Numerical Analysis
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Lecture -1
Introduction, Motivation

Good Morning everyone! I will be handling a course on Numerical Analysis and this course consists of say about 45 lectures. And but numerical analysis is a branch of mathematics that is concerned with the theoretical foundations of numerical algorithms for the solution of problems that arise in scientific applications. It actually covers the mathematics and the methodologies that underlie the techniques for solving scientific problems. For example if one wants to compute 1 sin 25 degrees s and he wants to use his calculator then he just presses some buttons on the calculator and then at the end he is able to arrive at the value of sin 25. Thus the calculator have a knowledge of how to compute sin 25 degrees. Actually the calculators are programmed to implement certain numerical techniques with the help of which one is able to arrive at the value of sin 25. So the computers are programmed to implement the numerical techniques that we develop in this course on numerical analysis.

Suppose that one tells you that he uses a computer to design an aircraft he is able to predict the weather or he is able to solve complex problems in science or engineering then he actually uses this computer to implement certain numerical algorithms which he knows to solve these complex problems in science or engineering. Therefore in this course we would focus on the development of such numerical algorithms and then try to solve some simple problems with the help of these numerical techniques. And we focus our attention on the following three issues namely.

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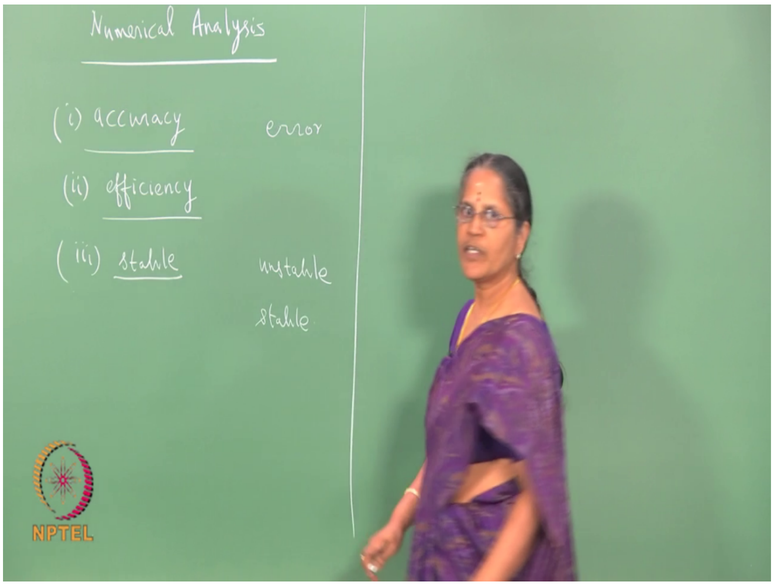

We shall be concerned with how accurate our numerical algorithm is? What is the amount of error that is incurred at each step of scientific computation? Can we control the error that is incurred? or how to control this error that is incurred at each step of computation?

And secondly we will focus on the efficiency of the numerical algorithm that we have developed. And thirdly we will also see is the method that is developed is a stable method? If small deviations are given to the input to the computer whether the outputs have large deviations if this happens then we say the method is an unstable method or corresponding to small deviations in the input the deviations in the output are also small.

Then we say that the method is a stable method. So our goal would be in this course to solve certain set of problems that arise in scientific applications. And to solve these problems we require certain techniques which are numerical techniques. So develop these numerical techniques or algorithms and implement them in solving our problems. So we will illustrate each of these techniques on the use of each of these techniques by solving some simple problems keeping in mind that we should be very careful about how accurate a solution is? How efficient our method is? And Is the method stable?

So numerical mathematics does not require a large amount of background but it does require a good knowledge about the background. So we shall start with some basic tools of calculus the mathematical preliminaries and then start dealing with the topics that we would like to understand. So in fact in this course we will be focusing our attention on about 5 important topics.

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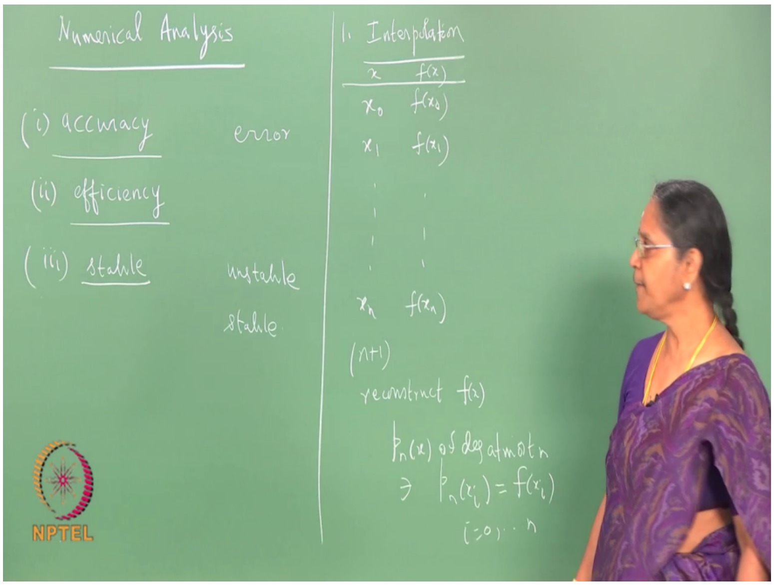

The first topic being interpolation, so let us see what is interpolation? Suppose you have some information about a function at a set of say n plus 1 points namely x 0 f(x 0), x 1 f(x 1), and so on x n f(x n). So you are given n plus 1 points and the corresponding values of the function so x is have independent variable and f(x) is dependent variable depending on x whose values are known at a set of n plus 1 discrete points.

I would like to reconstruct the function f(x) by seeking a polynomial p n (x) of degree at most n such that this takes the value at x i f(x i) for i is equal to 0,1,2,3 upto n. Or in other words I would like to get a smooth continuous curve namely a polynomial p n (x) of degree at most n that passes through the points x i, f(x i) in the xy plane. So if I do that then I have reconstructed this function f(x) which takes these values at a set of discrete points. Or one may also come across the following situation.

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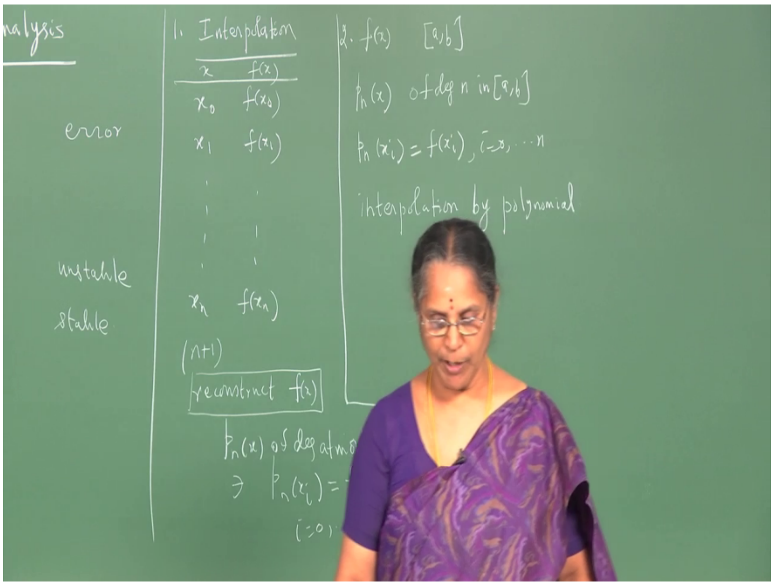

Namely I know the function f(x) say in some closed interval [a,b] all of whose values are known to me. I am interested in approximating this function f(x) by means of a polynomial p n (x) of say degree n in this interval [a,b]. Then I would see the polynomial p n(x) of degree n such that p n takes the value of f(x i) of i is equal to 0,1,2,3 of 2n.

So if I seek or if I determine such an approximation to the function f(x) by means of a polynomial p n(x) such that p n passes through the points x i and p n takes the value of f(x i) at the point x i then I have a polynomial that approximates this continuous function f (x) in the interval [a,b] whose values are all known to me in the entire interval [a,b].

When I do any of these right either problem 1 or problem 2 and seek a polynomial p n(x) that interpolates the function f(x) at a set of given discrete points then I am essentially doing right interpolation by polynomial.

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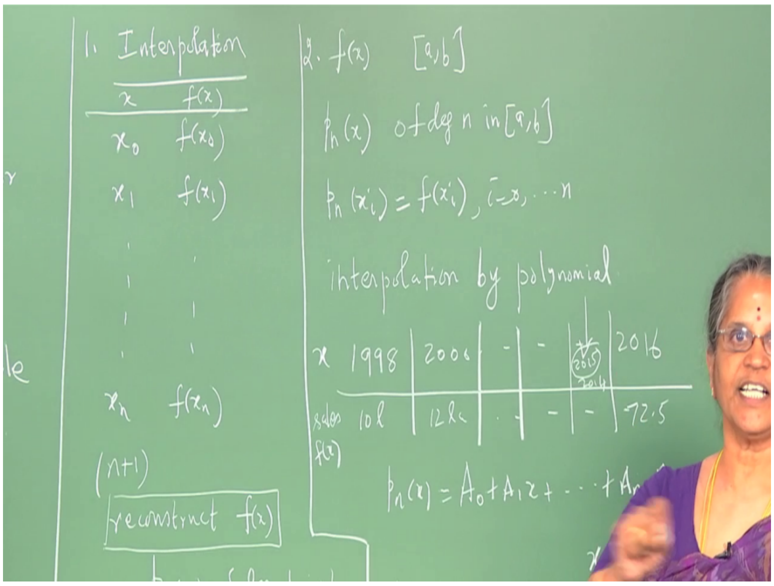

So for example say if I am given information about rainfall in every two years or the sales of certain commodity in every two years starting from 1998 to 2000 etc upto 2016. I know in these years what is the sales of certain commodity say it is some 10 Lacs this becomes 12 lacs and so on. So it has increased by say 72.5 lacs in these years one may be interested to know although the information is available one has no information about say what happens in the year 2015.

Information given here is in the year 2014 but one has no information in the year 2015. But for some practical purposes one may require the value or the amount of sales in the year 2015. Right then one can fit a polynomial p n(x) so this will be x and this will be the value of f at that x which are given to you.

So one can find an interpolating polynomial that interpolates this data and obtain a polynomial of the form say A 0 plus A 1 x plus etc plus A n x to the power of n. For any x that lies between 1998 and 2016 so one will be able to immediately obtain the missing information in the given data.

So the polynomial interpolation helps you to get the value of the function at any point x within the interval so we will be trying to get algorithms for the construction of such polynomials which are known as interpolation polynomials that interpolates the function at given set of points.

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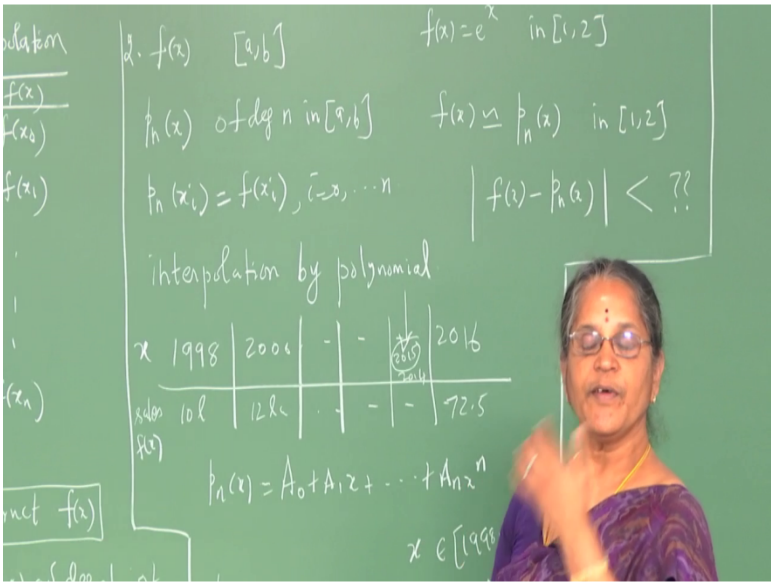

While doing this we do incur some error. For example f(x) is equal to say e to the power of x in some interval [1,2] and we are approximating this function by a polynomial say p n(x) in the interval [1,2]. And some result we would incur an error given by the difference between f(x) and P n(x). So we should be able to see what is the bound on the error namely absolute value of the difference between the two.

And see whether it can be made less than some quantity at any x which is in 0 in [1,2]. So that we will be concerned with the accuracy of the approximation that we have made using this polynomial interpolation. Next question is what happens if the points are equally spaced? or if the points are arbitrary located? So we should be in a position to construct polynomials depending upon what the given data is?

So we shall look into all these aspects while constructing interpolation polynomials. So that would be the first topic that we would focus on in this course. Now once we have constructed an interpolation polynomial than the natural question that comes to our mind is, ok I am given a set of information like this, will I be able to get some information may be about the derivative of this function at any x which lies between x 0 and s n that is the question or the second derivative the higher order derivatives.

Will I be able to get some information about the derivatives right for example if one writes down say the distances covered by an automobile at some intervals of time. I would like to know at what speed will this vehicle has moved between this time and this time. Which means I require the information about the derivative of the function values which are given to me namely the velocity v the rate of change of displacement or the rate of change of distance that is travelled.

So there are problems in which we require the approximation of the derivatives of certain functions whose values are prescribed in the form of discrete data. So how are we going to approximate the derivatives when we have the function values specified in the form of a discrete data.

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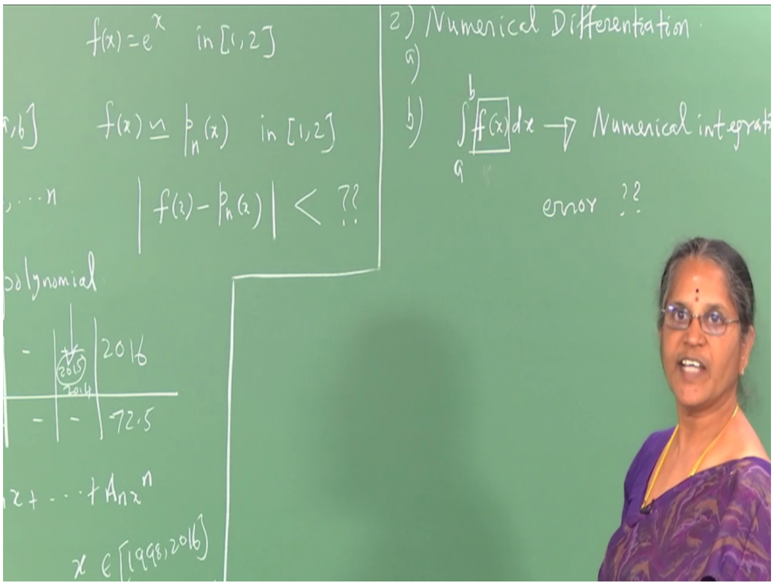

So these aspects will be covered in numerical differentiation. You may be given an integral of the form. A integral a to b f(x) dx and this function f(x) may be a complicated function or the definite integral may be such that you will not be able to apply any of the techniques that you already know to evaluate this integral. So in that case we should be able to get a value of this integral and we do this right in numerical integration.

As I said earlier we because we are integrating and evaluating it numerically there would be some amount of error that we would have incurred so we do take care of the accuracy in evaluating this integral numerically and we would also focus on the computation of what the error is and how to control this error and we get better and better methods. So that the error can be made as small as we please.

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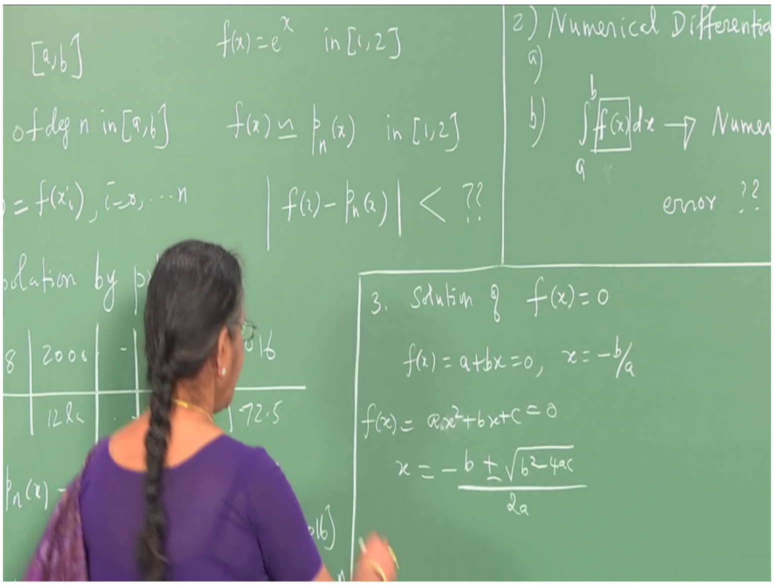

So these aspects will be covered in numerical differentiation and the integration then we move on to problems where we will discuss about the solution of either coincidental or algebraic of the form f(x) is equal to 0. And we all know that if f(x) is a linear equation of the form a plus b x is equal to 0, if suppose f(x) is a quadratic equation a plus a x square plus b x plus c equal to 0 then you know that this equation posses two roots given by minus b plus r minus root of b square minus 4ac divided by 2a involving radicals. So you will be able to solve a quadratic equation.

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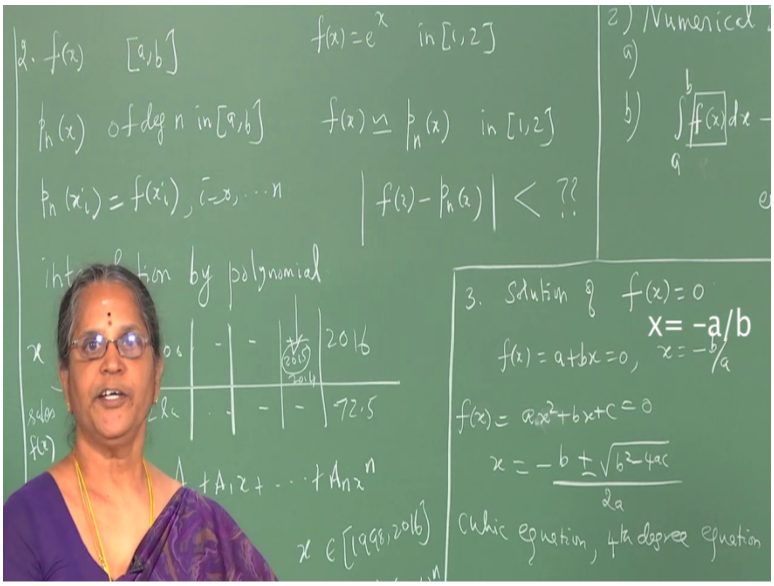

And it is known that a cubic equation 4 th degree equation which is called aquatic equation. These equations in general can be solved and the solution can be given in terms of the radicals under certain conditions. So one tends to think that it may be possible to solve equations of this form say if we restrict even to integer coefficients it may be possible to solve for example a fifth degree equation of the form say for x to the power of 5minus 2x minus 7 equal to 0. In terms of the radicals no the answer is no.

One will not be able to get a solution right of a queue of a fifth degree equation of this form in terms of the radicals and in general it is not possible to write down the solution in terms of such radicals right in the case of equations whose degree is greater than or equal to 5 and these are algebraic equations but we may come across equations which involve coincidental functions like log x c power c etc. the trigonometric functions the exponential functions and so on.

In f(x) then solving these equations analytically is (())(18:10) is a it's a difficult one so one has to seek the techniques or numerical methods for solving such equations numerically so we will be focusing on the techniques which are both direct as well as iterative techniques for solving these equations. And as I said we would perform a thorough analysis of the error if we do an iterative technique then we will be generating successive iterates for the roots of this equation f(x) is equal to 0. So we would like to see as the number of iterations increase whether the sequence of iterates that we have generated for the roots of this equation though they converge to the solution correct to the desired degree of accuracy that we want all these things will be dealt within the topic on solution of algebraic (())(19:15) equations.

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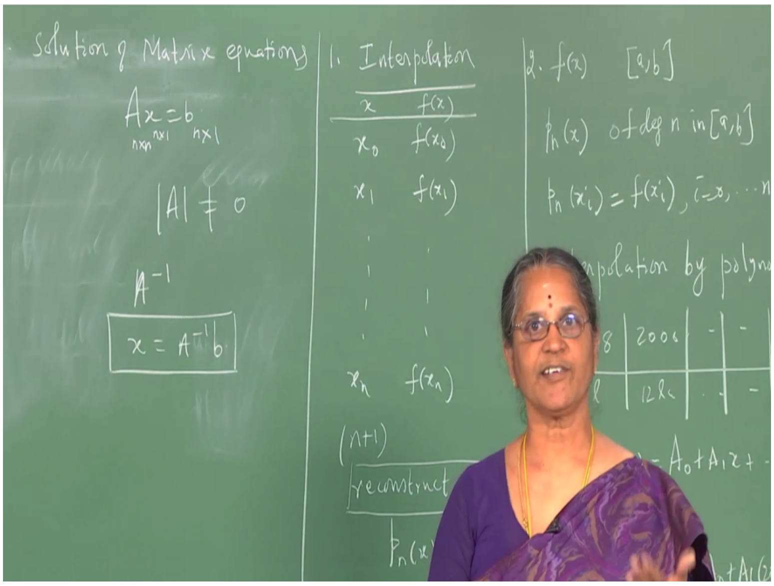

We will also focus our attention on the solution of matrix equations of the form Ax is equal to b. So just recall how do you solve this equation Ax is equal to b where A is a n cross n matrix and x is an n cross 1 vector and b is an n cross 1 vector. if determinant A is different from 0 alright when matrix A is invariable A is non singular determinant is different from 0 then in that case you will be able to compute the inverse of this matrix and write down the solution as A inverse b. this is how you have solved some simple matrix equations where A is a 3 cross 3 matrix or 2 by 2 matrix.

In your previous semesters or you would have solved them in school. Suppose A is such that it is a very large matrix in scientific applications when you solve a certain differential equation using finite difference methods or finite elements method you would come across large number of algebraic equations where namely n will be large and then computation of inverse is going to be a complicated 1 and therefore we need to find techniques or developed techniques so that we should be able to compute A inverse numerically.

In this topic we will develop a number of techniques namely both direct as well as iterative techniques for solution of matrix equations of the form Ax is equal to b. As I said earlier because we are numerically obtaining the solutions of these matrix equations errors will be incurred at each step and we will be focusing on the error analysis in this topic also.

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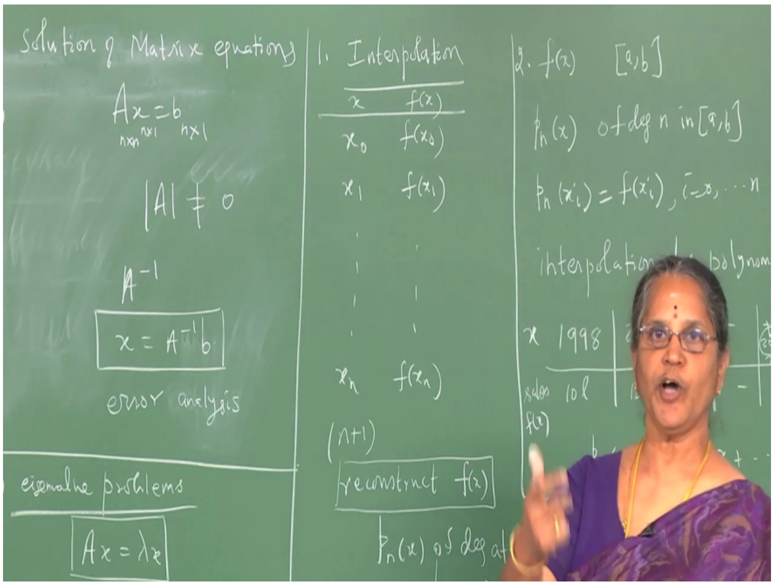

And once we do this then we also consider in this section the Eigen value problems we would be determining a scalar lambda a non zero vector x such that this matrix equation Ax is equal to lambda x is satisfied this is called and Eigen value problem and we will look at numerical methods for computing the most dominant Eigen value because that is very important in many applications. So our numerical method that we are going to develop will enable us to compute the most dominant Eigen value lambda and the corresponding non zero Eigen vector x such that Ax is equal to lambda x we will also be focusing on some results which will give us the location of all the Eigen values.

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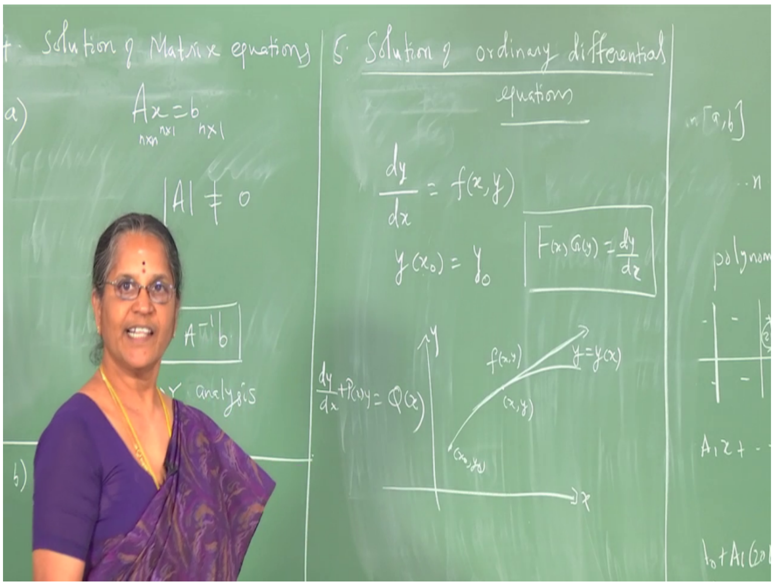

We will move on to the solution of ordinary differential equations that is going to be the fifth topic ordinary differential equations. So here again why do we have to solve them numerically is a natural question. See you would have solved in your school a simple first order equation of the form dy by dx is equal to f(x, y). What does this differential equation tell you and you would also have solved this equation subject to the condition that y (x 0) is equal to y 0. What does this differential equation represent?

To solve this equation what do you end up with. You end up with its solution as y is equal to y(x) which is a smooth curve in the (x,y) plane such that it passes through the point x 0 , y 0 because y (x 0) must be y 0 so it passes through the point (x 0 , y 0) and some additional information is given to us what is it? It tells you that at any point on the curve say x,y its slope is given by f(x,y) namely the tangent to the curve (x,y) has its slope given by the right hand side f(x,y).

So you look for a solution which is a smooth curve y equal to y(x) which passes through some point x 0, y 0 which is given to you and which has its slope or tangent to it any point x,y to have slope to be f(x,y). So you are looking for such a solution if suppose f(x,y) is such that it can be represented as a product of two functions say F(x) into G(y) that is dy by dx then you know you can apply a variable separable technique to solve this equation if f(x,y) is a homogeneous function of (x,y) again you know how to solve the homogeneous equation.

If the differential equation is an exact differential equation you know how to solve it if it is a linear Lagrange differential equation say of the form dy by dx plus p(x) into y equal to q(x) then again you know how to get an integrating factor make it exact and get the solution of this problem so there are special cases where you will be able to analytically solve the first order differential equation.

However if f(x,y) is non linear function of (x and y) and you are unable to get the solution of this differential equation analytically then you need to have some techniques by means of which you should be able to get the solution of this differential equation.

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So we would be determining in numerical solution of differential equations this smooth curve in the form of discrete values in the (x,y) plane we start with the point x 0, y 0 and we will obtain the value at some x 1 say which is at a distance of say h units from x 0 and then find out what this y 1 is. This y 1 will be determined in such a way that we know the slope of the tangent to the curve at this point to be f(x 0, y 0).

So we can approximate right the value of y 1 for example by saying that this y 1 is determined in such a way that it is the ordinate at the point x 1 such that it intersects the straight line passing through the point x 0 , y 0 and having slope given by f(x 0, y 0). And if I drop the ordinate at x 1 the point where it intersects is going to be give me the y 1 so I have the information about y 1. So I now know what x 1 y 1 is so I can compute the slope at that point which is f (x 1, y 1). And apply the same argument at the point x 1 y 1 and compute what x 2, y 2 is and so on and compute say the point x n , y n. Once I have information about these points I can join these points by a smooth curve and that will give me the solution y is equal to y(x) of this differential equation. But I have been able to compute this solution numerically. So we will develop different methods for computation of solution of such first order differential equations and applying these techniques and extending them it is possible to obtain solutions of higher order differential equations.

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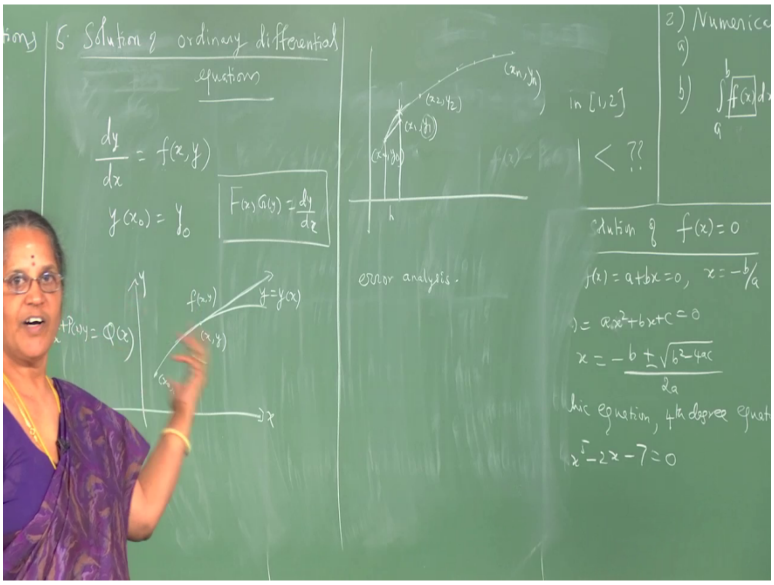

Again we will focus on how much of error that has been incurred at each step of all computation. So we will perform the error analysis here also. So these are some five important topics which will enable one to use these techniques and solve some simple problems that arise in scientific applications which may involve approximation of functions or approximation of the derivatives of the functions or solution of the algebraic or transidental equations or matrix equations for solution of differential equations.

So these are going to be topics that we would like to focus on in this course and as I said the knowledge for developing these algorithms is not large as far as numerical analysis is concerned. But it is important that one has a strong knowledge about what the background is.