**Differential Equations for Engineers.**
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**Lecture-63.**
**Uniqueness of the Boundary Value Problems for Laplace equation.**

So this is how we solve Laplace equation , this is how we solve the boundary value problem for the Laplace equation. When you have when you have this boundary data as the normal data, so it has to satisfy some necessary condition that net flux should be 0. But in this case, so far we have only given a solution, solution for this boundary value problem. So we have not shown the uniqueness of this problem, so we can see that, see if just looking at the uniqueness, if you consider Laplace equation in domain D, okay.

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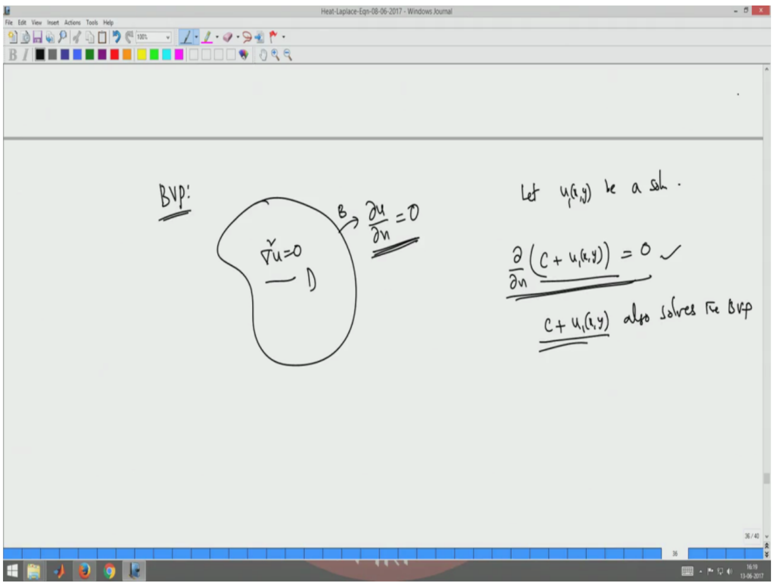


So gradient square U equal to 0 and suppose you provide the normal data, so do U by du n, so normal velocity, suppose it is given. Then you know that it has to suppose, such a way that, suppose let say some F of X, Y, okay, X, Y belongs to this boundary B, this is the boundary B, okay. So boundary is B, on this if you satisfy the necessary condition, so that is, let us say ds, S is along the arc over B equal to 0. Suppose the necessary conditions are also satisfied, okay, such that you have this condition is given. Okay. And let say F is 0 is given, so flux is, simply, let us take the simpler one. On the boundary you have a normal data, that is 0.

It is not really nonzero, it is simply, it is like you insulate this boundary, then what happens, those solutions, suppose you find the solution for some, the circular domains rectangular domains with the method that you have deviced, you can find a solution. But then if you add, once, let us say you call this let U of X, Y be the solution, okay, U1 be the solution, be the solution that you devised through this method, you calculate, that is your T one. You call this U1 of XY as a solution, then you add any constant, C plus you one of XY is also satisfying Laplace equation inside D and also satisfying, this was a normal, normal derivative of this, you see that this satisfies. Okay.

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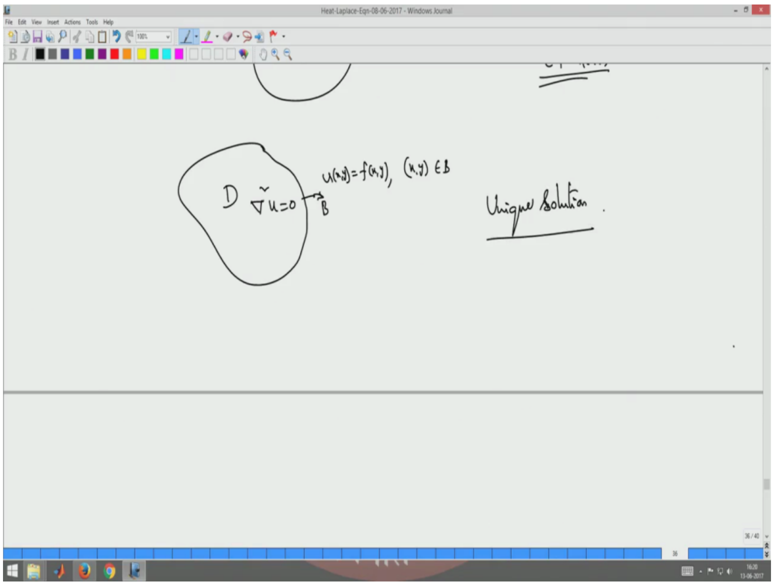


So that means clearly C plus, that means any constant times U 1 of X, Y also solves, also solves the boundary value problem. Okay. If this is your boundary value problem, if you have a normal data everywhere, so you can say that you have a unique solution. So clearly you do not have unique solution for this problem. But if you provide a normal, Dirichlet data, suppose you provide Dirichlet solely U is provided, so on B you provide U, U of, U of X, Y is given, there is some F of X, Y, okay, where X, Y belongs to B. And inside the D, gradient square U equal to 0.

So for this kind of problems you have unique solution, okay. So let us say you have this smooth domain or piecewise continuous domain, okay, so you have such a domain and you have this is your boundary. And we have something like maximum principle to show that for the Dirichlet data, you have a unique solution, okay. Solution, whatever reconstructed should be the only one solution.

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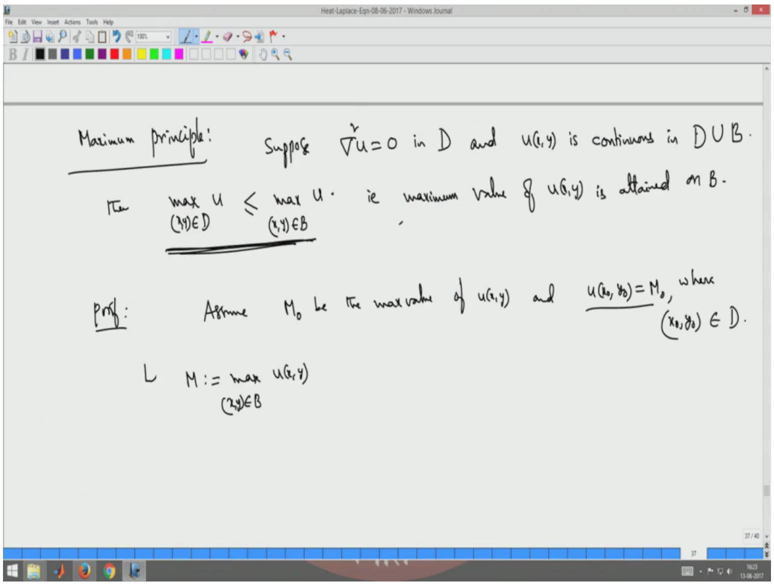


So 1st we write this maximum principle, so maximum principle is, it says suppose, suppose del square U equal to 0 in D, okay. And an U is, and you also assume that U of X, Y is continuous, continuous in D union B. So together covers including the boundary it is continuous function. Then maximum value of U, X, Y belongs to D, D should be less than or equal to maximum value of U where X, Y belong to B. So that means maximum exists, maximum value of U will be attained only on the boundary. On the boundary only you will find somewhere the maximum value of U, okay.

One of this, one of the points in the boundary, you will see that is the maximum value, that is the meaning of this, okay. That is maximum value, maximum value, maximum value of U of U of X, Y is attained on B. So that is the meaning of this inequality, we can see, okay. So maximum value of U inside should be less than or equal to maximum value of U on the boundary. So that means equality can be there because when it is a constant solution, so it is, suppose it is a constant everywhere, that means it is also maximum value is also on the boundary. So in that sense it is not an issue.

So this statement is still true. So we can prove this, simply just outline the proof, okay in few minutes, so that we can give this principle and using this principle one can show the uniqueness of this Dirichlet boundary value problem for the Laplace equation. So how do I do this? So I assume that lets mb small m b, 1st of all you assume, assume M0 be the maximum value, maximum value of U of X, Y and it is achieved and U of X0, Y0 equal to M0, this M0 where where X0, Y0 belongs to D. So I assume that it is not on the boundary, it is in D, inside D, so that is what is the maximum value.

And you also define what is M, M is the maximum value of U of X, Y where X, Y belongs to B. So simply on the boundary you call this maximum, maximum of, maximum value of U on the boundary is M, okay. So let us call this one, let this be this, okay. Then if I define, so I assume that the maximum, then if I define, so I assume that the maximum value of U is achieved only inside the domain D, okay, only inside somewhere it is defined. So now I define a new function V of X, Y, okay.

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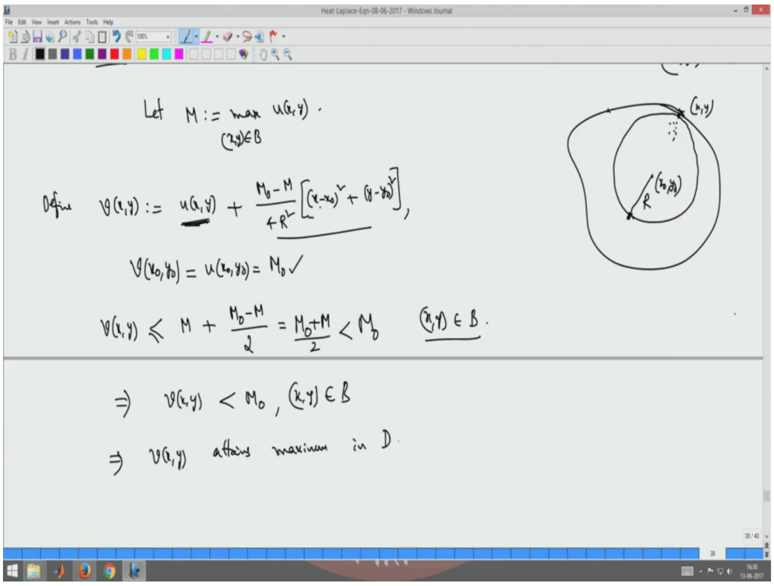


So this is equal to U of X, Y, this U of X, Y plus M0 minus M divided by 4R square into X minus X0 square plus Y minus Y0 square, so this is what is defined. Okay. Define this function, so what happens, this function if you define it like this, so what is this R, this R is, R is suppose, this is the boundary B, okay, R is the radius of, radius connecting. So you take any circle, it is entirely, this is a circle of radius R which is entirely in D, okay. So you take all the X, Y values like this, so if you want to have, so this is not just this one, this is for this all the points inside, this is the definition.

And you make one more circle here, this connecting to this point, at this every point of the boundary you make a circle, that is also defined. So for all these skew values, for all the values inside here, you have the same, some R, okay, that R you put it here, okay. So let us work like that, so for every point of the boundary, you make a circle, just inside the circle and with the radius R but it is entirely inside D, okay. And it is just touching that boundary, so instead only, so it does not matter, so it is just touching the boundary point B. So like that, so VX, for all VX I am here, so like that every point of the boundary, every point of the, every point of the domain D you can just have it.

So you can, you can have this one, so if you have any point you take, that you can always make a circle, that is just connecting, that is entirely inside D and just cutting one point of the boundary, okay. So like that to define this. Then you can clearly see that V at X0, Y0, same X0, Y0 is, when you put X equal to 0 and Y equal to Y0, it is simply U at X0, Y0. This is actually your maximum value of U, that is what we assume, okay. So that is what it is. And what happens when when you take X, Y, X, Y belongs to this B, the boundary.

On the boundary V is, you know that on the boundary U has a maximum value is M, U has a maximum value M. And here this will be and you see when you connect, so this is M0 minus M by 2 you take, okay, and 4, 4 actually 4 and this will be when you see this one, so inside this, I said this is always. I have less than or equal to N, when you take this point on the boundary, this is actually equal, equality comes into picture. R square, R square goes, this is this one, okay. So otherwise it is still less than, otherwise it is actually this quantity is bigger than this one.

So X minus X0 square plus Y minus Y0 square is always, say if you take here, this point, the 2 times actually, you have to consider any point inside the circle will be maximum X minus X0, okay. X minus X0 is R0, so this is, let us this is not, so let us let us make it again, so just to have this one, you have X0, Y0. X0, Y0 is here, you can always make a circle that is always contained inside, okay. So you can always create a circle and that touches the point of the boundary. This is what is your, this is with the radius R. So like this you can always move, you increase the circle if you want this to touch here.

But the only point is that it should be entirely inside this, inside this domain, okay, that is possible. So like that if you define, this will be, this will be, this will be actually much bigger than, this is less than, less than or equal to N this one and this is, this is on the boundary, okay. If, if you have this point on the boundary, that is this point, this is X, Y for the domain of B. And you have this one, so when you do this one, this point, only when X, Y is inside, is always , so this is always smaller than R square. So you can always make it less than 2R square, that 2R square, 2R square goes. So only thing you are left with is this one.

This is exactly equal to, this is exactly equal to M0, okay. So 2M minus M M plus M0, M plus M0 divided by 2, so that means this is always, because M is still smaller than M0, so this should be bigger than 0, okay. So you can still put less than here, so if you want, so it can be there, okay, this can be equal, this can be equal. So when you write this one, this is less than or equal to M and this also can be, if you consider this point, that may be equal, okay. In that case still actually it should be less than only. You will have 1 by 2R square, so when you consider this point for example here at this point, it may be the distance is 2R, 2R square, okay.

So this quantity is, this quantity is R square, R square goes and you have this distance is actually 2R square. So the 2R square goes, that can be, so that is actually equal to. For this point, at this point exactly, this distance is 2R square. So that goes, so that can be equal to this. Now if you do this equal to M + M0 by 2, because M is smaller than M0, that is what is the assumption, so this is the M 0. So it implies on the boundary, V of X, Y is less than M0, so it is always X, Y belongs to B. On the boundary it is always less that this M0.

So implies, so what is that you have achieved, so V at inside at some point M0 and all other points on the boundary it is M0. So implies it is B, the function V also achieves its maximum somewhere inside only implies V of XY achieves, attains maximum in D, so that is the meaning. So let us consider what is you just calculate if it attains its maximum in D, you just imagine a disk, you have a maximum value at the top. Okay. Imagine an umbrella, at the middle you have the maximum value.

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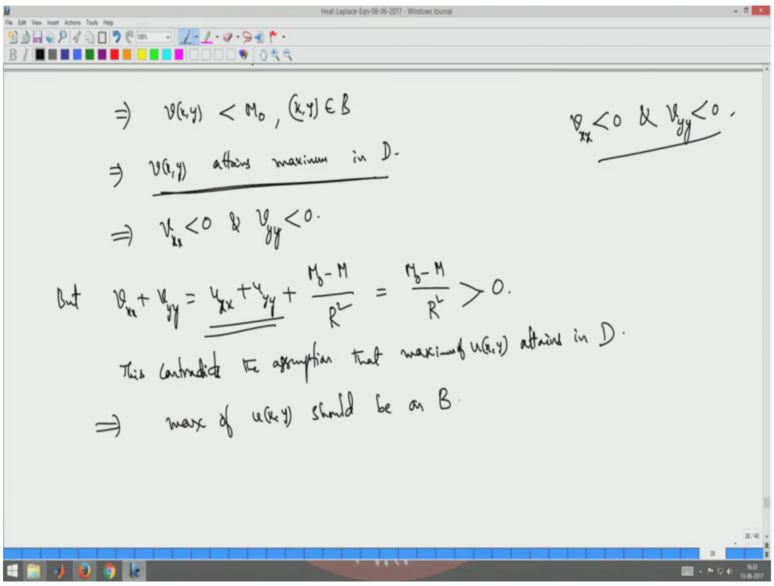


So if you have the maximum value V, that VXX, so you take the 2 derivatives, if you actually look at the slope of that, at that point to imagine like this, you have a maximum. You consider this flow, slopes are actually decreasing. So that means V X X should be less than 0 or in any direction if you go, it is a two-dimensional object or actually VYY should be less than 0 if this is to be true, okay. This implies VXX is less than 0 and VYY should be less than 0, this is remaining, okay. If it attains maximum only inside, this is what should happen. Okay.

So this implies but you can now calculate, you define this V, VXX plus VYY if you calculate, from the definition of V, you see that U UXX plus UYY plus, now if you can differentiate this twice with respect to X, that you get 2 and here also you will get 2. So you will get total, so 2 + 2, 4 okay. So 4, 4 goes, finally you get M0 minus M divided by R square. So if you calculate this derivative, so you do this one, so this is, we know that this, this U satisfies the Laplace equation, this is 0. So this means M0 minus M by R square. But M0 is bigger than M, this is if you assume that M0 is actually inside, M0 is bigger than M.

M is maximum value of U on the boundary full suffer this is always positive. So this is what is the contradiction, this contradicts the assumption. Okay. What is the contradiction here? But this is positive, okay. But you see that, so but when you calculate, this is positive. This contradicts the assumption that maximum value of U X, Y belongs to D, okay, is actually or simply write maximum value, the maximum, maximum of U attains in inside D. If you assume that, so there is a contradiction. So if you do not assume this that it has, that means it has to, if you assume that, if maximum value of U at inside D, then you got this contradiction, okay.

So like this you, the proof goes like this. So this is the proof, okay, so that shows that U attains that implies U should be, the maximum value of, maximum value of, maximum of U of X, Y should be on D, the boundary, okay. So, now you consider, you work with minus U, let minus U you take. Then minus U actually satisfies gradient of minus U is 0 and minus U is also can be, you can give the , okay. Minus U also satisfy the Laplace equation in minus U is also continuous. If U is continuous on the boundary, and minus U is continuous on D union B.

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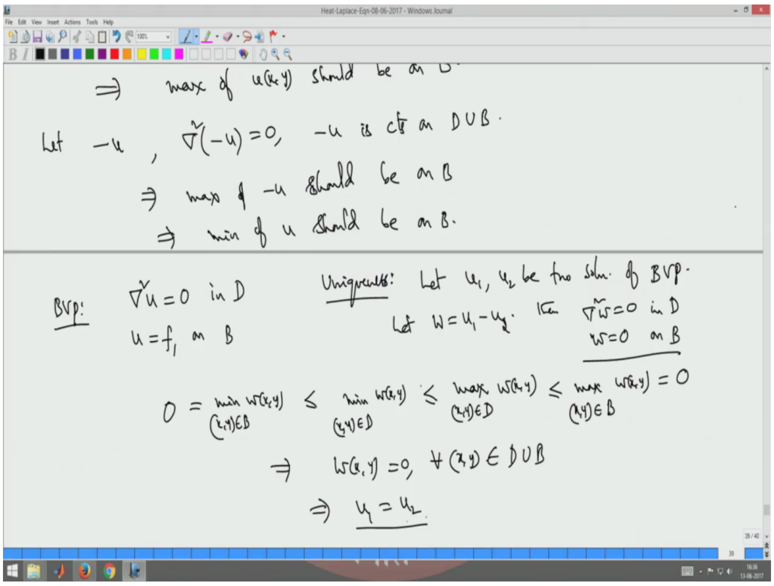


So implies maximum value of the maximum of minus U should be on B, implies minimum of U should be on B. So not only maximum of U is attained on B, on the boundary, minimum also should be attained on the boundary. So that means, so that is what is the maximum, minimum principal. If you apply now for the Dirichlet problem, so the Dirichlet problem is Dell square U equal to 0 and U is given some f1, okay, on the boundary on B and this is in in D, okay. This is the boundary value problem with take, now you can show the uniqueness, uniqueness you can apply this maximum minimum principal.

So let u1, u2 be normal 2 solutions, 2 solutions of boundary value problem. Sorry we have 2 solutions, so W to be, you take it as as usual, U minus U2. Then del square W is 0 and W equal to, on the boundary in D, on the boundary this is 0, okay. This is what it has become. Now the boundary value problem is this, now because the maximum value of U is 0, it should be on the boundary, boundary to 0. So this is less than equal to maximum value of, okay. So maximum value of, maximum value of W of X, Y, X, Y belongs to D should be same as, this should be less than or equal to maximum value of W at X, Y, X, Y belongs to B, on the boundary.

But we know on the boundary it is 0. And similarly this should always be less than or equal to minimum of W of X, Y, XY belongs to D, minimum value should always exist on the boundary. So you have one more inequality here, minimum, minimum value of W of X, Y, XY belongs to B and this we know it is 0. So implies W of X, Y should be 0 for every X, Y belongs to D union B, okay. Implies what is W, U1 equal to U2. So you have uniqueness. So in this way you can actually show that Laplace equation with the Dirichlet data is having only one solution, that is unique solution. This is what we have devised, this is what we have determined by the method of operation of variable for certain domains, okay.

So this is how you can actually, so this so far we have solved typical equations, wave equation, heat equation, Laplace equation, we have provided on, we have provided uniqueness also for all the 3 equations. Okay. So we will see this, so this is how we solve these equations and its uniqueness. The only thing that I have not touched is the solutions depending on the initial data. You perturbed the initial data little bit, still the solution is also will not be going to much deviated. So that means you perturb little bit the initial data, you just changed little bit and you, solution also should be changed, going only little.

So if you change only the initial data with Epsilon change, and you can, so you can also have a solution, the solution difference with and without perturbation, the solution difference will also be under controlled. So this is the continuous dependence on initial data, initial and boundary data, so that also can be done but there is little analysis involves, so we have not that in this course, okay. Thank you very much.