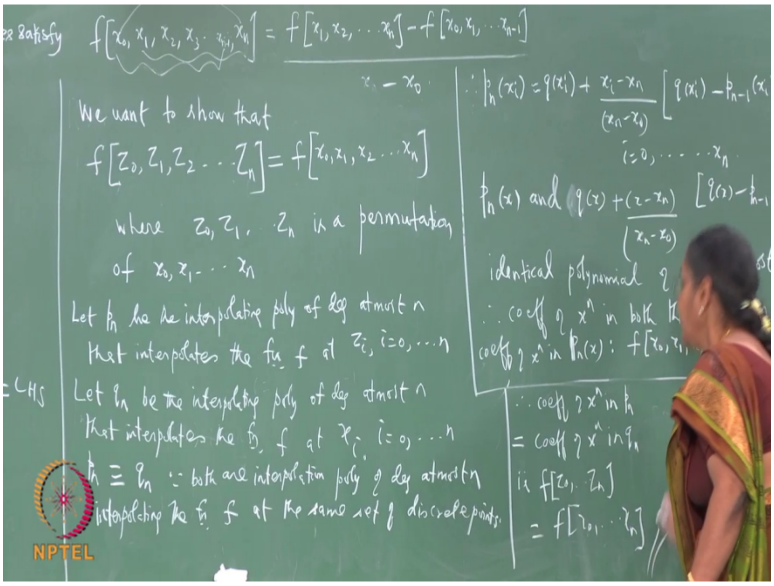


So p n is identically the same as q n since both are interpolation polynomials of degree at most n interpolating the function f at the same set of discrete points. So therefore coefficient of x to the power of n in p n must be equal to coefficient of x to the power of n in q n that is f [Z 0 Z 1 etc Z n] must be the same as f[x 0 x 1 etc x n] and that proves the second property that it is symmetric in its argument.