Numerical Analysis
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Lecture No 6
Part 2
Error in Interpolation-2

So we shall now consider the case when the given function f(x) is approximated by a quadratic polynomial which we de0e by P 2 of x, so our theorem tells us that f(x) + P 2 of x = n + 1 derivative, what is n + 1 in this case, how many points should be given to you? Information at 3 points is required, so that we can represent the function by a polynomial of degree to which is P 2.

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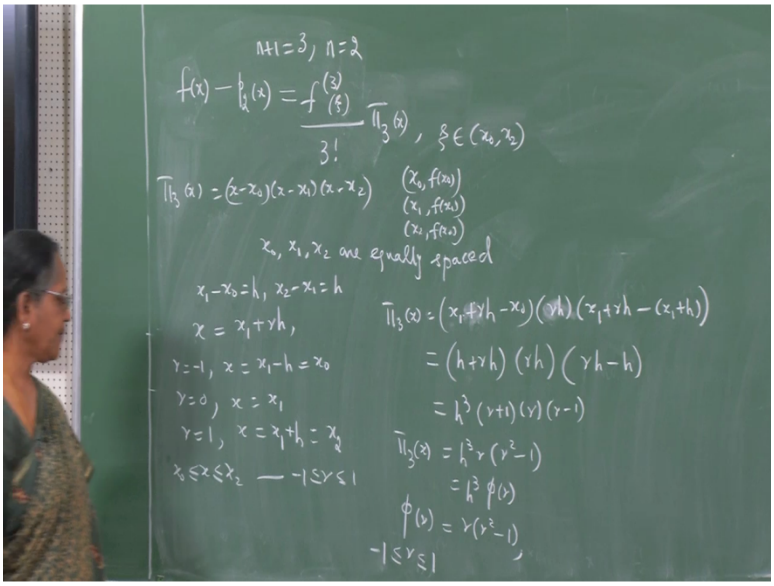

So f(x) + P 2 of x is the n + 1 derivative that is the third derivative at sum Xi divided by n + 1 factorial into Pi n + 1 of x. What is Pi 3 of x? It is x + x 0 into x + x 1 into x + x 2. So information is given at x 0, f of x 0, x 1, f of x 1 and x 2, f of x 2, so this Xi belongs to the interval x 0 to x 2. And what is P 2 of x, P 2 of x is the quadratic Lagrange interpolating polynomial that interpolates the function at the set of discrete points x 0, x 1, x 2 and what do you know about x 0 x 1 x 2? X 0, x 1, x 2 are equally spaced. So we are determining the error bound in case the interpolation points are equally spaced. So in this case I shall take my x 1 to be such that our x 1 + x 0 is h, x 2 + x 1 is h. So any x, I take this to be x 1 + r h. So when r is + 1 what do I get? X is x 1 + h and what is it that is x 0 so I am at x 0.

Whereas when r is 0 my x is at x 1 and r is 1 my x is x 1 + h and that is x 2, so as x runs between x 0 and x 2, my r runs between + 1 and + 1, so I shall write down this Pi 3 of x as, what is Pi 3 of x? It is x + x 0. What is x? x is x 1 + h or x is x 1 + rh + x 0 than x + x 1, so x + x 1 is rh, so I shall directly write down that and then I have x + x 2. So what is x 2 x 2 is x 1 + h, so x is x 1 + rh + x 2 which is x 1 + h.

So this gives me x 1 + x 0 which is h so h + rh into rh into x 1 gets canceled so I have rh + h. So taking the factor h from each of these I have h cube into r + 1 into r into r + 1. So h cube into r into r square + 1 and that is Pi 3 of x, so I shall write this as h cube into Phi of r, what is Phi of r? Phi of r is r into r square + 1 and r runs between + 1 and + 1, so r is such that it ranges between + 1 and + 1.

So now let us look at the properties of Phi of r, when we know the properties of Phi of r, from here we can get some information about Pi 3 of x and that will help us in finding the error bound because what is the error bound? Modulus of f of x + P 2 of x is the maximum value of third derivative of f at Xi in the interval x 0 to x 2 by 3 factorial into modulus of Pi 3 of x. What is modulus of Pi 3 of x? It is h cube into modulus of Phi of r. So we require the properties of Phi of r, what is Phi of r? It is r into r square + 1.

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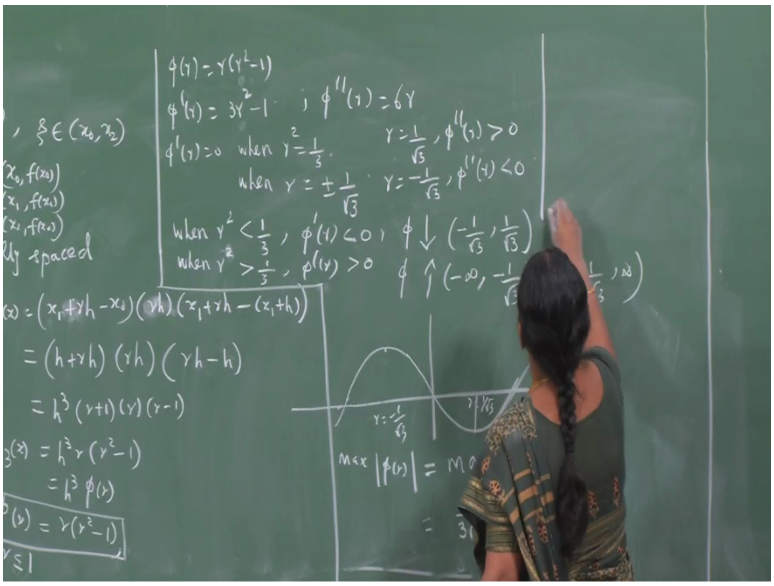

So let us find out Phi dash of r, so Phi dash of r will be 3 r square + 1 and it vanishes when r square is 1 by 3 or it vanishes when r = + r + 1 by root 3. So now let us see what happens when r square is less than 1 by 3. When r square is less than 1 by 3, then here when r square is less than 1 by 3, we have Phi dash of r to be negative. On the other hand, when r square is greater than 1 by 3 Phi dash of r is going to be greater than 0. So this tells us that Phi decreases in the interval + 1 by root 3 to + 1 by root 3 and this tells us that Phi increases in the interval + infinity to + 1 by root 3 and 1 by root 3 to infinity.

Phi double dash of r is going to be 6 r, so at r = + 1 by root 3 Phi double dash of r is positive and at r + 1 by root 3 Phi double dash of r is negative therefore, Phi attains minimum at r = 1 by root 3 and attains a maximum at r = + 1 by root 3 and it decreases between + 1 by root 3 to + 1 by root 3 and on + infinity to + 1 by root 3 and 1 by root 3 to infinity it increases. So the graph of this function is such that and Phi of r is 0 at r = 0, so at r = + 1 by root 3 it attains a maximum, and at r = + 1 by root 3 it attains a minimum, in this interval between + 1 by root 3 to + 1 by root 3 it decreases and beyond this it increases and therefore, the maximum of Phi is attained at this point. So we compute the maximum value of Phi which is attained at r = + 1 by root 3.

So maximum of modulus of Phi of r for all (()) (9:51) between + 1 and 1 is going to be maximum of modulus of r into r square + 1, so r is + 1 by root 3, r square is 1 by 3 + 1, so that gives you 2 by 3 root 3, so when I know maximum of modulus of Phi of r, I can write down what is the maximum of Phi 3 of x, when x runs between x 0 and x 2, so it is going to be h cube into 2 by 3 root 3. So we shall make use of this here and obtain the error bound in quadratic interpolation.

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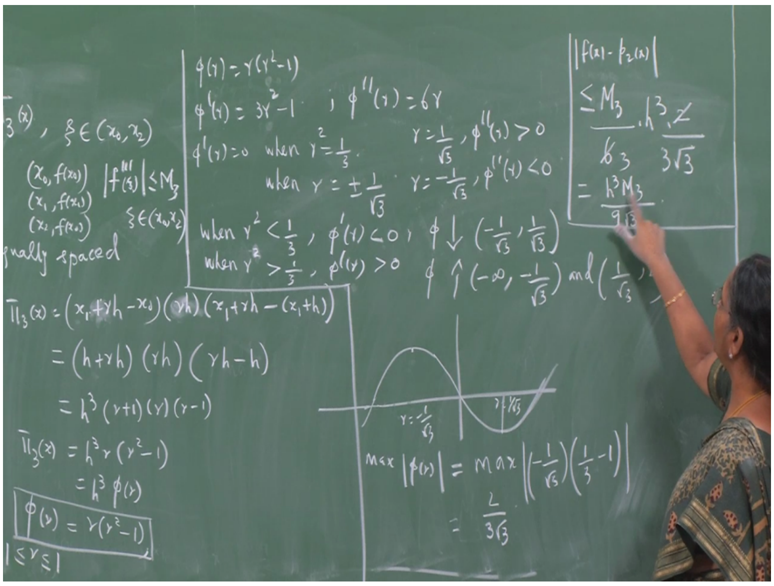

So let us work out the results here, so modulus of f of x + P 2 of x therefore is less than or = M 3, what is M 3? Modulus of the third derivative of f is less than or = M 3 for Xi in the interval x 0 to x 2. So m 3 by 3 factorial that is 6 into maximum of Pi 3 of x, Pi 3 of x is h cube into Phi of r. And have the result here so that gives you h cube into 2 by 3 root 3, so this is h cube into M 3 divided by 9 root 3. So the error bound in quadratic interpolation is such that modulus of f of x + P 2 of x is less than or = h cube into M 3 by nine root 3. So this gives you the size of the bound on the error when you approximate the function f of x by a quadratic interpolating polynomial.

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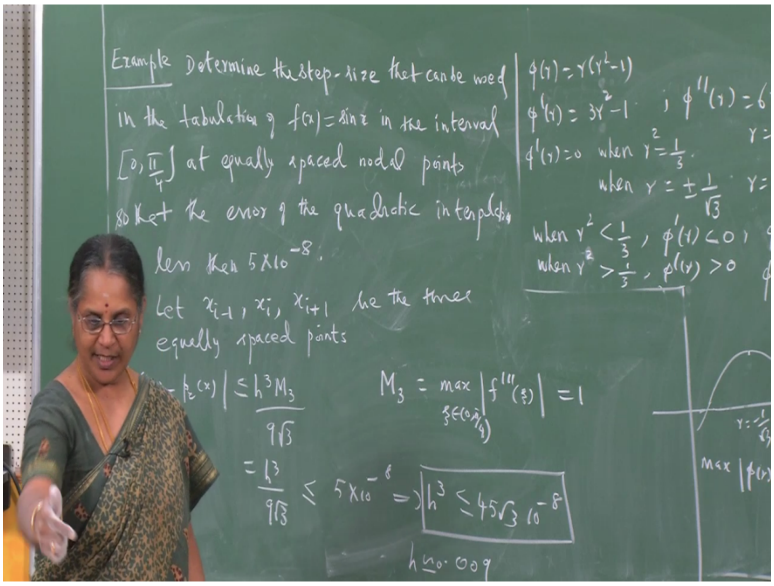

So we shall again illustrate this result by means of the following example, let us consider this example the question is obtained step size h that can be used to tabulate the values of the function f of x = sin x in the interval 0 to Pi by 4 at equally spaced nodal points, so that the error of the quadratic interpolation is less than 5 into 10 to the + 8. So let us take the 3 equally spaced points to be x i + 1 x i, x i + 1 be the 3 equally spaced points, which we want to choose from the table of values of sin x and then interpolate the function sin x by a quadratic polynomial that Interpolates the function at a set of these 3 points. We have already computed the size of the bound on error namely, modulus of f of x + P 2 of x is less than or = h cube into m 3 divided by 9 root 3.

So, now I want to find out the step size h such that x i is x i + 1 + h, x i + 1 is x i + h, so that I satisfy the requirement given in the problem. So I require M 3 and that is the third derivative of sin x in absolute value and that will be less than or = 1 in the interval 0 to Pi by 4. So M 3 can be determined and that is the maximum of modulus of f triple dash of Xi for Xi in the interval 0 to Pi by 4. So the first derivative is cosec, second derivative + sin x the third derivative is + cos x and the maximum of that in this interval is going to be 1 so M 3 is known. So I want h cube by 9 root 3 into 1 to be less than or = 5 into 10 to the + 8, so h cube must be less than or = 45 root 3 into 10 to the + 8 and if you evaluate h and that turns out to be approximately 0.009.

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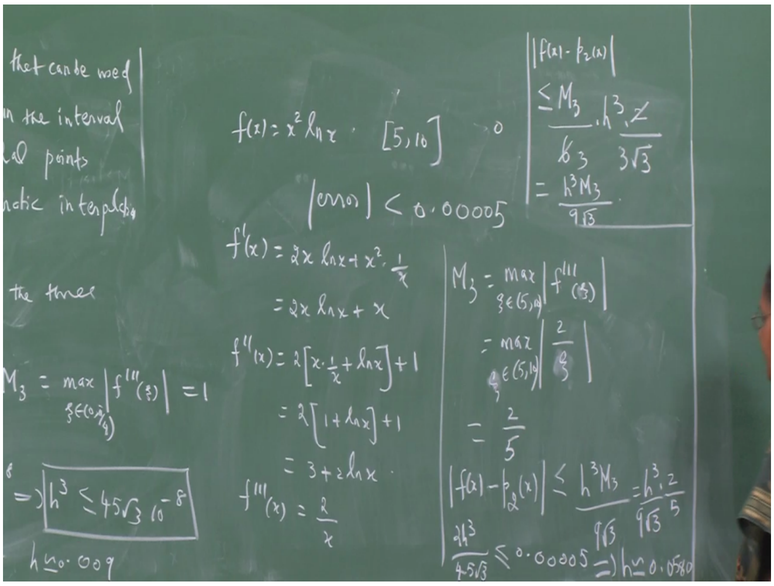

So I shall give you a0her problem and you can try to work them out in the as a home work. Suppose f of x = x square into log x the question again is determined the step size that can be used in the tabulation of this function in which interval? In the interval 5 to 10 at equally spaced nodal points, so that the error in quadratic interpolation can be less than 0.00005 that is what the question is. So maybe we can work out the details quickly so that that is a0her example that illustrates our result and I can give you more problems similar to this in the assignment sheet. So what do we want? We require f dash of x that will be 2 x into log x + x square into 1 by x. So it is 2 x log x + x we require f double dash of x so it is 2 into x into 1 by x + log x into derivative of x + 1, so it is 2 into 1 + log x + 1, so it is 3 + 2 log x.

And I also require the third derivative because I want M 3 so that will give you 2 by x, where does x belong to? X belongs to the interval 5 to 10 and what do I want? I want M 3 which is maximum of modulus of f triple dash at Xi for Xi in the interval 5 to 10. So we require maximum of modulus of 2 by x for x lying in the interval say 5 to 10. Since f triple dash at Xi I can write this as Xi and this also as Xi. So what is the maximum, where is it attained? The maximum is attained at the lower limit, so it is 2 by 5. So we can now use the inequality for the error bound and write down that modulus of f of x + this quadratic polynomial is less than or = h cube into m 3 by 9 root 3. This is h cube into M 3 is 2 by 5 9 root 3 and what do I want? I want 2 h cube by 45 into root 3 to be less than or = 0.00005 and that gives h to be approximately 0.0580.

So, if you take the step size h to be such that h is approximately 0.058 and chose 3 points x i + 1 x i, x i + 1, which are equally spaced such that x i is x i + 1 + this h, x i + 1 is x i + this h, then if you approximate this function x square log x in this interval by a quadratic interpolating polynomial, the error can0 be greater than this value that is what the result says.

So summarizing what we have done in this class, we obtained the error in interpolation and then obtained the error bounds for the cases when we approximate f of x by a constant polynomial a linear polynomial or a quadratic polynomial for the cases when the interpolation points are equally spaced. We derive the Lagrange interpolation polynomial for points which need 0 be equally spaced and also obtained the expression for error in interpolation for that case and then obtained the error bound for the special cases when the points are equally spaced. We shall continue with a0her way of developing interpolation polynomials when the points are 0 equally spaced and we use the concept of divided differences and we shall consider this in the next class.