Differential Equations for Engineers.
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Lecture-7.
Non-Exact Equations -Finding Integrating Factors.

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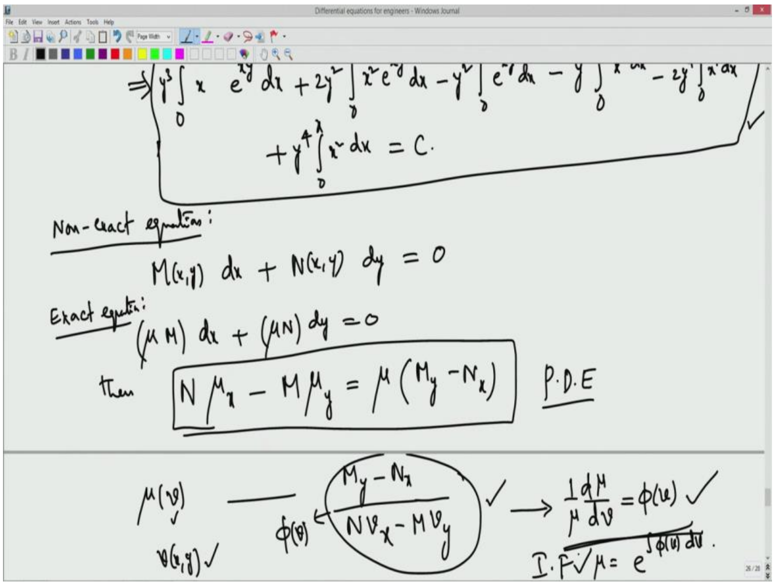

So what we had is non-exact equation like this M x, y dx plus N x, y dy equal to 0. So when you are given this non-exact equation, if you want to make it an exact equation, you multiply it with some mu, okay, some mu function, mu function of x, y into M dx now mu N dy equal to 0. So mu is unknown, you do not know priorly. If you multiply some function that makes this an exact equation, exact equation. I multiply some mu of x, y so that this equation after multiplying both sides with mu is, now the 2nd, the 2nd equation when you multiply mu, this is what it becomes.

Now this is exact, you want this to be exact, such a mu I. So what exactly we have seen is, it satisfies, mu satisfies some differential equation, some ordinary differential equation, that is, then mu satisfies, what you have seen already that N into mu x minus M into mu y equal to mu of mu into My minus N next. So then mu satisfies this ordinary differential equation. So this is actually not ordinary differential equation, this is actually PDE, this is a partial differential equation.

Because you know only ordinary differential equation, if at all you know, you know the methods to solve ordinary differential equation as of now. So you want ordinary differential equation, that too whatever you know so far, okay, with these methods can you solve it. Okay. So if you look for mu, if you want mu to be function of x only, if you want that mu to be only function of x, okay mu as a general thing we have done mu of v, okay.

If mu v, then what you have to check, if it is mu of v, so you just have to check this function M y minus N x divided by N vx minus M vy okay. If v you want x, then you put that x in the place of v and you verify this quantity. It is function of v, that means function of x only, then you can go ahead and solve that equation. What you have is d 1 by mu D mu dv equal to whatever you get here as a function of v. If I call this some function of v, so you have sanction of v.

So this is variable separable method, you can solve this equation and you can find mu, okay. So what is required is, if you are looking for mu as a function of, as a v, v can be in specific form. v can be x or y alone or v can be function of x, y like x minus y, x into y or x by y or x square plus y square, you can take any complicated form, say sin xy, sin x by y, everything, you can choose whatever may be your y, your v, you choose it as some function of, but this is fixed, you fix this v and then verify this quantity, M is known, N is known, v now it is known, once you fix it, so you verify this quantity and after verifying this should be function of, this whatever you have chosen v, some function of v, then you solve this, you come and put it here and you solve this for mu, mu of x, y, v is anyways function of x, y, v.

So these are variables, you do not see this is mu and v, these are the 2 variables d mu by mu equal to Phi v into dv. So what is the solution? Mu equal to e power integral Phi v dv. This is your integrating factor. Okay. So this is what exactly you have done.

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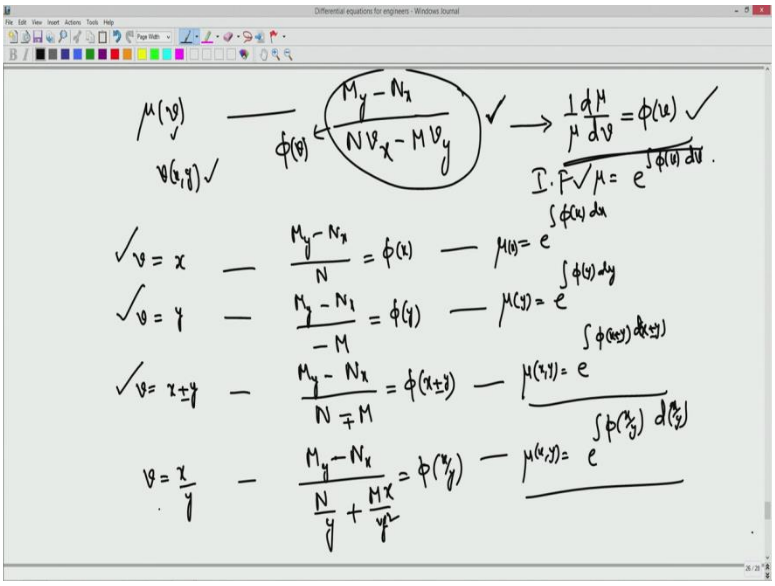

So far, if I take x I will just right at one place for standard forms, what you have to check. If my v equal to x, then what you need to check, My minus Nx divided by N has to be Phi of x, then mu e power integral Phi x dx, functions of x. Okay. If v you take as y, what you have to, verify, My minus Nx divided by, put v equal to x, v equal to y, dy by dx, there are 2 different variables, independent variables, okay. v of x, y, you are differentiating, v is only you, you are taking as only function of y, it is not function of x, okay.

So v is only function of y and you are different shooting with respect to x, 0. So this is 0, so you have minus M, vy is 1, so derivative is 1. So this is, if this has to be function of y because Phi v, v is y, so it has to be this, then only, then your mu y, simply go ahead and write it as Phi y dy. Okay. Otherwise any general v of xy, for example now I write, I will give you list. I list v is this and then you can go ahead for other forms, okay.

So v equal to, let us say, I will just give you x plus y, if you want, if you are looking for in this form, then what you should check is My minus Nx, dow M by dow y minus dow N by dow x divided by N into vx is 1 minus M into vy is also 1, Dow v by Dow y is 1. This should be, this I verify that this should be function of v, that is v is x + y, x plus y. Then what is my mu x, y, is equal to e power integral Phi of x plus y into whatever may be is a variable x plus y. Okay.

This is like D, D of this, D of x plus y, if it is like this only, I can integrate. So this is what is my function of x, y, integrating factor, like this you can go on. I will give you one more, v equal to x, if I take x minus y, this will become plus, okay, then this will become minus, x minus y plus or minus, plus or minus, plus minus, okay. So x by y if you, if you are looking for v as a function of x by y, what you have to verify again here, so you have mu My minus Nx divided by N into vx is N by y minus minus plus M by, Mx by y square. Okay.

This whole thing should be some function of x by y, then only you can get your , integrating factor x, y as e power integral Phi of x by y D of x by y. Okay. I can easily integrate. Okay. So this is how certain forms. If you are, but you do not know, given equation, which form you have to try. So mostly you always check, if it is not exact, you check these 2, maybe you have time, you can check this and if it works, it is fine. I will give you examples where you can use other forms as an exercise, okay.

And I will give you the assignment, I will give you certain equations where I give you hint that I give you come I ask you to choose this v, some form of v so that you can make the equation exact, okay. Otherwise we do not know how to exactly, we do not do exactly how to find such a v so that the non-exact equation becomes exact, okay. So most standard method is this is what you see, v equal to x and v equal to y. So you always verify, this one or this one will work. Okay.

Most of the text books, these are the things, but this is a procedure that works, this is how it works, this is how you make many, many equations in different forms, many equations you can, many non-exact equations, you can convert into exact equation and then solve it. Solution, solving exact equation, you know the solution procedure, you simply integrate M with respect to x, keeping y as it is and N you integrate from, from y0 to y and keeping x, fixing x as x0, okay, equal to constant, that is the general solution, that is straightforward, okay.

So we will move on, we will try to, this is how, these are the methods you have, standard methods you have to solve general first-order ordinary differential equations, okay. But if you see, if you have first-order equations as a linear equation, okay, that means you should not have any powers of y or y dash, then it is called linear equation.

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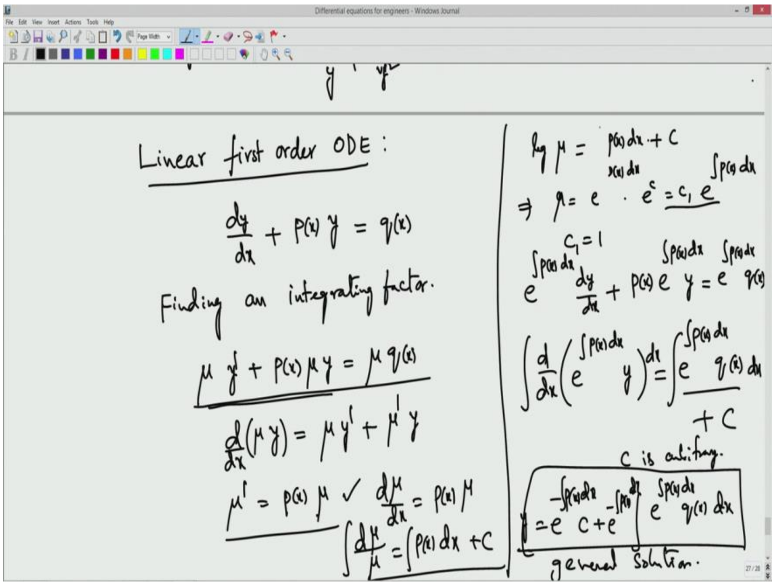

So the standard form of linear equation we can write, so standard form of linear equation, so linear equation, linear first-order ODE. So you can put it in the standard form as dy by dx plus some px, known quantity, function of x into y. So you can see y dash is here, y as it is, you do not have square, you many hours of y and y dash equal to, it can be some function of x, some Qx, this is what you have, px and Qx. This is the standard form of linear first-order ordinary differential equation. How do we solve it?

Now you know, you have seen how you convert exact equation into, non-exact equation into an exact equation, you multiply some function that is called integrating factor. So by multiplying an integrating factor, you reduce that equation into some derivative of something so that you can integrate. So here also you try to do the same. So this is a simple first-order linear differential equation. This comes in many, practical use, whatever, many times you come across these equations, linear equations.

After simplification of your problems, somewhere small, so it is standard to know, you should, you should know how to solve these linear equations and its various forms. Okay. So today we will do this linear first-order ODE. So what is the method is as usual, so integrating factor. So finding, finding an integrating factor. How do you find this integrating factor? Again so as usual you multiply mu y dash plus px mu y equal to mu Qx, mu Qx, okay.

This is only function of one variable, so I put simply mu x, mu x, mu of x, okay. So let us, I I do not know really exactly what is, y, mu is a function of x or not. So I simply multiply some mu, okay. So this is equal to, so if I multiply, so the left-hand side, left-hand side I expect some form like derivative of. So you see that mu y, so y into mu. If I say, if I come if I write like this, if I write left-hand side like this, then what is the left-hand side. Mu times y dash plus y Times mu dash. Mu dash means I am doing with respect to d dx, d dx of mu y is this, okay.

So I want to write this left-hand side as this. In that case what I got is mu y dash, that is M, but mu dash y is P into that mu into y. So my mu dash should be equal to px into y, px into mu, okay. So mu should satisfy this ordinary differential equation. So this you know how to solve, what is mu dash, d mu by dx equal to px into mu. So this I know how to solve. D mu by mu equal to px into dx. I simply integrate and get the constant so that will determine my mu. Okay.

So you will have log mu equal to integral px dx plus C, okay. So this will give me mu equal to e power integral px dx into e power C, that I call it, some C1 e power integral px dx, both are arbitrary constants, C or C1. Okay. So this is what I got. So P is function of x, integral is also function of x, exponential of that, so it is mu is a function of x. So this is how you get your mu. If you want after multiplying the left-hand side if you want in this form. Okay.

So now I go and replace my left-hand side, so given equation I multiply mu with this. If I multiply this, I take C1 equal to1, if I multiply e power integral px dx into dy by dx plus px into e power integral px dx into y equal to e power integral px dx into kx. I multiply both sides of the equation. So the left-hand side, with if I multiply this with any, fixing my C, C1 as 1, this becomes derivative of d dx of mu e power integral px dx. Okay into y, mu is this and y as it is equal to e power px dx into Qx. So this is known.

Q is given, this is, this I calculate because P is given and I have like this. So this is like a derivative of something equal to something known, some function, I can integrate both sides, how do I integrate this, so I simply integrate with respect to x both sides, dx, dx, so we will get some constant C. Okay, C is integration constant, C is arbitrary. This is your solution, so this implies e power integral px dx into y equal to C times, C plus integral e power integral px which is a function of x Qx dx.

So this is your general solution, so if you bring this exponential, the other right-hand side, what it becomes is equal to e power minus, minus px dx. The integral px dx and we have to write it here as well. So C plus e power minus integral px dx into this, okay. So this is what is your general solution, this is now you get a general solution of given linear first-order ordinary differential equations, okay. So we will do some examples, so we will see how it can be done.

So one remark here. When I am doing this integration, I did the indefinite integration, so when I did this, integrating factor, I did indefinite integral, simply integral. So you can also do by fixing some x0 to x, then this C will not come into picture, okay. So C will not be there, if you do the definite integration, C1 will be there, instead C will come as mu value at x0. So mu is unknown, so you are looking for mu which is an unknown function. If you fix it as some point, some arbitrary point x0, that will be arbitrary constant mu at x0.

So it will come as an arbitrary constant at the end. But you will have integral from x0 to x, x0 you can fix, that should be in your domain. So if it is, nothing is given, you can take, x belongs to full R and x0 you can fix it as 0. So you can take it from wherever in the domain, in the domain you take your x0 and you can integrate. So sometimes it is useful, sometimes directly, indefinite integration is useful, okay. So as and when, whichever is easier we will do in the problems.

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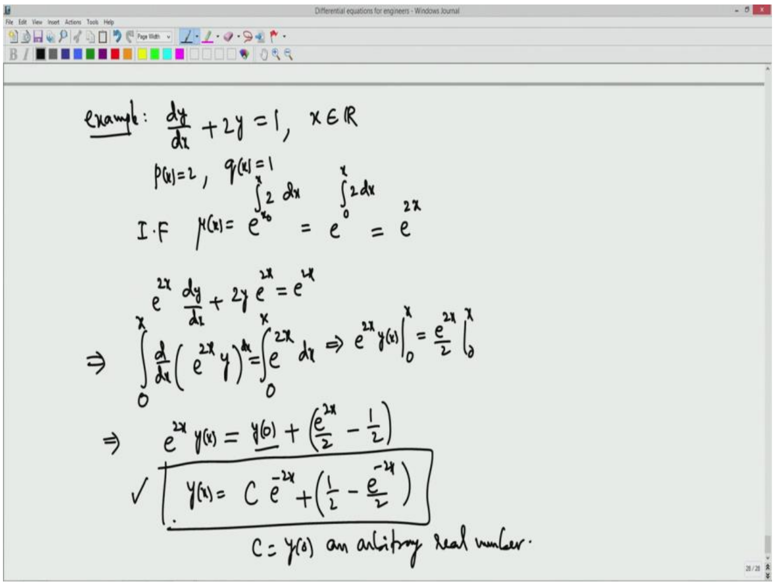

So let us, I will give you some examples to solve this, solve these linear equations, you take simple linear equation. So let us take simple example, which is dy by dx plus 2y equal to1. So you see that y and y derivative, y dash y are simple, they are in, they do not have powers, power is only, so okay, the degree is 1. So this is the linear equation whose solution if you are looking for, so what is that P is, P of x equal to 2, constant, Q of x equal to1, which is also constant. So integrating factor is mu of x which is equal to e power integral P is 2 dx, okay.

So you can now, this is always I think good practice to fix your domain. So x0 to x, okay. So if you do like this, what does this become? So choose x0 equal to some domain, some point in the domain, any point you can choose. So if nothing is given, 0 is also part of the domain, it belongs to R, you can choose x0 as 0 to x 2 dx. Okay so what is this, this is equal to, simply 2x, okay. So e power 2x is your integrating factor, so you multiply e power 2x both sides of the differential equation.

So dy by dx plus 2y e power 2x equal to e power 2x. So this left-hand side, immediately I can write d dx of that you into that if e power 2x into y. You see this, e power 2 x dy by dx, one derivative, if you differentiate one, you will get 2y is power to x, okay. So this is equal to e power 2x. Now again you integrate both sides from 0 to x. 0, I take it as, x is arbitrary, 0 is part of the domain. So both sides you do it from this, I do not have to write, I do not have to write arbitrary constant, okay.

Integrate both sides from 0 to x, so what you get, so you will get e power 2x yx, you substitute these limits equal to e power 2x by 2, substitute this. So this will give me e power to x y x minus e power 2, 0, so that is 1 into y at 0 equal to e power 2x by 2 - 1 by 2. Okay. So if I take this y0 on the other side, this becomes plus y, this is what you get. So what is y, y is the solution, y0 is the arbitrary constant, okay. y is the unknown function, so y at one point, y at 0, because it is unknown function, it is arbitrary, it can be anything, okay.

This is your arbitrary constant. So if you say yx equal to, my solution, y at 0 into e power minus 2x, bring this e power 2x this side, so this becomes one by 2 minus e power minus 2x by 2. This is what you have. So where y at 0 is not known, okay. When you have like this, so this is your general solution because y0 is arbitrary constant. If yx is the solution you are looking for in terms of itself, itself at one point, that you can give, so that is still arbitrary. Okay. So you can write this as a constant where C is y at 0, arbitrary real number, okay.

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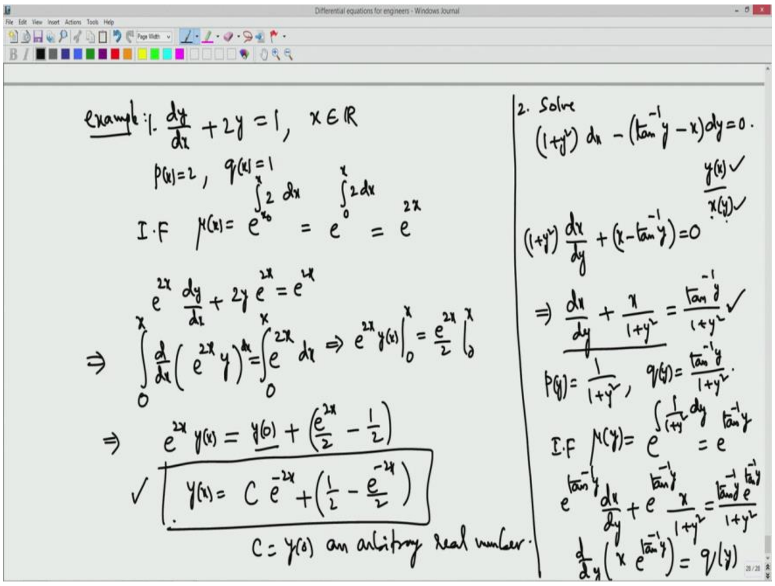

So this way you can get the general solution of the given linear partial differential equation of first-order, okay. So let me give you now complicated one, then I close. Little complicated, so we will see, I will write, I will take one more example here. So 2nd, solve, if I ask you to solve this equation, 1+ y square dx minus tan inverse y minus x dy equal to 0. If I ask you to do this, you see by looking at the equation, you have y square, y is tan inverse y, so it is not linear equation.

You can expect to solve by other method, it is not separable method, it is not homogeneous, you see this function is not homogeneous, this function is not homogeneous, so homogeneous you cannot reduce, reducible to homogeneous, it will not work. So exact, you try, I do not know, so you may have, may not work. So best way is, as you say, the domain is not given. So x is, this function is, x is defined. So you view this as a y, y as independent variable. So suppose solution is yx, okay, suppose this function y is invertible.

So I can write invertible function is x, y. So in that sense once you invert this function y of x and you write it as x of y, y becomes independent variable, x becomes dependent variable. So now you view that way, the x is a dependent, dependent variable. So you can see that, dx is, so you can write dx by dy 1+ y square minus, so plus x minus tan inverse y equal to 0. So this becomes dx by dy plus x by 1+ y square equal to tan inverse y by 1+ y square.

So I view the equation as though I have a solution that is invertible. So once I have an invertible function x of y, the equation, same equation which is, which you can view as, y and dependent variable and x as independent variable, now with this invertible function, I can also view as dx by dy equal to something. So that is what is I have. So now x of y, x of y is this, so we will solve this equation. So now, this is linear. This is the linear equation, dx by dy, in place of x, you have y, okay, linear equation how is it, earlier, dy by dx equal to px into y equal to Qx, instead of x you have y now.

So this is a linear equation, P is 1 by 1 plus y square, it is not px, it is going to be Py here, it instead of x, you have y. Q y is tan inverse by 1 + y square. So you simply integrate, so what is the integrating factor, integrating factor mu of, now it is a function of y, e power integral 1 by 1 plus y square dy. So I do not do like a linear, so this I know because 1 by 1 plus y square I do need to integrate from some fixed number 0.

Because it is invertible from R to R, y of x, it is, you can invert it x to y or y to x, you can go, means, that means x takes all the values and y also takes all the values. So you can choose your indefinite integral or definite integral here. 0 to y can choose or simply because you know the integration, I am not doing it. So here is actually e power tan inverse x, tan inverse y, okay. This integral value is tan inverse y.

So if you multiply this, what you get is d dy of e power tan inverse y into x. What is this one, this is exactly, this is left-hand side. Okay, after multiplying this. After multiplying, if you multiply this integrating factor. So let me write, if you integrate, if you multiply to this e power tan inverse y into dx by dy plus e power tan inverse y, x by 1 + y square equal to tan inverse y, e power tan inverse y divided by 1+ y square.

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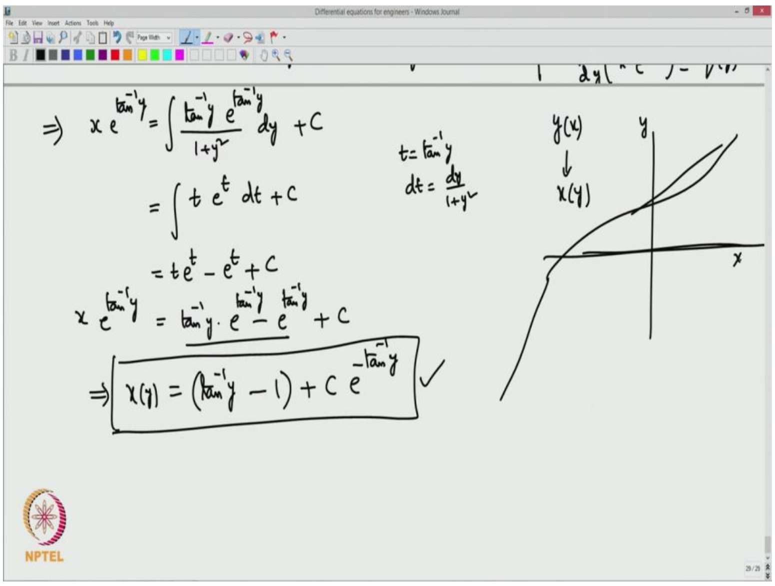

So the left inside is simply derivative of d dy, okay. So d dy of x times e power tan inverse y, equal to, I have same right-hand side, okay, it is called some function Q of y. Okay. So, so this implies, you can integrate with respect to y. If you integrate both sides with respect to y, what you end up is x times E power tan inverse y equal to integral tan inverse y e power tan inverse y by 1 + y square dy plus C, C is an arbitrary constant, okay.

So this you can take as integral, tan inverse y you can take it as T, one variable, so e power T d, if T equal to tan inverse y, dt is dy by dt by dy is 1 by 1 plus y square, y dy to bring into the side you can write dy as dy by 1 + y square, this to I can write it as dt plus C. This is equal to, what we have, so that e power dt minus e power T, this is the solution, so plus C. So what is T, T so substitute back, we have tan inverse y into e power tan inverse y, minus, so e power tan inverse y plus C is my x into e power tan inverse y.

You bring it the other side, so this will give me x, x is now dependent variable, it is a function of y. x of y equal to tan inverse y into e power tan inverse y and if you bring it the other side, e power minus tan inverse y, that goes, minus1, okay, so this whole thing into e power, tan, minus tan inverse y is this, plus C times e power minus tan inverse y. So this is your general solution, okay. So you view assuming that you have an inverse, you have a function y of x that is invertible.

So that means you can write as x of y. You change the independent variable to dependent variable, that is possible only if you can go from x to y and y to x, it is invertible back, okay. So you have something like this, so if you have e power x or simply some curve which is defined for all values of x, for all values of x it is defined, this is x, this is y. So you can also view this as like this. x as a function of y. So it is invertible map, okay. I hope you know.

Then once you have this, so you can easily see whether you have, you can invert this. So this is the general solution, so this is the general solution of the given equation. Okay. So you can, you can simplify, if you really want as y, you want to view y as a function of x, you can do that. How do we do, you can write, invert this , this implicit relation, okay. So inversion you can also get y of x, y in terms of x, okay. So this is how you can solve a linear first-order ordinary differential equations.

So we are, I have given 2 examples, you can also do some more like this, next we will do some equations which are not exactly linear equations but they are close to the linear equations, that can be made them into linear equations by some way, okay. So we can reduce them into the linear equations and follow the same procedure which you have done for the linear equations to find the general solution, okay.