Differential Equations for Engineers.
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Lecture-9.
Introduction to Second-order ODE's.

Discussing about the remarks and the methods that we have so far devised. So exact equations, you know how to solve it, so you cannot solve all the equations by these methods only, so you have to verify whether, whether it belongs to, whether certain method works through that equation or not. Only if you make sure that it can be, the method will work for such an equation, so some conditions you are verifying, so that, so that you can attack such a method to solve the problem. Exact equations for example you just have to check dow N by dow x, dow N by dow x, okay.

Dow M by dow y equal to dow N by dow x, that condition of exactness you have to verify. And if it is not same, you have to verify certain forms that are given for different V of x, y values, V of x, y function, if it is x, V of x, y is x, V of x, y is y, V of x is of special form like x minus y, x by y or xy, in that case you have a certain form to verify. If it is that, whatever the, you verify in the expression, it should be function of that V, in specific form of V. Specific function of that form of V. then you know how to get the integrating factor by multiplying the integrating factor you can integrate the equation and get the general solution of the equation.

This is how, if given, you are given an equation, you just have to see what method you can apply, just by looking at the equation sometimes, sometimes smartly by rearranging you can directly integrate. Okay. these are, these other methods, standard methods that are available for to solve the first-order ordinary differential equations.

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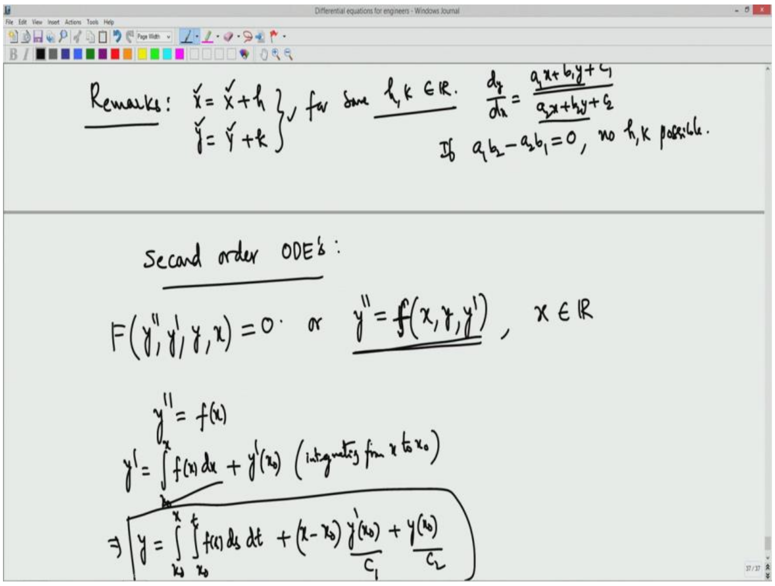

Now second-order, so let us move onto second-order ODEs. Second-order ODEs, how do I solve it? So y double dash, again what is the general form, F of y double dash, y dash, y, x equal to 0. So this is the general form where F can be anything, in an explicit form, in an implicit form. In explicit form you can also have an explicit form that is you, you solve for the higher derivatives y double dash which is equal to some small F, this is also arbitrary function F which is, when you write like this, x, y, y dash.

This is also general solution where, generally question of second-order because highest derivatives is 2nd derivative, you have this. So these equations, if it is a general F, you do not know, you may not, you may not be able to solve. So you have to go for, so there is a question of existence, existence of the solution itself in the domain it is defined. If the domain actually, x is the independent variable, so that is only the domain is R. Y is the dependent variable, y take the values, real values, y dash and y double dash also take the real values. So that is the solution.

So not all equations can get easily solvable, certain forms but certain forms of equations, some ad hoc methods we have to solve the second-order equations, I will give you one or 2. Okay. So in standard form, one simplest second-order, second-order or higher-order for that matter. So if I have y double dash equal to only function of x, I do not have y and y dash terms, I know simple integration. So y dash equal to integral x0 to x, F of x dx plus y dash at x0.

I do one more integration from both sides, okay, this is by integration, integrating, integrating from x to x0, this is what it gives. One more time if you do, you get y at x equal to x0 to x, now if I use this whole thing, whatever I have here is the function of t, x0 to t, F of s ds is function of t dt plus, your integrating from x0 to x, so x minus x0 and y dash into x0, because it is constant, integrating from x0 to x and you have here y, y at x0 minus y at x0. So y at x minus y at x0. So that I bring it here. So we have y at x0.

So this is the arbitrary constant, this is because, this y is unknown function, y dash has also an arbitrary constant. you have, if you call it C1 and C2, so this is your general solution. I have a solution y, I have a solution y, that has one arbitrary constant C1, another arbitrary constant C2 which is given like this. Okay. So this is how any, any order for that matter, so one N derivative is equal to, if I have function of x, N integrations gives me the general solution. Okay.

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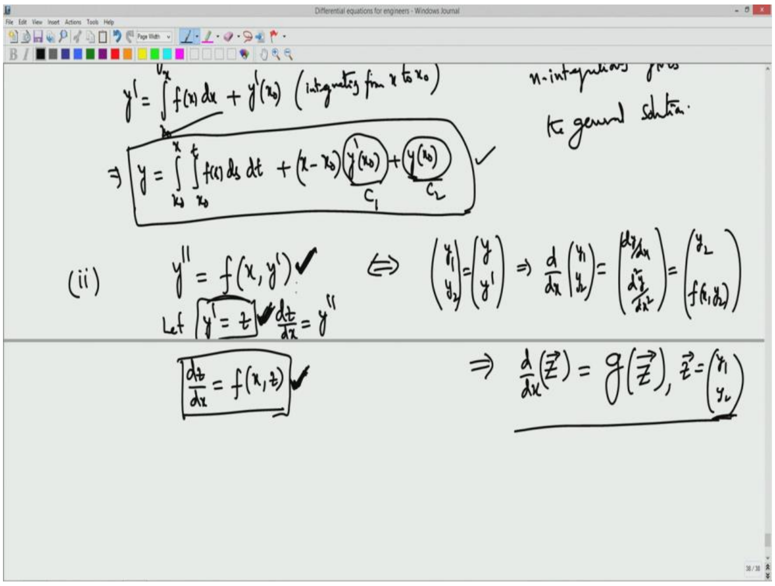

So this is a trivial, trivial form of F for which, if it is function of x, I can simply integrate many times and get the solution, okay. So what are the different form, another form that you can work out. So if I have y double dash equal to F of x, y dash. there is no y, the right-hand side function, that does not involve the terms of y, I have only y double dash and y dash. In this case let me choose, y dash equal to some variable z. Okay. dy by dx equal to z, z is a function of x, okay.

So I replace this dependent variable y dash as new dependent variable z. So then dz by dx is y double dash. So left-hand side I can write dz by dx, right-hand side I can write F x, z. So this is the first-order ordinary differential equation. If I know how to solve this, then I can find my z and then I go back and do one more ODE y dash equal to z. Having known z from this ODE, I can put it here and I solve this 2nd ODE to get my y, that is the general solution of, that is how you solve the second-order equation.

So you have somehow found, you solve the second-order equation by solving 2 first-order equations, okay, simultaneously. So this gives me a feeling, so I can convert this into a system, that is possible, it is equivalent to this cannot equation, we can actually write, you write it as y1 as y, y2 equal to y dash, okay, I am trying to put this as a vector form. So you try to write this, you try to make vector which y and y dash, okay. Then what happens to d dx of this vector y1 and y2. Which is equal to, in vector if you differentiate, each component you can differentiate.

This is equal to dy by dx and dy dash by dx, okay. dy1 by dx, that is dy by dx, dy 2 by dx, that is equal to d dx of y double dash, that is actually d square y by dx square. So this is dy by dx is nothing but y2. So this is my y2 and this is equal to d square y by dx square is F of x, y dash, that is y2. Okay. So you can, you can see, this implies, you can see d dx of some z, it is a vector form is equal to, this whole thing is like a some function, some function, let us call this some G of z bar.

Where z is, z bar is y1 and y2. So you can convert second-order equation into first-order system of equations, okay. So like that you can convert any N th order equation into an equivalent first-order system of equations or first-order, first-order equation for the vector M by, M by 1 vector, it is like this, these are N Rows than 1 column, N Rows and one column. Why we are seeing this because in the solution method, what you have with this equation, when you put this, in the special form, when I write y dash equal to z, then what you have is 2 ODEs.

If you simultaneously, you solve them simultaneously, this equation and this equation, that is how you get the solution of this. That means these are your simultaneous equations, if we solved, that is equivalent to solving second-order equation, okay. So we have not interested in that way because you do not know how to solve the system of equation, simultaneous equations, these simultaneous equations here, they are easier to solve, so that is why we are, basic idea is there.

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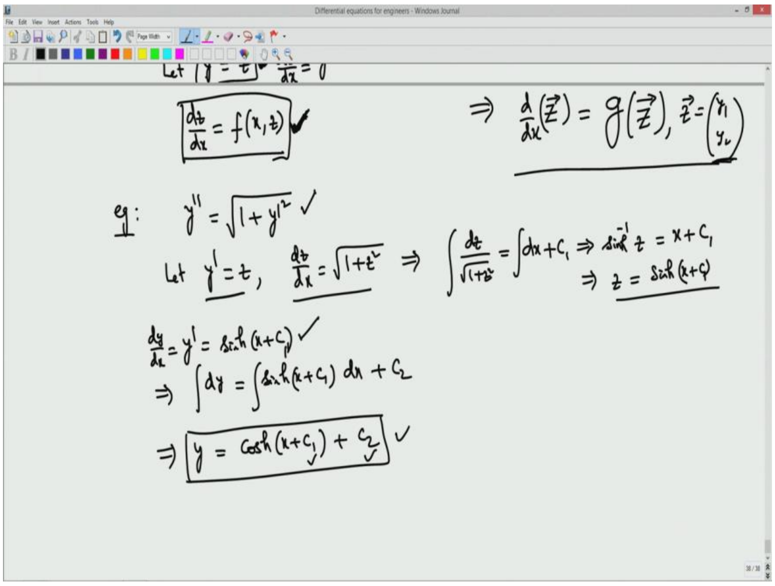

So the second-order equation we reduce into 2 simultaneous equations first-order which can be solved, okay. In this special form they can do that. If I have y, that cannot be possible, okay. So I will give an example. For example, you try to solve y double dash equal to square root of 1 plus y dash square. I do not have y, I have y dash. So y dash equal to z, then your dz by dx which is equal to square root of 1 plus z square. Now I know how to solve this. Okay. then I can solve this.

First let me solve this, this implies dz by square root of 1 + z square equal to dx. I can integrate both sides and write the arbitrary constant. So this implies, this is actually Sin inverse Sin hyperbolic inverse of z equal to x plus C. this implies z equal to Sin hyperbolic x plus C. Okay. So now having known your z, you come back and put it here, y dash equal to Sin hyperbolic x place. Let me call this C1 so that you have one arbitrary constant C1.

So now here I am trying to solve this equation, I am trying to solve the 2nd equation, okay. the first equation by replacing z, having got z from the 2nd equation, that is general solution, so I bring it here and put it. So now I integrate directly, this, so this equation is simply d this is dy by dx equal to Sin hyperbolic x plus C. So this you can write dy equal to Sin hyperbolic x plus C1 into dx. Now the variables are separated, so you can integrate both sides plus C2, arbitrary constant.

So this will give me y equal to cos hyperbolic x plus C1 plus C2. So this is your general solution of the second-order equation this with 2 arbitrary constants, you can easily see 2 arbitrary constants C1 and C2. If it is a 2nd order, you will have 2 arbitrary constants, okay.

So it is called general solution because it involves 2 arbitrary constants, the solution involves 2 arbitrary constants C1 and C2. So if you see, the equivalent system is equivalently, second-order equation if you equivalently write it as the first-order equation for the vector, like here, so what is the general solution?

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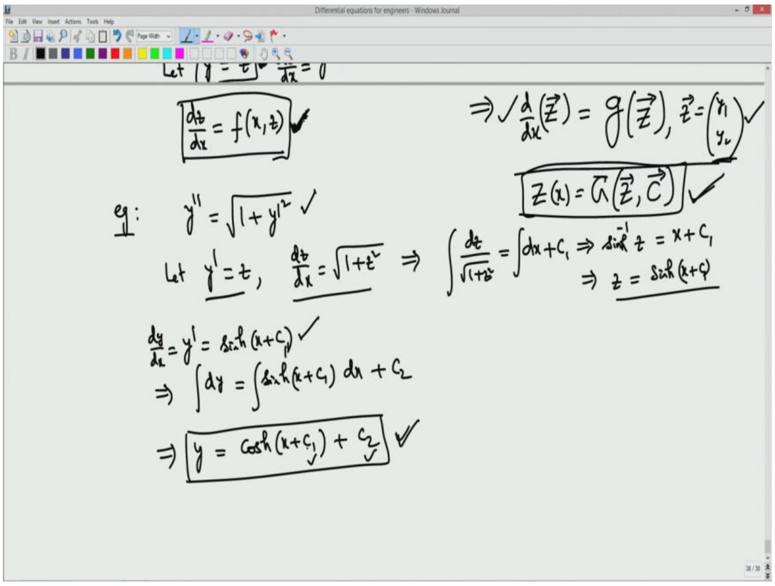

General solution you will get z of x as some form of some, let us say some, some known function, once you solve this, some G of z bar, what is the constant here, constant means the solution z bar at certain point. Z bar is a vector, z bar is the vector, z bar at certain point is y1 at 0, y2 at 0, let us say. These are y2, that is z at zero, that is simply a vector. So you have a vector as a constant. So this is how you get the general solution of that.

So you have 2 arbitrary constant C1 and C2 in the second-order equation, the equivalent case also you have 1 arbitrary constant C but that is a vector. Since vector is having 2 elements, so 2 by 1 vector, so you have 2 arbitrary constants here. So they are actually equivalent, okay. So this is the general solution for first-order vector equation, this is the general solution of the second-order equation, scalar equation with 2 arbitrary constants C1 and C2. Okay. So with this, we have in this special form, you can solve it, you can solve the second-order equation.

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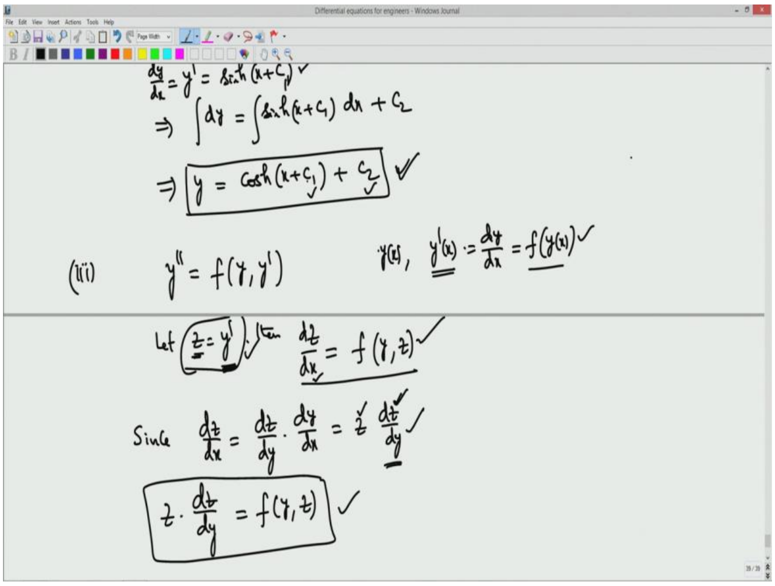

This is an example. Let me give you one more type where you can solve. So 3rd form I will give you so that you can solve, so y double dash if I have F y, and y dash. If I have like this, can I solve the second-order equation? So again as usual if I choose, let z equal to y dash, then z dash equal to, that means dz by dx equal to F of y and z. Okay. And see I have both x variable and y, y is that here. So I cannot hope to call this but I solve it if I presume that what happens, you look at this one.

Since dz by dx, I can write it like dz by dy into dy by dx where dy by dx is z. So z into dz by dy. Since this is dz by dx equal to z into dz by dy, if z is a function of y, if I can assume, that means I have y dash, that is function of x. As a function of x, which is z, if y dash, that means you have a solution y, x whose derivative is y dash of x, this is equal to, this is also a function of x, right. y dash is dy by dx, y is, y is, so this is, this is a function of y of x, if you can write, some function of y of x.

If the derivative is function of y of x, then I can, that means I can see that y dash is function of y, that means z is a function of y. If z is a function of y, dz by dy makes sense, okay. So that I can replace dz by dx here into z into dz by dy. z is now depending on y. Okay. Because z is actually y dash that is actually function of y, okay. Such a thing if it exists, you have a solution yx whose derivative is actually function of y of x. If such a thing exist as a solution, we assume that it exists, then I can write like this.

So that I can replace dz by dx is this, equal to F of y, z. Now it is, I could remove x variable in the domain. So I have dz by dY F of y z. So this is the first order ODE for the dependent variable z with an independent variable Y. If I solve this, I can get my z, again go back and from this you can solve for Y. Okay. So this also I will give you one example.

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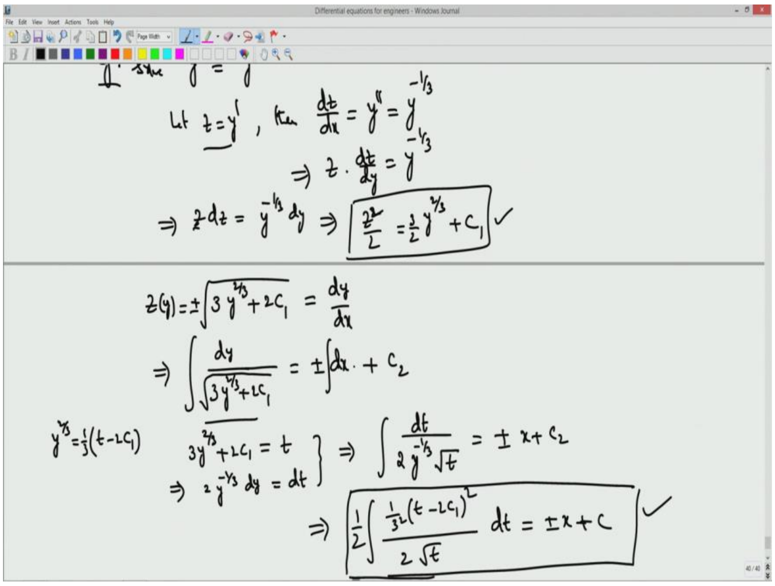

So solve Y double dash equal to Y Power 1 by 3, one simple example, okay. So I have Y here, one simple example. So you can replace z by Y double dash dz by dx, let z equal to y dash, then dz by dx equal to y double dash which is equal to y Power 1 by 3. Clearly dz by dx is function of y, so I can write dz by dx equal to z into dz by dy equal to y Power minus1 by 3. So this will solve dz by, so z dz, I can integrate directly, so y Power 1 by 3 dy.

So how do I integrate, so this will give me z square by 2 equal to y Power to buy 3 by 3 by2 plus arbitrary constant. So this is my general solution. So from this I get my z, z of y which is equal to 3y Power 2 by 3 + 2C, okay plus or minus square root of this, this is my z, okay. z, z square is this, so you have z is plus or minus square root of this. This is equal to dy by dx because y dash is z. So now I can solve dy by 3y Power to by 3 + 2C under root equal to plus minus dx. this is what is the first equation becomes.

So this you can solve it by integrating, okay. So you can integrate this easily, how do I integrate? So you can integrate plus some other constant C2, we call this C1, C1, there become C2. So how do I integrate this? You can integrate by taking 3 power y Power 2 by 3 + 2 C1 as some t. this will give me 2 by 3 3, so 2 y power 2 by 3 - 1 is minus1 by 3, okay, dy by dx, sorry, dy equal to dt. Okay. So this will give me, this integral as I can write dy as dt by 2 times y Power 1 by 3, so from this you can get your y power to by 3 equal to t - 2 C1 by 3. Okay.

So this square will give me 1 by 3. y Power 1 by 3 into root t. So now I have to replace 1 by y Power 1 by -3 as, so this is equal to plus minus x plus C2. So this will give me, this I have to rewrite, so 1 by 2 out, y Power 1 by 3 is, y Power 1 by 3 is, 1 by 3T -2 C1 square, 3 square divided by 2 into root t dt equal to plus minus x place. Now this you can, this is simply function of x, is a polynomial, diverted by. So this you can easily integrate and get the solution.

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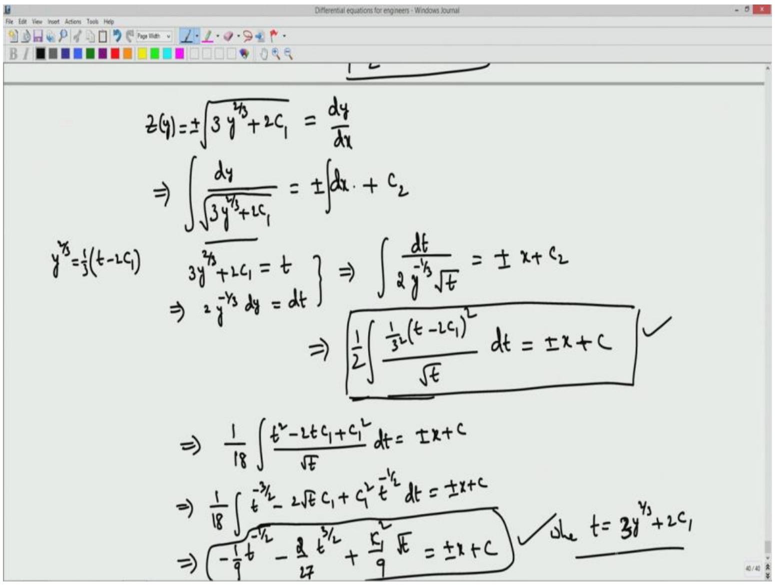

Okay. So we do this really, so this implies 1 by 4, 36, so there is no, only one t, so 1 by 18, so integral t square dash 2t C1 plus C1 square by root t dt equal to plus or minus x plus C. So this will give me one by 18 t square 2 minus half - 3 by 2 - 2 root t into C1 plus C1 square plus t power minus half dt equal to plus minus x plus C. So finally you get t Power 3 by2 -3 by 2 + 1 is minus 1 half, okay. -2, so 1 by 9 minus 1 by 9T Power half, t Power 3 by 2 into 2 by 3, 27, okay, + 1 by 18 C1 square t Power under root into 2, so 9. Okay, equal to plus minus x plus C.

So you have C and C1 arbitrary constants. So that will solve the problem. Okay. So this is your solution where t is, you replace t as 2 y Power 2 by 3 + 2 C1. So you have functions of, in terms of y and x as the general solution for your second-order ordinary differential equation, okay, is given as this. So these are the ad hoc methods, if it is your right hand side is in certain form, only in the specific form you can solve it, otherwise you do not know how to solve it, okay.

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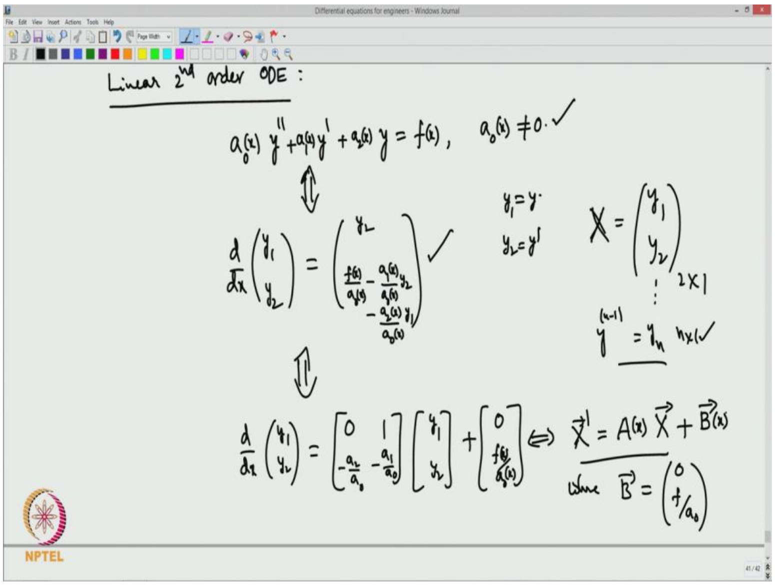

Or, like in the first order ordinary differential equations, second-order linear ordinary differential equations can be solved. Okay. So linear, linear second-order ODE, we solve this, so we go to Linear. So it is so far any form, now it is only linear. Okay. So, so what is the form. So once you say linear it should be, it should have only y, y dash because the second-order, you should have only y dash and y double dash, okay. Let me put it y dash here, y, yes.

So like this I should have only these terms, okay, and their combinations, their combinations I should have which should be equal to some function of x. So whenever you have dependent variable F x, so you may write this F x is the right-hand side, whenever I have a certain form, if it is linear, I should have only y and y dash, y double dash, not its squares, not its powers. Okay. So what is the, how you can fill it up. So only its combination. So sum, sum are addition, so I just multiply some A0 of x, A1 of x, A2 of x equal to F x.

So this is the form you have, this is the second-order linear ordinary differential equation where F is, F and A2, A1, is A0… A0, A1, A2 and F x are given functions, okay, functions of x only. And the moment you write this, higher order derivative coefficient is A0 of x, that should not be zero. So that means your domain should have values, all values of x, where A0 of x should not be zero. Okay. So that means you cannot have a domain in which you have A0 of x is0, okay.

So how do I solve this equation? So this is the linear second-order ODE, before we solve these equations, we try to convert this into system of equations. Okay. So like we have, I just explained earlier, here also you can convert this equivalent, equivalent first-order system of equations, that is y1, y2 as a derivative equal to, what do you have, so what is your y1, y1 is y. y2 is y dash, okay, so you have dy1 by dx which is dy by dx that is y dash which is equal to y2. So I have y2 is the first order, first equation.

Dy1 by dx is y2. What is your y2 dash? d dx of y2 is y, d dx of y2 which is y double dash. y double dash I can get it from the equation, that is Fx A0 of x minus A1 by A1 x by A0 of x, what is y dash? y dash is y2, that is y2 minus A2 x by A0 of x into y. What is y, y is y1, so we have y1. Okay. So this is, this is the system of equations. You have the 2 equations, 2 coupled equations in ym by 2. So this is actually equivalent to d dx of y1 and y2, you can split it in as matrix times y1 and y2 plus some vector which is zero, Fx by A0 of x. Clear.

If I multiply something so that I should get my y2. So I should have 0, 1. And here A2 by A0, minus A1 by A0. You multiply so that 2^{nd}, after multiplying this, this into this, this is what we get. Plus if you add this part which is known, this is the right-hand side. Okay. So you have something like equivalent to, some x dash, x is a vector which is y1 and y2 equal to some matrix A which is depending on x because A0, A1, A2 are functions of x into x, again matrix is the vector. So this is your linear system, the right-hand side is actually some B, is a vector. Okay.

This is the vector, this is a vector. Where B is zero and F by A0 which is also a function of x, okay. So this is how you can convert any second-order equation into first-order system. This is called system, we do not know how to solve this system of equations. But what we do is in the course we try to give you methods to solve these second-order equations and same way you can try to pose this to any higher-order equations. Okay.

This is the second-order, higher-order equations you will have, instead of my 2^{nd} derivative, you will have Nth derivative, so you have Nth order differential equation in the same way, in the same line you can give Nth order differential equation, ordinary differential equation which is linear and its equivalent system. So in this way we can have, we can define Nth order linear ordinary differential equation by the same way. So you can have instead of y double dash you have Nth order derivative, you have the next term is N minus 1 th derivative and then up to zeroeth derivative, that is y and equal to right-hand side F.

All these dependent variables y, y dash up to y N derivatives should be linear and their coefficients should be some function of x and then you can combine them with addition which is equal to S. this is the most general form of Nth order differential equation that can also be solved by the methods we devised for the second-order equation in the same way. The move is just the same, you just follow the methods of second-order, you can apply for the higher-order. Small change will be there, okay.

So the Nth order differential equation, higher-order differential equation you can also write equivalently as system, here second-order you have got x as, x as, where x is actually vector y1 and y2. So it is a 2 by 1 vector. So if it is Nth order equation, you will take y1, y2, yN, so it is N by 1 equation, will go up to y N, y1, where y1 is y, yN is actually y N minus 1th derivative. N minus 1th derivative of y is your yN. Like that you can go on, you can make your x and you will get a N by N matrix. X is N by N, in that case it will be N by N vector, B is also N by N vector.

So like that you can make a system of coupled equations, N coupled equations that you can represent as system of first order equation for the vector N by 1th vector, N by 1 vector, so we will try to give you elementary methods how to solve second-order equations and then we will try to generalise for Nth order equations.