Differential Equations for Engineers.
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Lecture-8.
Linear First-Order ODE and Bernoulli's Equation.

So, linear equation is solved, you know how to solve the linear equation now. So we will now today, we will try to solve some equations that can be reduced to linear equations. One form of the equation that can be reduced to linear equation is called Bernoulli's equation. So we will try to solve the Bernoulli's equation, I will give you how to convert that into a linear equation or general equations that can be reduced to linear equations, if, possible, okay. So and then we will try to solve some problems and then see how we can attack, we can find the general solutions of those equations using the methods that we have device. Okay.

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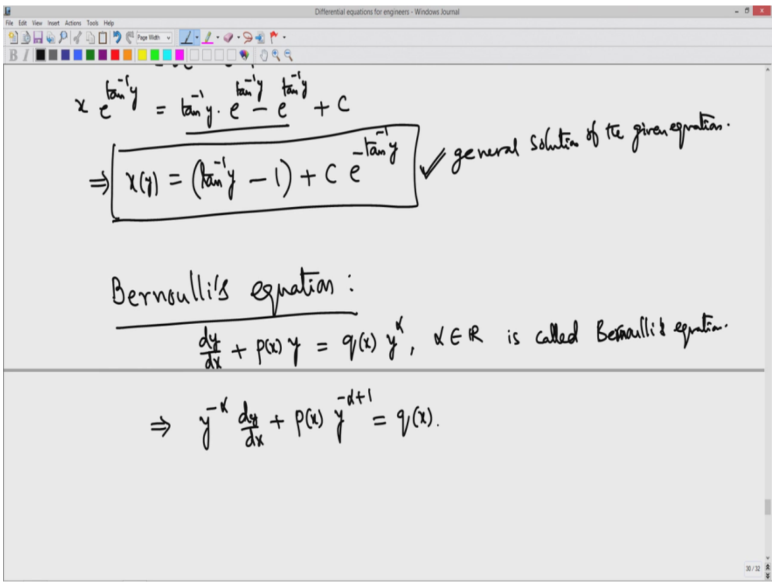

So let us start with Bernoulli's equation, so we will say, we will start with the equations that can be reduced to linear equations, one such form is the Bernoulli's equation, Bernoulli, Bernoulli's equation. So what is the equation, the equation looks like this. If you try to solve, dy by dx plus p x y equal to Q x, this is a linear equation. So if I have some function of Alpha, so if I have some Alpha here, so function of y, so y power something, so this is called, for have a linear equation, only right inside is involved in Q x into some by power Alpha, Alpha belongs to real, I called Bernoulli's equation, is called Bernoulli's equation.

For Alpha equal to 0, it is a linear equation, otherwise Bernoulli's, can reduce this equation into linear equations. How do we do this, we simply divide, so if you want this to be a linear equation, right-hand side should be a function of x, so you divide this y power Alpha both sides, so we get by doing that, we get y power minus Alpha dy by dx plus P x y power minus Alpha + 1 equal to Q x. So, so what is the difference, this is now looks like right-hand side of, Q x on the right-hand side is only function of x, left-hand side, what we did for the linear equation, we want the left-hand side to be derivative of something, something in exact form.

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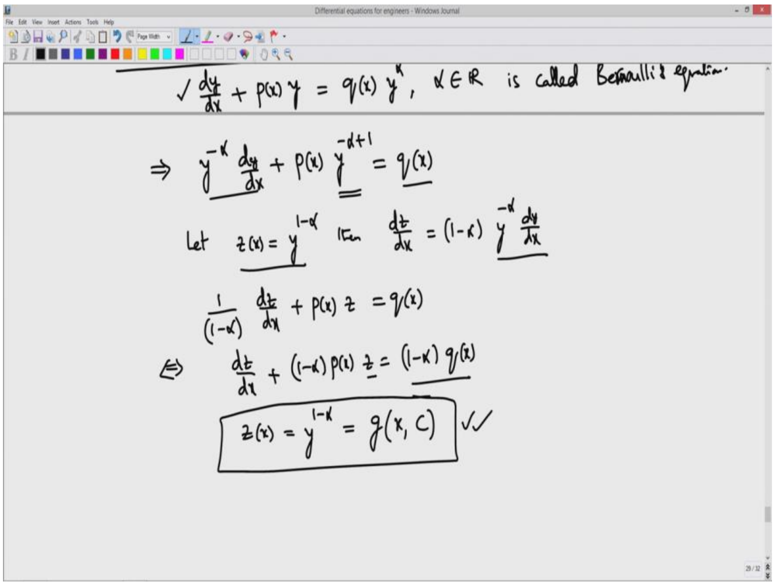

So by multiplying some integrating factor. So here by doing this, you can easily see that, so we will, so if it is a linear equation, you know what is that integrating factor. Right now, because you have a multiplication factor, so y power minus Alpha here, so you can easily see that if I choose this as some z, then you will see what will happen. So let us make this substitution, let, otherwise the function of x, so let we call this z of x as a function of y power one minus Alpha.

Then dz by dx equal to 1 minus Alpha times y power minus Alpha dy by dx. So you can see that I can replace with one by one minus Alpha into dz by dx. Okay. So this I replace with one by 1 minus Alpha dz by dx plus Px and this I replace with z equal to Qx. Now I have a linear equation can easily see. So this is equivalent to dz by dx + 1 minus x into, 1 minus Alpha into P x into z equal to 1 minus Alpha times x. So Alpha is constant, so one minus Alpha into Qx is a function of x, now the dependent variable y becomes z which is given by this transformation.

So now this is a linear equation, you know how to solve this. So once you solve this linear equation, so you can get your zx. Finally that should be equal to y power one minus Alpha G some function of x and C. this G will be known ones you solve this equation. So that is how you find the general solution of a Bernoulli equation, this with, by reducing it into a new equation, by exploiting its solution we can find the general solution of the Bernoulli's equation, okay.

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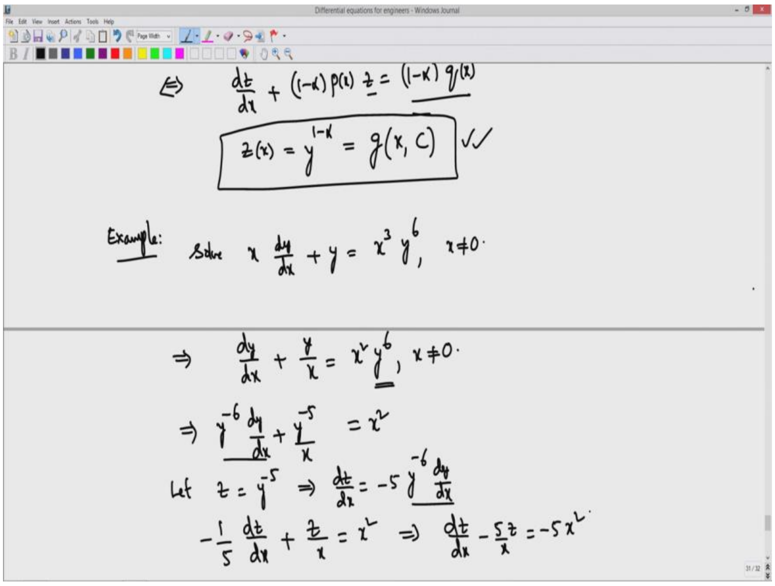

So we will do some examples. Let us do some example. So we have example. I will give you simple example let me choose. Solve x dy by dx plus y equal to x cube y raised to the power 6. So what should I do, so to make it, this is clearly Bernoulli's equation, function of x, so we can easily see, so easily we can rewrite like Bernoulli's equation by dividing it. So because, see whenever you have a high order coefficient, so this is higher derivative coefficient is x, this should not be zero, okay.

So if this is zero, it is a singular point because if I divide this and if I rewrite like this y by x equal to x square y power 6. So if I divide with x, because x is not equal to 0 I assume, so I can divide it. If it is zero, you cannot divide. So if x equal to 0, at that point if I divide, if it is, if zero is a point of the domain, I cannot solve this equation. So it is not, so this is the domain, the domain should be, some domain that does not include zero, okay. So now this in the form of Bernoulli's equation, we will choose, you now, we divide this with, you want right-hand side to be only function of x to make it x square.

We divide all thing with y power 6, so we have y power - 6 dy by dx plus y power -5 by x equal to that. Now choose the transformation z equal to y power -5 that makes it dz by dx equal to minus 5 y power -6 dy by dx. Now this I can replace here, this, this terms, both are same, so we can replace with minus 1 by 5 dz by dx plus y power - 5 is z, z by x equal to x square. So this is, this implies dz by dx plus, sorry minus, I simply multiply - 5 both sides, so we have - 5 by xz equal to minus 5x square.

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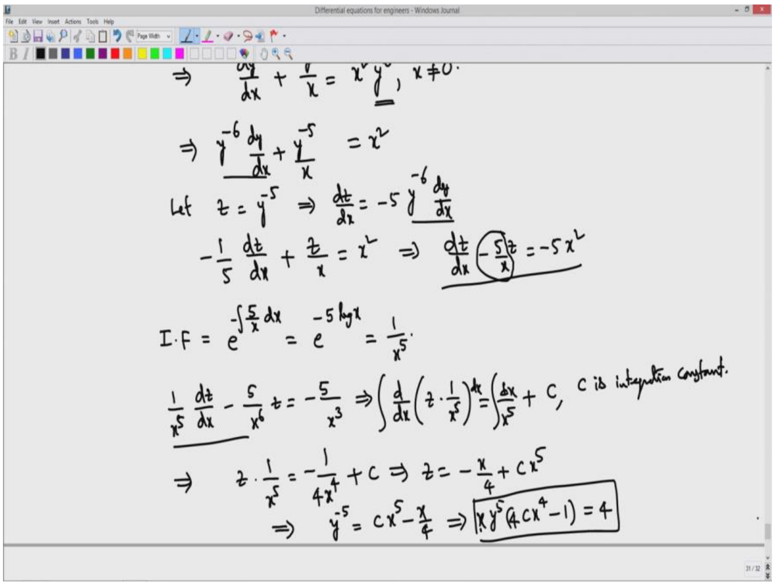

So this solution you can see, how do you solve this equation? So this equation you can solve by multiplying an integrating factor, because it is original equation, integrating factor is e power integral, this is my E. So e power integral -5 by x dx. So this is nothing but e power -5 log x, this becomes one by x power 5, okay. So if I multiply this with 1 by x power 5, both sides of the equation, then it becomes 5 by x power 6 dz equal to, I multiply this with 5 by x cube. So this is what you get.

This implies, so the left-hand side, now I can rewrite this as derivative of d dx of z times 1 by x power 5, you can easily see that. So this is exactly is this, right. So dz by dx into this plus z into d, d dx of 1 by x power 5 is nothing but minus 5 x cube. So this is equal to 1 by x power 5. Now integrate both sides, you integrate both sides, so that we can have some log C here, some constant, arbitrary constant. C is integration, integration constant is arbitrary constant.

So this implies, now we can easily solved, so this with respect to dx is, this with respect to dx, we have this, so we have simply z times 1 by x power 5 equal to, what is this, what is integration by x power 5? x power 1 by 4 - 4 by x raised to 4, okay. By 4, this is how it is.

yes. - 4 power x power 5. So this is the integration for x power -5, okay plus constant. So this implies, z equal to, x power 5 you bring into this side, so you have x by 4 + C times x power 5.

Now what is the z, you can replace your z, z is y power - 5. So y power - 5 equal to Cx power 5 minus x by 4. So this is your solution. So if you really want simplified way, y power 5, 4Cx power 4 - 1 times x equal to 4. Okay. So this is, this is your general solution, simplify this into this. Okay. So this is the general solution with an arbitrary constant C. this is how we can solve the Bernoulli's equation. It is Bernoulli's equation, so you can solve like this.

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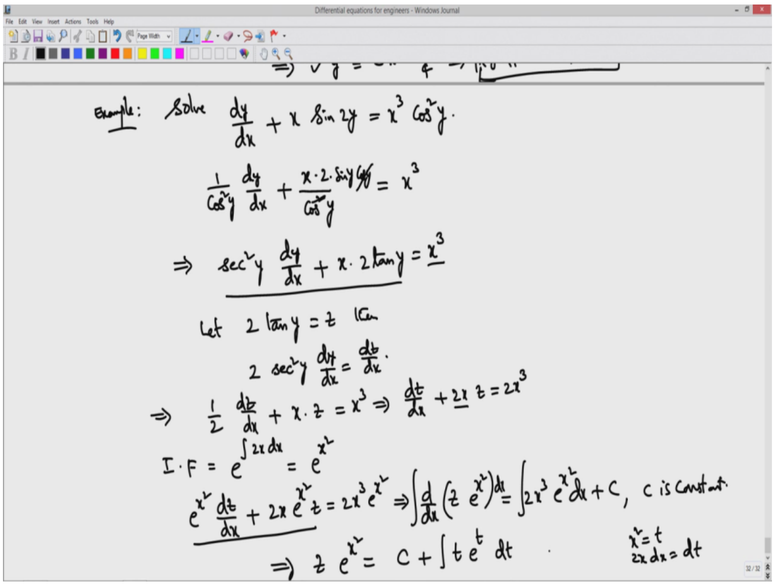

I will give you one more example, so one more example we can do so that you get familiar with, so we will solve, solve an equation that does not really look like a Bernoulli's equation but something similar, okay. So then we do, when we follow the same procedure, how we make it an exact equation, so it will work. this is not exactly the form of Bernoulli's equation that we can solve the same, apply the same technique. So if you solve this equation, dy by dx x times sin 2y.

So I do not have y here. Sin 2y single function equal to x cube cos square y. So we solve this equation, though it does not, it is not even near close to linear equation or Bernoulli's equation. So the main idea is to try to do it, so if you, right-hand side, if I make it only function of x, I have 2 divide with cos square y. So 1 by cos square y dy by dx plus x by x into, this you expand Sin 2y that is denominator cos square y because I divide it into 2 Sin y cos y. So one cos y will go, so this actually becomes, so this is secant square y dy by dx plus 2x tan y equal to x cube.

So you have this form, I divide the right-hand side, I make it a function of x, now the left-hand side. Left-hand side if I follow the same technique, that dependent variable, so what is the dependent variable you? So I will write 2x equal to x into 2, so dependent variable is simply the function of x. this is P x into whatever that dependent variable you want to be here if you want a linear equation. So let us apply, so let 2 tan y equal to z.

If I do this, can I replace the other term, so can I, can I do, can I make it linear equation, okay? then 2 dy by dx, so secant of tan, d dx of tan is secant square of y dy by dx equal to dz by dx. So that implies, then the question becomes, the received first term becomes 1 by2 dz by dx plus x times z equal to x cube. Now we have got the linear equation. So it becomes dz by dx + 2 xz equal to 2x cube.

This is the linear equation, now you can solve it, now I am applying the integrating factor that you can calculate with e power integral, this is your P procedure of solving the linear equation 2x dx. So this becomes e power x square. this is your integrating factor, okay. So whatever the integration constant, you can (())(14:58) because it comes as C1 which is equal to e power C that you can take it as one while doing the integrating factor, okay. Integrating factor, this is how you calculate.

So we multiply this both sides, e power x square into dz by dx plus 2x e power x square z equal to 2x cube e power x square. So this implies, the left-hand side, the whole left-hand side I can write d dx of z times e power x square equal to 2x cube e power x square. this implies, now you can integrate both sides with respect to x and you can make a constant. Where C is an integration constant, so it is constant, an arbitrary constant, okay. So this gives me z times e power x square equal to C plus, this will be if I replace x square by t, that will become, choose x square equal to t.

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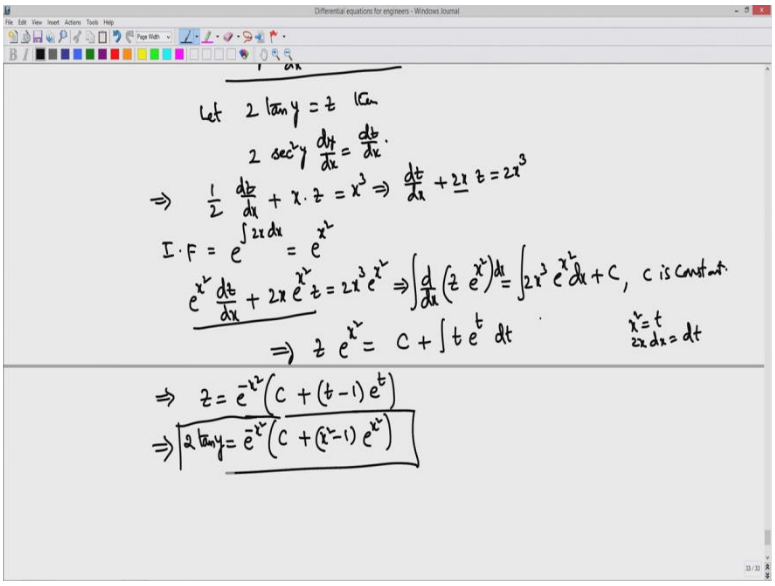

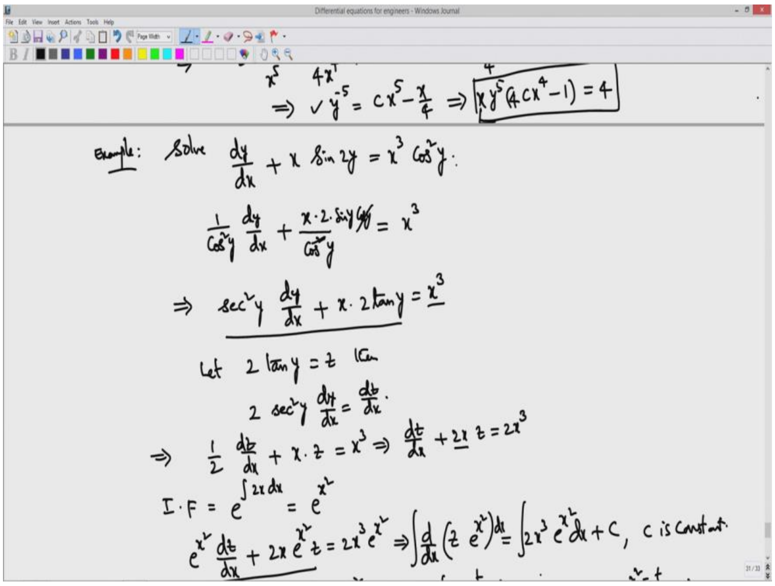

So you have to x the x equal to dt. So one to x dx I replace that dT remaining is x square that is t, e power x square, that is e power t. So this is what is the solution. this implies z equal to e power minus x square times C plus this you know the solution of t into e power t. So you have E, t into e power t minus e power t. So t – 1 e power t. Let me write t minus 1e power t. So this is exactly, so now I can replace by z by new variable, old variable 2 tan y, 2 tan y, e power minus x square C plus t I replace with x square, x square - 1 into e power x square.

So this is your general solution of the given equation, this is function of x, okay, function of x and y. So y is solution, you can satisfy implicitly this relation. So this is the general solution, okay. It is the general solution of the given equation okay. This is how you solve, even if it does not look like Bernoulli's equation, we can try making it, right-hand side function of x and then, and then you make the left-hand side, try to write by, replace, by choosing some dependent variable, okay.

Okay, so here I have chosen x into some function of x into z you want. Right. If you want this to be a linear equation, you should have the 2nd term should be Px into some z. Okay. If it is not y, the whole thing if you replace with z, can you make my first term as something so that the whole term together is like derivative, it becomes a linear equation, okay. That is what we have done, so like this you can solve some of the, sum of the first order equations that does, that does not, even though they are not linear equations we can solve differential it is like, it is like Bernoulli's equation or equations that are close to linear equation.

So I described the method and then gave you an example. If you are given a problem, then how do you know which method to apply? So we will now choose, we will start with certain set of problems and then we will see how we will, how to, how to solve them one by one, okay. So we will choose one problem, okay.

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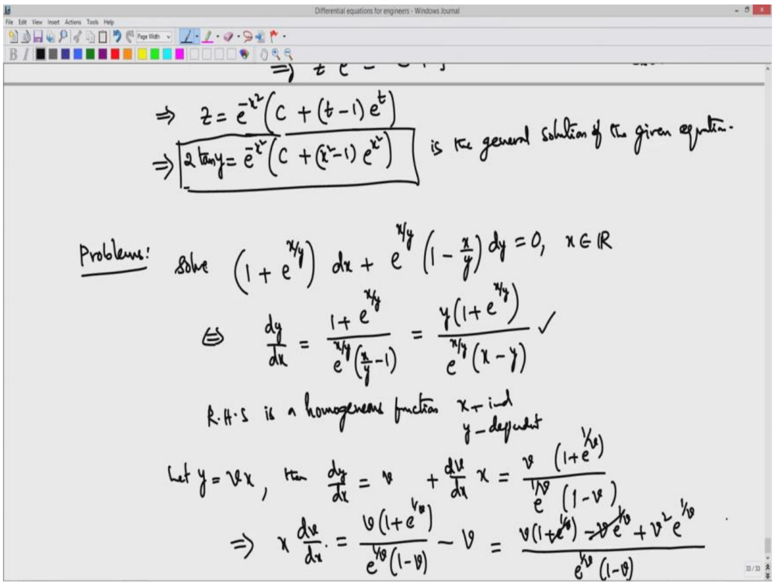

So let me do certain problems. So let me do one, first one, solve this equation, let me write 1 plus e power x by y into dx into e power x by y into 1 minus x by y dy equal to 0. If I have in this form, can I make it separable, so I just have to see the equation. So the equation is defined, well-defined, so it is defined for every x belongs to a real, that is what we have to see. Even if you write dy by dx, which is in written form, so if I write dy by, this is equivalent to dy by dx which is equal to 1+ e power x by y by e power x by y into x by y minus1.

So this is how it is given, so. So it is not even, so you can clearly see that, it is not easy, is it homogeneous, is it homogeneous? So you cannot, you cannot make its first of all, you cannot, you can see that it cannot be separated as a function of x into functions of y. So next method you know is the homogeneous method where the right-hand side, if it is a homogeneous or not. So this is actually equal to y into 1 + e power x by y divided by e power x by y into x minus y.

So clearly y divided by x by y is a homogenous function 1 plus e power x by y, e power x by y. Okay. So this is equation, so actually it is a homogenous but sometimes, since this example, for example in this, even though this is a homogeneous function, if you actually follow the homogeneous, method will be complicated. So you can easily see, when that will be a curve, you will see the difficulty. So it is actually homogenous equation, this right-hand side is a homogeneous function. Okay.

So you are looking for F of x, y, sorry, homogeneous function, x is the independent variable, y the dependent variable. Okay. So this is what you have. So if it is homogeneous, you can easily say put x equal to K L x, y equal to Ly. So what happens, e power x by y equal to e power x by y. So there is nothing, no change. Numerator finally, when you put y equal to Ly, L comes out, so L Power one. Right. Similarly if you put it here, in the denominator, same thing is coming out, when you replace x by L, y by Ly, x by L is, y by Ly, L comes out, and we get L Power one. So L Power one L Power one, so we get L Power zero.

So this is the homogeneous function of degree zero. Okay. So it is clearly homogeneous function, we can go ahead. So if we choose yy equal to vx, v is the new dependent variable, then let, if we choose y equal to vx, then dy by dx, left-hand side I replace with v dx plus dv by dx, so v plus v by dx into x equal to this right-hand side, that is y, y is that of vx, okay and this is 1 + e power v divided by e power v, e power 1 by v Rather, okay, e power 1 by v into x minus y is vx.

So you can cancel this v, you can cancel this x both sides, so I can write like this. This is only a function of v and x. So this is equal to, so dv by dx into x equal to1 plus e power one by v by e power one by v into one minus v minus v. So this is equal to, so v into 1 + e power one by v minus v into, so that divided by e power one by v into one minus v minus v e power one by v into minus minus plus v square e power 1 by v.

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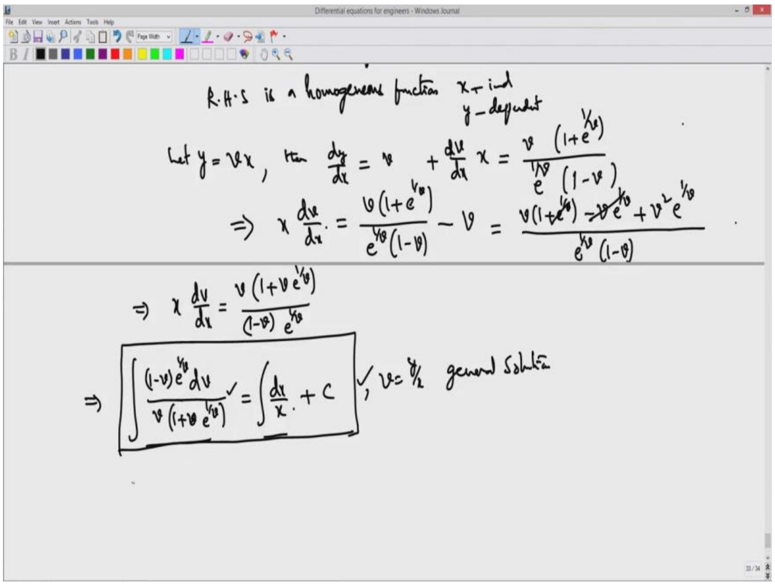

So you can cancel this term with this, so we have finally, what we have is, so this implies we have x dy dx equal to v is common, so we have 1+ v, e power one by v divided by 1 minus v times e power one by v. So this is what we have. So finally if you can write dv divided by dv, dv is, bring all the v on one side. So we have 1 minus v e power one by v divided by v times 1+ v e power one by v. So this is, I brought everything on one side.

So we have dv, this is equal to dx by x. So we can now integrate both sides and write like this. So this integral I have but this is a little difficult to have. So if you want, you can go ahead and say that this is, if I can evaluate, whatever the value as a function of vx and finally you replace with v by, where v is y by x is the general solution, because this should be the, implicit function, implicit relation that has a solution y. If you are looking for y as a function of x, this satisfies this implicit relation with an arbitrary constant C, it should be the general solution of the given equation.

But you have done only half work. So you could not evaluate the integral, this may not be the reason. So you might end up sometimes, sometimes some equations, sometimes some integrals you will not be able to evaluate. So if you cannot, you just evaluate, so whatever the value of the integral, it should be function of x, if it exists, okay. And then, that together with the whole thing, with the integral, so for example we have this, this should be the general solution.

So if you want in explicit form, if you want in explicit form, this equation, same equation, we will try to attack in a different way. So what the best way is, x is a independent variable, y is the dependent variable, okay. So you have an, now finally you have found the general solution as an, as an implicit relation, yx satisfies, y and x are satisfied by this, y, x is satisfying the implicit relation, okay. If you think that, suppose there exists such a function y, x, there is a solution. So that is the solution curve for the given equation.

And suppose you have a function yx and that can be invertible, if there is invertible function yx, you have an invertible function for yx, select x of y, that satisfies, if, **if** that is the case you can make your differential equation in terms of, you can make x as dependent variable and y as independent variable, okay. So if you, at the, looking at the equation here, you try to make the equation as that dx by dy, then your life may be simpler. Okay.

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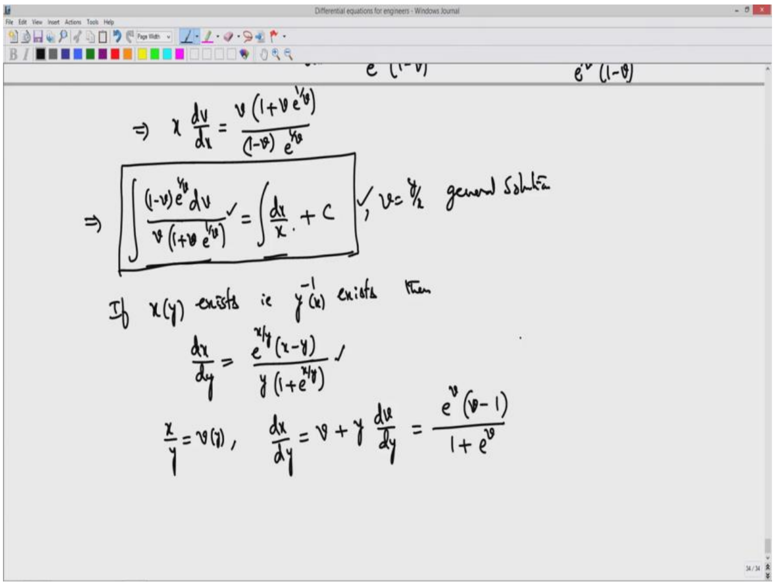

So let me write dy by dx as, if x of y exists, we do not know really. If x of y exists, that means there is y inverse of x exists, okay. So you have an inverse function for yx, that exists. So then, then I can write that, so if it exists, using that x is a dependent variable, so you write the equation as dx by dy, so that will become e power x by y into x minus y divided by y into 1+ e power x by y. So this is what we have. So y is the independent variable, x is the dependent variable. Now you can see 1 by F, if F is a homogenous function, 1 by F is also a homogenous function.

So this is a homogenous function with degree zero, it is clear. So I use instead of y by x equal to v, we use x by y equal to v of y. Okay. So because, why v of y? v, where v is a function of x because x is a function of y, y, y is the independent variable, so v should be function of y. Okay. So if we do like this, we will get dx by dy equal to v plus y times dv by dy, same procedure which is equal to, dx by dy now, I know this one, this is e power vx minus y, so you can write x by y, y you take it out, 1 minus v minus 1.

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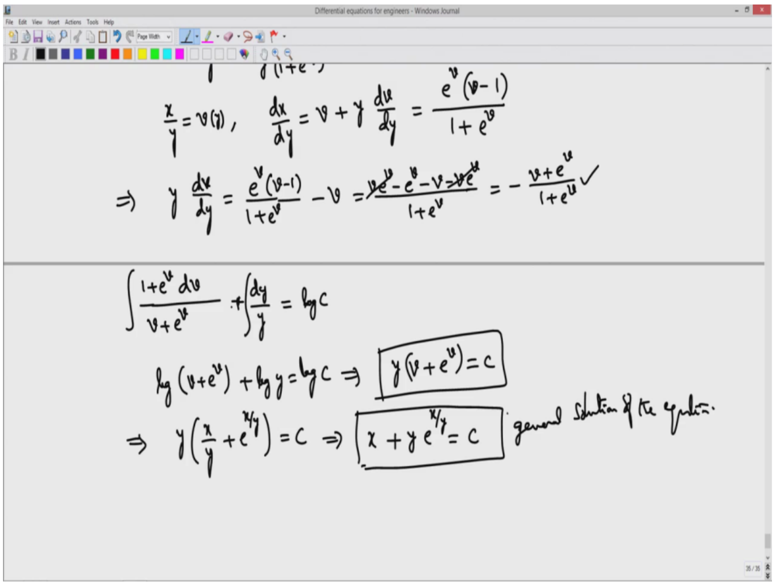

y, you take it out. So we have x by y minus1, so that is v - 1, so y comes out here, so y, y cancels, so still, denominator you only get e power v. This is easier to deal, okay. So this implies, this implies y dv by dy equal to e power v by 1 + e power v into v - 1, e power, minus, v minus1 minus v. So this is equal to one plus e power v, e power v into v, that is the first minus e power v is the 2nd term and now if you cross multiply this, you get minus v minus v, e power v.

So you can see this gets cancelled, what you are left with is v plus e power v by 1+ e power v with a negative sign. So this is what you want to solve. So now these are separated, where v and y are variables are separated, bring all the v variables one side. So you get 1+ e power v by v plus e power v okay. And then equal to plus dy by y equal to 0. I brought this this side, 1 plus v power to dv divided by this, dy, so dy by y is the right-hand side minus dy by y, I bring it to the left-hand side, this becomes zero. this is the one.

Now I can integrate both sides and finally write the right-hand side as some arbitrary constant which I am calling log C because if C is an arbitrary constant, log C is also an arbitrary constant. Log mod C, so log C is an arbitrary constant, so you, I wrote this because we can simplify this in much easier way. So this is, this now is easier. Is a simply log of v plus e power v is the anti-derivative of this integrand. So my integral value is this plus log y equal to log C.

So this implies y into v plus e power v equal to C is a general solution of which equation, this separable equation. Now go back and put your variables y into v is x by y plus e power x by y, okay. So equal to C. So this gives me xy, sorry, simply x, y y cancels, plus y e power x by y equal to constant. This is the general solution of the equation, okay. General solution of the given equation. So this is in a clear form. Although I did not get in an explicit form, this is also an implicit form, whereas better form, but implicit form than earlier way, okay.

So if you look at the, sometimes you have to look at the inverse function as so it exists, you continue to write, you continue to solve the problem for the dependent variable, new dependent variable x as the inverse function, that is x, x as the dependent variable and solve it and get the solution and whatever x that satisfies the final general solution which is in implicit form is the general solution, okay.

So this should be invertible both ways, okay. Whatever If x, if such x satisfying this, which has an inverse, it will have an inverse, that will satisfy my earlier solution form, earlier explicit form general solution, which is the general solution. Okay. So 2 different ways you can do. So you have to choose the best way so that you can simplify things and get in a good form, final form should be nicer.

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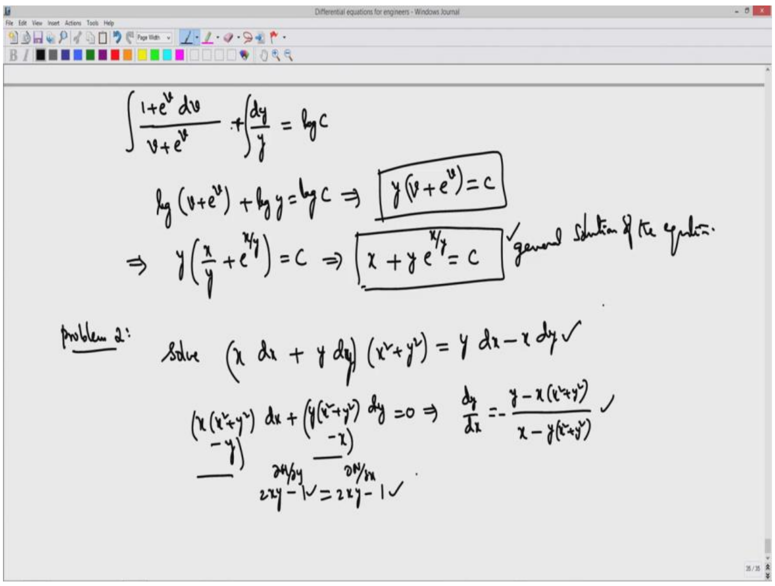

So this is how we solved this problem. Now let us solve one more problem, problem 2. Let us pick up some equation, if we have like this, sometimes you may have to do like this, x dx plus y dx, x dx plus y dy into x square plus y square equal to y dx minus x dy. How do we solve this equation? I do not have any hope, even if you write the x dy, you cannot make it a, you cannot make it a separable form, okay. We can see that.

So this is a homogeneous, still, so you want to see that, try to write it, so x into x square plus y square into dx plus y into x square plus y square into dy, so this minus, so this minus y is your dx, this minus y, this minus x is your dy. So this is equal to 0, so this will give me dy by dx equal to y minus x into x square minus y square divided by minus x minus y into x square minus y square. So it is not, clearly, this right-hand side is not homogeneous function, so we cannot apply the method.

That means we cannot reduce this, we cannot reduce this in non-homogeneous into homogeneous, we cannot reduce. You have to see now exact form, if it is exact or not. If it is an exact, so dow N by dow x is y square, dow M by dow y is, what is dow M by dow y, this is my M, this is my M, dow M by dow y is simply 2 xy minus1. So dow N by dow x which is 2 xy minus1, so actually exact equation. So this is how you calculate. So dow M by dow y, dow N by dow x, they are actually equal, so you can actually solve by an exact equation if you want.

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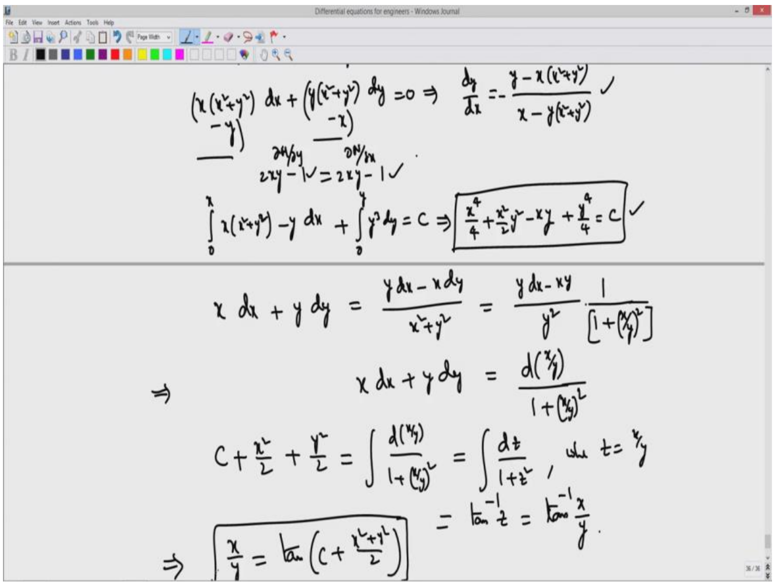

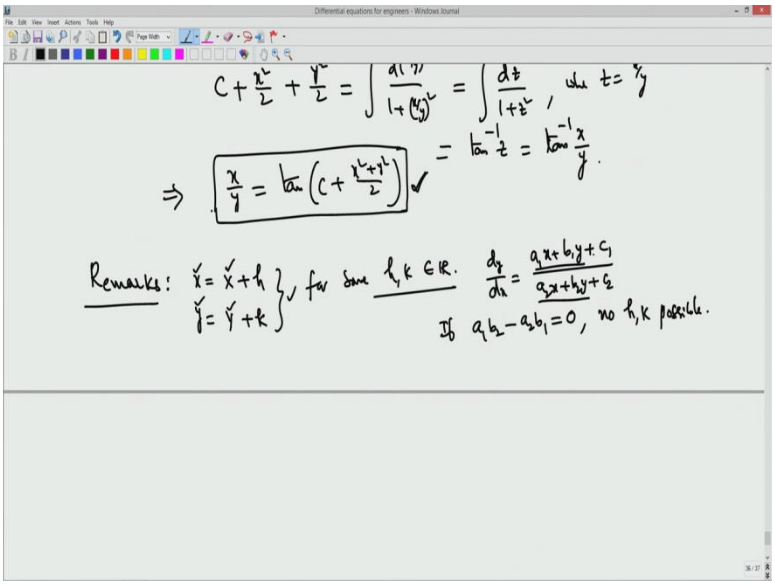

This is an exact, right. So this is, you do not have to make this M dx plus N, N dy equal to 0 form and see that is an exact. Sometimes by looking at the equation you can just arrange, rearrange it in such a way that you can just directly see that it is, it can be integrable. How do I see this, you can directly see that. So now we will write the general solution. So once you know that it is an exact, you know how to solve this, an integral from zero, so there is no issue, 0, 0 is in the domain. So you can integrate from 0 to x, x into x square plus y square minus y dx.

Plus only terms, so 0 to y, put x equal to 0, that makes it only y cube dy equal to constant. So this will give me x cube, so x power 4 by 4 + x square by 2y square minus xy, that is the first integral, okay. And then the 2nd one is this y power 4 by 4 equal to constant. This is your general solution, this is how you get your general solution. Okay, by the exact equation method. Directly looking at the equation, so let me rewrite this equation in this form. So x dx which I can integrate, y dy I can integrate equal to y dx minus x dy divided by x square plus y square.

So this I will rewrite like y dx minus x dy, I take y outside this xy plus y square, y square, so that what I have is 1 divided by 1 plus x by y whole square. Okay. This x square plus y square I wrote, y square I take it out, I wrote 1 + x by y. Okay. Both are same. Now I can integrate both sides, so you can easily say right hand side, this is the derivative of x by y divided by 1 by x by y whole square. Choose x by y as z, when it is like dz by 1+ z square.

So the left-hand side is x dx plus y dy. Now I integrate, they are in the exact form. So this is if I integrate, this is actually d of x plus y minus x by y by, what is this one, this is the right-hand side is, okay let me write. So you can integrate both sides, x dx you can integrate x square by 2 + y square by 2, another integral. So here, this is simply integral d of x by y by 1 + x by y whole square which, which is equal to integral dz, dz… So this is like you can x by y you can replace by z, this is like z square where z is x by y.

So this is equal to tan inverse z. Okay. So right-hand side is tan inverse z which is in the xy variables, this is x by y. So I have got now finally tan of x square plus y square by 2 is equal to x. So this is the plus I have, because when integrated I have the arbitrary constant. So I have the arbitrary constant. So it is not really, you cannot do this. So if you, if you put your arbitrary constant here, the left-hand side itself, the left-hand side also integrated. So I have arbitrary constant here. So that makes it x by y, I apply both sides tan, so you have tan of C plus x square plus y square by 2. Okay.

So this is your general solution, it involves an arbitrary constant C. So this is the one-way, directly you can integrate by rearranging the terms suitably in such a way that you can directly integrate, okay. Although these 2 forms look different, actually they form the general solution. So one C, C, if you view some C here, that will be another C here. So for different C value, you have a solution, okay. So this is how you solve the general solution of certain equations.

So that will give one more direct kind of equation. So one simple form, now I will choose some other thing. So some remarks I would like to make on the methods so far. So separable form, that is a direct method, if it is, if the right-hand side, F of x is separated, you know how to solve it, it is homogenous, you know how to solve it, non-homogeneous equation that can be reduced to homogeneous equations, you know how to solve it. But when you do the non-homogeneous conversion to homogeneous equation, what you have is, you use the transformation x equal to x plus H.

x is the old variable, capital X is the new variable. Small y is equal to small y plus K. these are the, x, y are the, small, small x, y are old variables, capital X, capital Y are new variables. you should have, this is the transformation for some H, K belongs R. If it exists, only when they exist, you can proceed. Only when they exist, you can proceed and solve, you convert this into homogeneous equation, otherwise what happens, if you cannot find, when can you not find such a unique H and K or no H and K value such that that makes the equation an exact equation.

You can reduce a non-homogeneous equation to a homogeneous equation with this transformation provided you have such a H, K values. Those H, K values you can do is I have dy by dx which is equal to A1 x plus B1 by plus C 1 divided by A2 x plus B2 y plus C2. Okay. If A1 B2 minus A2 B1, this is the determinant, if this determinant is zero, I will not be able to find my H, K. No H, K, no H, K possible. Okay. So this you have to be careful, so in that case, you simply, because that means if this is zero, this term is same as this term, it is just a constant multiple of other.

So you can replace that as x plus y, you choose as some new variable and reduce that equation into in that new variable. So we will do that example later.