Differential Equations for Engineers.
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Lecture-62.
Laplace Equation Over Circular Sectors.

Welcome back, in the last video we have seen, how to solve Laplace equation in a annulus regions, so that we have seen the boundary value problems inside a circle or exterior of the circle as a special cases of this annulus regions. Also mentioned, today we will have, today what we do is we will consider the sector, part of the circle, that is a sector, so in an annulus region we consider. So in an annulus region we consider the sector, so that means that R is between R1 and R2 and Theta is between from Theta 1 and Theta 2. So this is the general domain.

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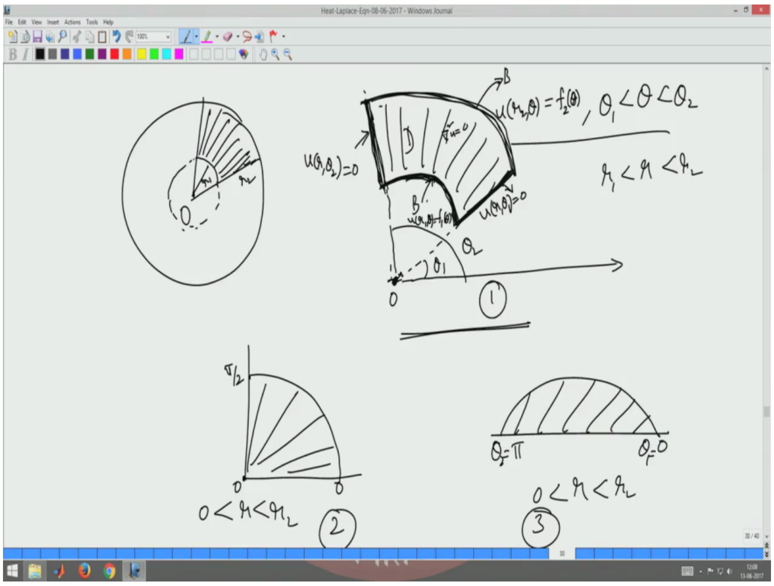

So we actually see that, so we will just look at the figure. So if you see that this is the figure we see, so Theta 1 is, so Theta 1 is this, 1 is the problem one, so 1 is the problem, so this is the domain of the problem. So U is inside, the whatever we have here, so this is the part of the annulus region, so if you consider, this is your domain d, so in this domain d, you have a Laplace equation satisfied. So that means gradient square U equal to 0. And anything that at R equal to R1, you have so you have, you can give the boundary data, so you can think of curved domain like this as a plate. So initially, so when it reaches the steady-state, so and the temperature reaches steady-state inside this domain, so you can see, you want to see the temperature distribution.

And provided, the boundary, at the boundary, you have this temperature is maintained U at R, Theta, R1 Theta is some F1 of data. So it only depends on Theta because R1 is fixed along this curve. And similarly on this outer circle, so you can provide this data as a temperature, you can fix, you can maintain distance with all along this curve line. So if you want to solve this problem by the separation of variables method, that, it is attracting we have mastered so far. So we need 2 boundary conditions, so you look for , so this is the data is given on for all variable and R1, R equal to R1 and R equal to R2.

So what you get is, you extract, try to extract Sturm Louisville problem, so in this case what we see is, earlier for the annulus region, it is already, it is already intrinsic so you have for Theta variables, you have a Sturm Louisville problem and we do not use any boundary condition because it is a complete circle, so you have Theta, that is actually same as 0 to 2 pie so you have intrinsically, have a periodic Sturm Louisville system. But here if you want the and you see that for Theta variable, when you extract the Sturm Louisville problem, so you have 2 more boundaries, this is the one boundary and this is another boundary, that this Theta equal to Theta 1 and Theta equal to Theta 2.

So at Theta equal to Theta 1, you have to provide zero boundary conditions. So that is, , temperature you maintain at zero and here also you maintain, some other zero temperature here. If we give this as a dirichlet data, we can expect a Sturm Louisville problem, so regular Sturm Louisville problem for Theta. So that is what we will see, so we will try to solve in this domain, so we write it as a problem, so we will try to this as a problem, okay. So we just rewrite, redraw this area, so what you get is this.

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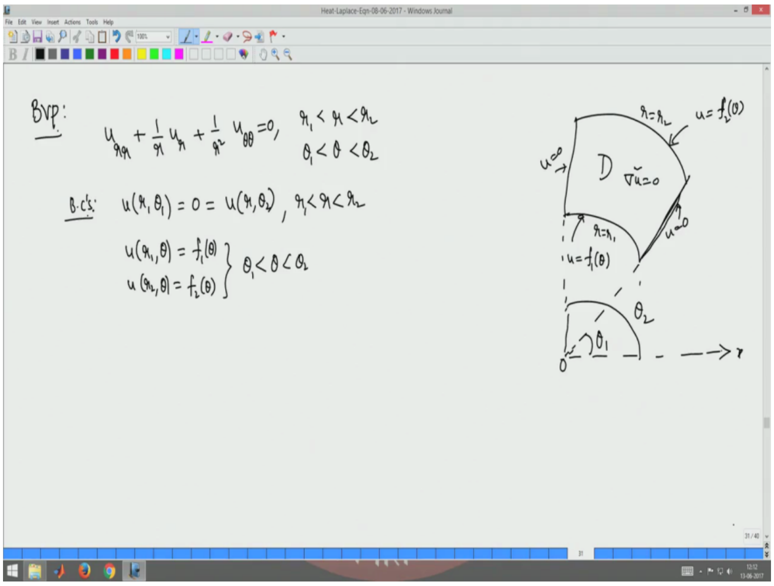

So this is your part of this annulus region, so you have zero here. So if you say zero, this is zero and this is your R1, R equal to R1 is this curve when you have this is, this is x-axis and you have this makes Theta as Theta 1 and this makes Theta 2, this line, okay. And this is R equal to R2. So on this boundary you have U is 0 and on this boundary also U equal to 0, okay. And only here, you have to provide this data, so U is some function of Theta, so that is F1 of Theta and here U is, you can give, give some data F2. So inside this domain, this is your d in which you have, this satisfies my Laplace equation.

So let us write this U X X, U Ah, so this Laplace equation in polar coordinate, it is a curved domain, so circular domain, so we can have Laplace equation in polar coordinates URR +1 by R UR +1 by R square U Theta Theta equal to 0. So the domain of the differential equation is partial differential equation is R1 and R2 and Theta is between Theta 1 and Theta 2. So this is a partial differential equation, this is a Laplace equation, this is domain. Now you have to provide the data, so you can give the boundary data, you have U at Theta 1 R Theta 1 equal to 0 and also U at R Theta 2 is also zero.

So this is the boundary condition, boundary conditions and you can also give other boundary conditions like U at R1 Theta, this is F1 of Theta and U at R2 Theta, that is F2 of Theta. So here Theta is between Theta 1 and Theta 2. So here R is between R1 and R2. So this is your bound data, you want to solve this problem, okay. So this is a boundary value problem for the Laplace equation. So how do we solve.

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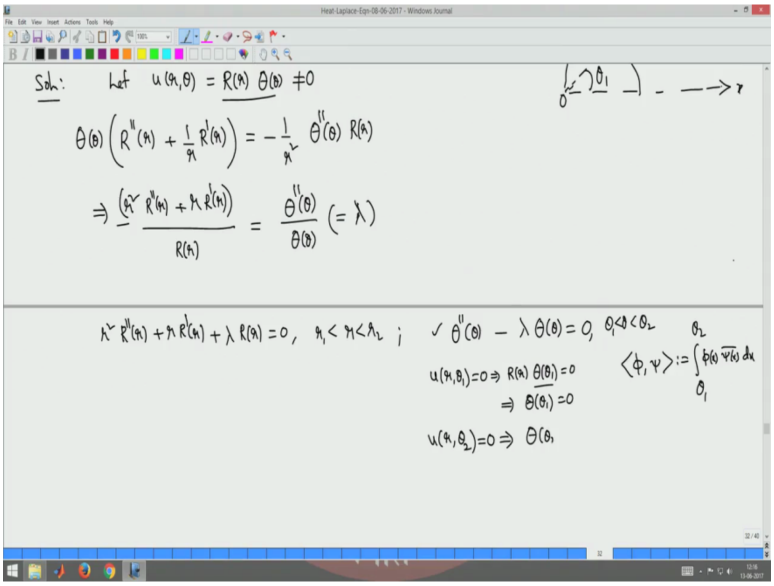

So solution procedure which you know, we start with separable form of solution, that U of R Theta, as you look for R of r, capital Theta of Theta as a nonzero solution. If you look for, if you substitute, what you get is R double dash of r + 1 by R – of r, okay with Theta of Theta that comes out as Theta of Theta as a, if you substitute into these 2 terms, that is equal to minus1 by r square Theta double dash of Theta into R of r. So we substitute into the remaining part and take it the right-hand side. So this is what you have, so this implies, now you divide with both sides with R of, U of R Theta, that is R of r, Theta of Theta.

Theta of Theta goes, so what you are left with is R double dash of r + 1 by R R – of r divided by R of r, bring this minus here and this you write it as, now the right-hand side becomes Theta of Theta and you can bring this, you can write this one by R square, one by R square times Theta of Theta. Okay. Now we can bring this R to the other side, so you have R square here, and this becomes R. So this is what is this. Now left-hand side is function of R, right inside is for sure of Theta, so it should be constant. Okay. Unless 2 independent variables are same as a function, okay, unless there, so those expressions should be constant.

So this is what you have, so this is what you get. So the Laplace equation becomes, the 2 ordinary differential equations, one is R square R double dash of r plus r times R double dash of r plus lambda times R of r. So this is Euler Cauchy equation, this is between r is r1 and r2. This is one and for Theta, so Theta double dash of Theta minus lambda Theta of Theta equal to 0. This is in the self adjoint form, you can easily see as 1, P is one, Q is zero equal to lambda times W is one, so your dot product you can define the dot product as, what is the domain, Theta domain is Theta 1 to Theta 2, okay.

So Theta is between Theta 1 and Theta 2. So for that reason the domain if this and you have phi X sigh X bar the X, okay. And this is the dot product you define. So, now you apply this boundary data, so boundary conditions are U at r Theta 1 equal to 0 which gives me R of r into Theta and Theta 1 is equal to 0. So that means this constant should be zero, R of r can as a function cannot be zero, so this implies Theta of Theta 1 is zero. So this is a boundary condition for this ordinary differential equation. Similarly you get R2 Theta, Theta Theta 2, so other boundary condition, R Theta 2, so homogeneous boundary condition you will get Theta of Theta 2 will be zero.

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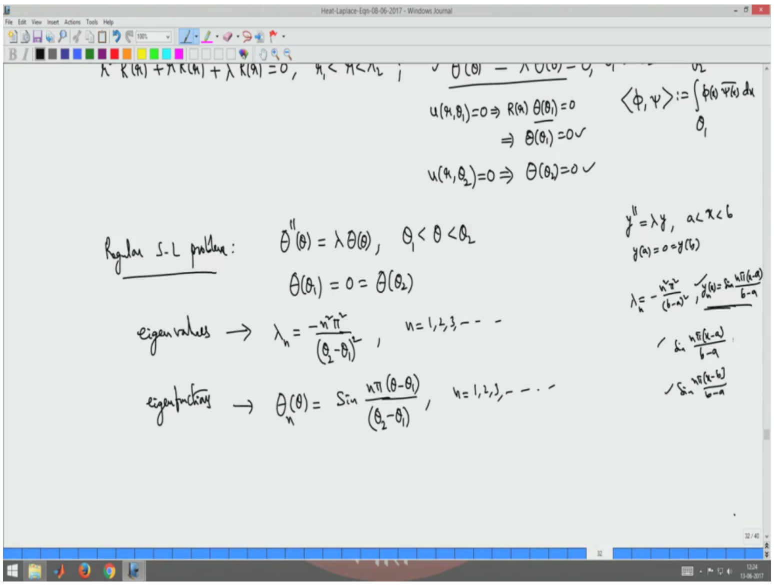

So these are the 2 boundary conditions for this, so what you abstracted is a regular Sturm Louisville problem, so this is the ode with this boundary condition, this is regular Sturm Louisville problems. So you can find the solutions, okay. So let me write, regular, regular Sturm Louisville problem, problem is for Theta, so Theta, double dash of Theta equal to lambda, Theta of Theta, this is Theta is between Theta 1 to Theta 2. And what you get is Theta of Theta 1 equal to 0 equal to Theta of Theta 2. So what are the eigenvalues and Eigen functions?

Eigenvalues and eigen functions. So what you get, so what are the eigenvalues and eigen functions, if you actually see earlier, we have actually solved this Sturm Louisville problem and from which you can see that the eigenvalues are, eigenvalues you can see that lambda is, I will just give you lambda is lambda equal to minus n square pie square, minus n square pie square divided by Theta 2 which is bigger than Theta 1, so Theta 1. So this is what you have as square, okay. So so these are your Eigen values and Eigen functions are, so what you get as Eigen functions, these are your eigenvalues, n is running from one, 2, 3, onwards and Eigen functions, what we see is, this depends on n.

So I am labelling this eigenvalues as lambda M and if you call this Eigen functions as Theta ns, Theta n of Theta as sin n pie by , this is n pie, Theta minus Theta 1 divided by Theta minus Theta 1. Theta 1 is small, so divided by Theta 2 minus Theta 1. So these are your Eigen functions. Okay. So we just look back, so V have solved this for Y double dash equal to lambda y, lambda Y. Theta is, X is now between, why is a function of X between a to B and you have Y at n equal to 0 at Y equal to B. For this, what we found is lambda equal to minus n square pie square by B minus a the whole square as your eigenvalues.

And Eigen functions are Yn of X. These are simply sin n pie X minus a divided by B minus a. Or also, sin n pie X minus B by B minus A, that is also possible but that is simply constant multiple of this, so we can take this as Eigen, dependent. If you take X minus B here, sin M pie into X minus B by B minus a, if you take that function, that is all simply constant multiple of this function. So in that sense both are, if you consider, what I am saying is n pie X minus A by B minus A and sin n pie, X minus B divided by B minus a, both these functions are Eigen functions but they are constant by (())(14:01). Okay.

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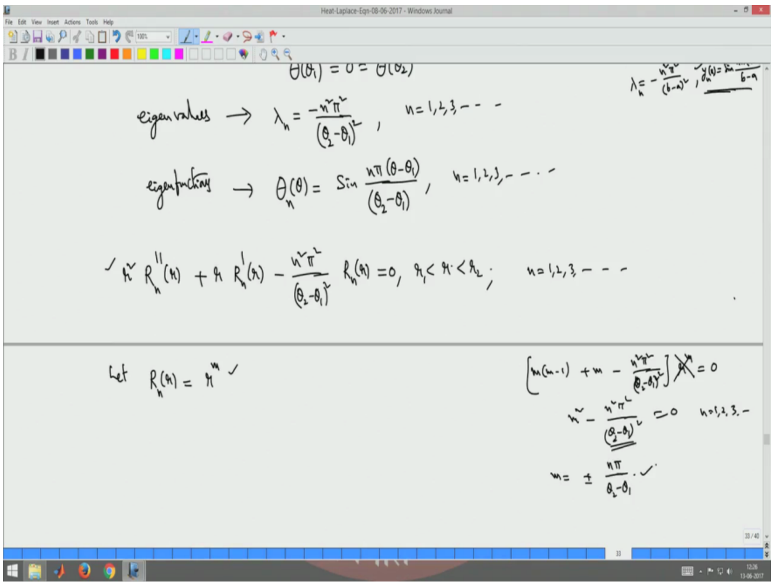

They are actually dependent, so you can consider anyone of them, so I choose this one. So in that sense if you do it here, that is what is written here. So these are eigenvalues and Eigen functions for Sturm Louisville problem. So for n is running from 1, 2, 3 onwards. Now you look at this Euler Cauchy equation, in other words for this lambda n, so if you write that R square, capital R, for each n Labelling as Rn, R double dash of r Plus R times R – of r plus lambda is now minus n square, minus n square pie square by Theta 2 minus Theta 1 whole square into Rn of r. That is what is equal to 0, Theta R is between R1 in R2.

So this is Euler Cauchy equation, so for each n, n is running from, n is running from 1, 2, 3, onwards, you have these many numbers of ordinary differential equation. So you have Rn of R, we know, if you can look for solutions of the form R power n, okay. If you look for r power n is the solution, then what you see is, you get M into m minus1 plus plus n minus n square P square by B, Theta, Theta 2 minus Theta 1 whole square. Okay. And this into r power n equal to 0. So if you simply substitute this into this equation, what you get is this, so you looking for nonzero solution and so that means M square minus n square pie square by Theta 2 minus Theta 1 whole square equal to 0.

So that gives me, since n is running from 1, 2, 3 onwards, this cannot be nonzero, so you have 2 different limits, so plus or minus n pie by Theta 2 minus Theta 1, the 2 roots, n is running from 1, 2, 3, onwards. Because zero is not involved, so you do not have repeated roots. So the solutions of, so I can write directly, so we have 2 repeated roots, so the general solution is, so the general solution is, I can write the general solution now, if you know 2 linearly independent solutions, one is positive, another one is a negative route. So we have An is the constant and R power n.

So R power n pie by Theta 2 minus Theta 1, this is one. And then plus Bn R power minus n pie by Theta 2 minus Theta 1. So this is, these are, this is the general solution of this equation, okay. So for this, this is true for every n. So what you get is the solution now I know R and of Rn Theta n of Theta. So you can make a product and write your solution.

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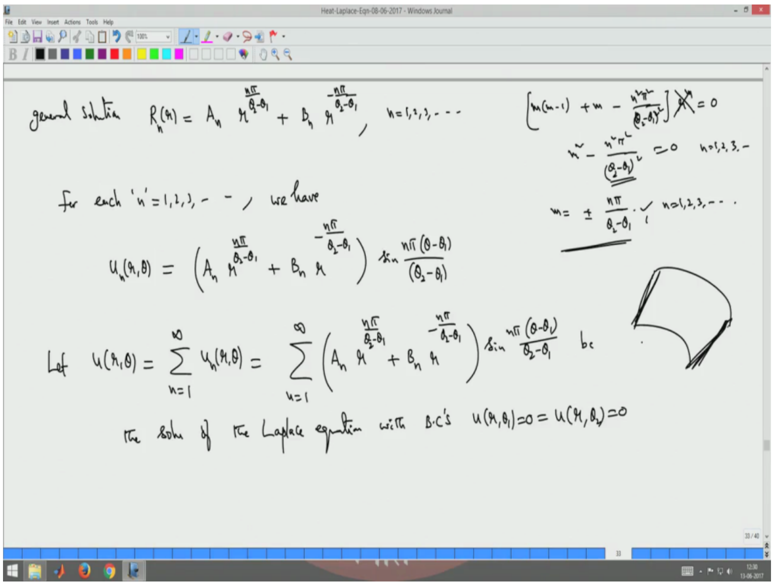

So I write the solution, Un of R Theta for each n, for each n, which is running from one, 2, 3, onwards, we have, we have the solutions. So U 1 of Theta is R n of r An times r power n pie by Theta 2 minus Theta 1, to the power, okay, so B n times that e power minus n pie by Theta 2 minus Theta 1. This into and what is Theta n of Theta, so that is sin n pie, you can write this sin n pie Theta minus Theta 1 divided by Theta 2 minus Theta 1. So these are, this is a function of R and Theta, this is not exponential, this is r power, r power that, okay.

So for each n, this is the solution that satisfies Laplace equation inside the domain and the 2 boundary conditions, okay. So this, these boundary, at these boundary, these are the boundary conditions, those are boundary conditions 1 and 2, okay, what we have written is boundary condition 1. For each n, I Un satisfied this equation and the boundary conditions here. So still it is not satisfying this thing, the other 2 boundary conditions. So you superimpose that U of XT, U of R Theta be the superposition of all these solutions, n is from 1 to infinity, Un of R, Theta. That is I can make the superposition, here is to take a constant here, that constant into this constant will be another arbitrary constant, so well, so that is why I did not write here.

So n is running from 1 to infinity, so you have arbitrary constant is An r power n pie by Theta 2 minus Theta 1+ B and r power minus and pie by Theta 2 minus Theta 1. This into Cn n pie Theta minus Theta 1 divided by Theta 2 minus Theta 1. Since this is what is the, let this be the solution, the general solution, okay. Solution of Laplace equation with boundary conditions, boundary conditions one and 2, okay. Boundary conditions like U at r, Theta 1 equal to 0 also U at r Theta 2 equal to 0. So satisfying these 2 boundary conditions, okay. These are things, so that satisfies this.

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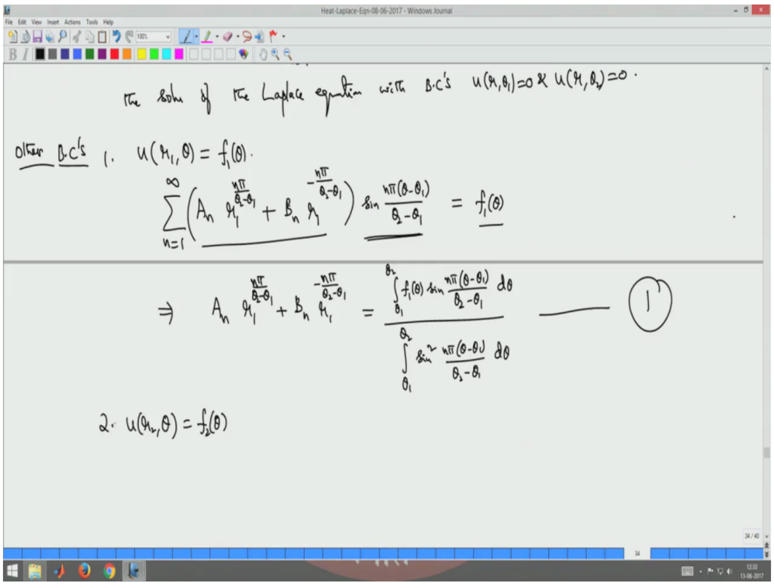

Now you want to, you also need to satisfy other boundary conditions, apply the other boundary conditions. Boundary conditions, other boundary conditions, the boundary conditions, what are those coach and U at r1 Theta equal to U at r1 is F1 of Theta, so you apply that. So if you apply, what you get is the, you apply that, so U at r1 Theta is, you have this summation, n is from 1 to infinity An, what happens, r power, your to replace, now instead of r, you have r1 power n pie by Theta 2 minus Theta 1, this is simply a constant plus Bn r power n pie by Theta 2 minus Theta 1. So this is also r1, substitute in place of r r1, Theta.

Now this sin n pie, now you are putting Theta as it is, so you have this, these are Eigen functions, Theta minus Theta 1 divided by Theta 2 minus Theta 1 equal to a nerve Theta. So now these are your Eigen functions, so you can make a dot product both sides with these Eigen functions, so that you can get An times this constant, this whole constant r1 power n pie by Theta 2 minus Theta 1 + Bn r1 power minus n pie by Theta 2 minus Theta 1. This whole arbitrary constant I can get it as integral, now Theta is between Theta 1 to Theta 2, that is a dot product, when I make a dot product f1, with this, with this Eigen functions.

So what you get is f1 Theta into this bar of this function, so that is, these are all real valued functions, so bar does not matter. So n pie Theta minus Theta 1 by Theta 2 minus Theta 1 d Theta, that is the right-hand side. The left-hand side what is the main is only the integral Theta 1, Theta 2, so itself. So that is sin square, sin square of n pie Theta minus Theta 1 divided by Theta 2 minus Theta 1 d Theta. Okay. So this actually we can actually calculate, so that is half of, so you will evaluate this, okay, I will not evaluate this now exactly.

So this is one equation, so let us call this one equation, okay. Now we apply the other boundary condition, U at r 2 Theta equal to f 2 of Theta, this is one and this is 2, so that is the remaining 2 boundary conditions. 1st we apply, so everything is same, so you get the same equation with r1 is replaced with r2, f1 is replaced with f2. So what you get is, you get by the same procedure you get A n times, in place of r1 you get r2, r 2 n pie by Theta 2 minus Theta 1+ Bn r 2 power minus n pie by Theta 2 minus Theta 1. This is equal to, on the right-hand side instead of f 1 you put f2, so Theta 1 to Theta 2, f2 of Theta sin n pie Theta minus Theta 1 by Theta 2 minus Theta 1 d Theta, this is one integral.

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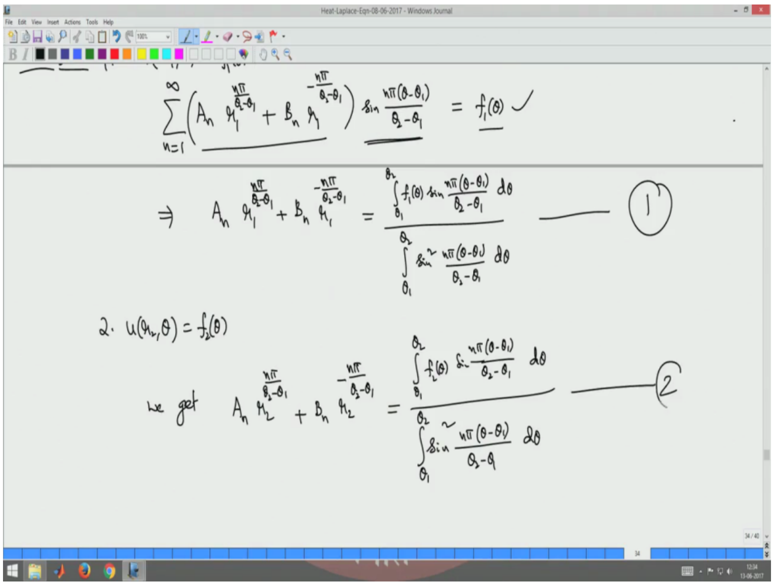

Divided by other integral, there is a constant, Theta 1 to Theta 2, so sin square n pie Theta minus Theta 1 divided by Theta 2 minus Theta 1, this is for the sin, sin of this sin square d Theta. This is equation number 2. Okay. From one and 2, so you can easily see that, so what you require in this solution, only Ans and Bns are known, okay. So if we can get Ans and Bns from this, these are all knowns, An, Bn, An, Bn with constants, these are constants. So this right-hand side, whole thing is one constant, this is another constant. So you are unable to linear equations 1 and 2 which for the unknowns Ans.

So solve 1 and 2 to get An and Bn, that is it. So once you get An, Bn, you substitute your, so this is the general required solution. So what I am putting as a, in this box is the general solution, this is the solution, this is the required solution, okay. So get this, once you get these An, Bn and substitute here, that will be the general solution of, that will be the solution of the boundary value problem that we have considered, okay. So that is how you solve this, this is the (())(24:45) something exactly similar to what we have done for the full semi-annulus, full annulus region that I have solved in the last class video. Okay.

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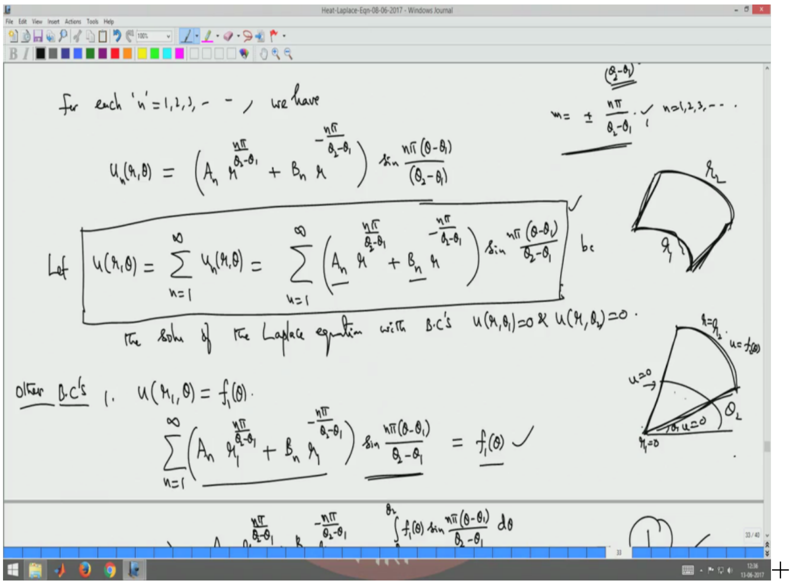

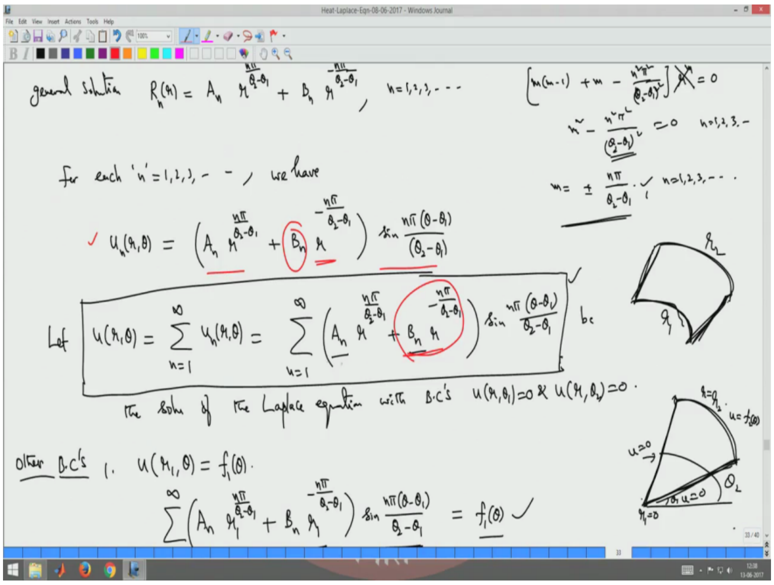

So once you know the solution here like this, now I can make, I can make many things. So this is the r1 and this is r 2, okay. So now the next problem, so what are the problems you can expect out of this is consider, suppose you have a, suppose you have 1st and foremost is this one. So you try to make r1 equal to 0, if you do that, r2 and this becomes effective, simply factor. So you have r1 is 0 and r2 is, R equal to r2 is something else, okay. R equal to 0 here and this is 1. So here you also need to, this is a boundary and this is the boundary in here. You provide here, so U equal to f2 of Theta and U equal to 0 here and U equal to 0 here. So U at R of r, Theta 1, this Theta 1, this is Theta 2.

So you have Theta 1, Theta 2, you have homogeneous boundary conditions. Then the solution is what, then these Eigen functions are all same, Eigen functions are fine, so they all same, they are all same, no change here, okay. Because Theta1, so ordinary differential equation for Theta is same, and the boundary conditions, these 2 boundary conditions if you fly, you get the same Sturm Louisville problem and you get the eigenvalues and Eigen functions. Only thing you have to see is this equation becomes 0 to, I will write it in red, so 0 to R is between 0 to R2, so this is what you will see.

And because of this, but you see if it is a plate or sector kind of plate and you maintain these temperatures here and in this curved boundary, at 0 it cannot be infinity. So if R is 0, R equal to 0, you cannot have an infinite solution, okay. So when you write this, when you find the solutions, when you see that this is a solution, this is a temperature for each n, and 2, because it has to be bounded at R equal to 0. At R equal to 0, this is going to be infinity, so this, that means Bns are 0, Bns will become 0 and implies, so you superimpose all the solutions. So finally you get the solution to Un is only this one and this one, okay. So An times r power n pie by Theta 2 minus Theta 1 into sin, this Eigen function.

So you superimpose all of them without these Bns, okay. These Bns the avoid and what you get is only one sum, An times of r1 into r power of this into sin of this Eigen function. So you apply only the remaining boundary, boundary is only now this one. So this boundary if you apply, that is U at r2 of Theta equal to f2 of Theta. So without this, so you get the 2nd equation. So 2nd equation without this implies you can find your An. This is a constant, An is simply, this you can divide, so that is what is your An. So once you know Ans, you know everything here, okay, that will be the solution, solution of this sector power. Still this is general sector.

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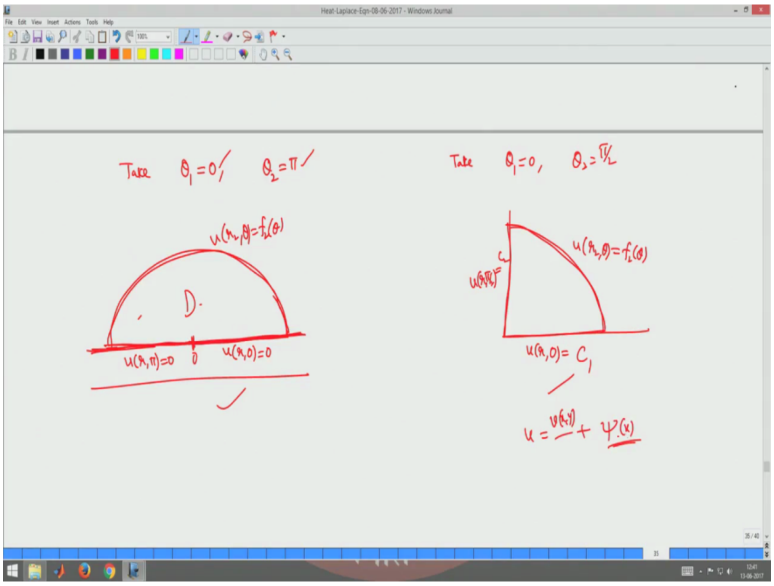

Now the 3rd problem you can look at this, you can also consider, you can now simply take Theta equal, Theta 1 equal to 0 and Theta 2 equal to pie. What is this region, this sector becomes, this sector, full sector becomes now, this is simply, this is like a semicircular region. This is your domain d, you provide U at r2, so let us say that r2, Theta is f2 of Theta on this circle and now this is 0 and you have Theta. So you have Theta, this is exactly, you have r, Theta 1, that is 0, so that is 0 equal to 0. So here and here you provide 0 boundary conditions, U at r, pie equal to 0.

So the temperature you maintain here as a 0, that is I am seeing as 2 boundary conditions for Theta equal to 0 and Theta equal to pie. Okay. Between, r is between 0 to r2. So if you do the same thing procedure is same, only thing, so here you can directly get An as this, only difference is Theta 1, Theta 2 you take it as 0 to pie. If you take 0 to pie, only change in this reduced solution, okay. For this problem, for this problem you take simply Theta 1, Theta 2, you change, you will get, you will solve this problem, okay. And now if you take, if you take Theta 1 equal to 0 and Theta 2 equal to pie by 2, so you, what you can solve this in the quarter playing, it is a factor in the quarter plane.

So that is also you can solve, so that is also you can solve, so this is how you can solve these kinds of curves curved linear, curved domains. Here Theta U at R, Theta 1, that Theta 1 is 0, so 0, and you have to provide 0 conditioner, so that is R pie by 2 equal to 0. If you are given this problem, so the result is exactly the same, same problem you can solve. So Ans, only thing is Ans will contribute, so the solution will be this without this term. So you have Ans,

Ans are actually calculated from this, from the 2nd equation, okay. Only thing is Theta 1, Theta 2 you replace with 0 to pie or the 2 pie by 2, to solve these 2 different problems. Okay.

So like that only in certain domains you can solve this problem is, only curve in this polar coordinates, only some circular domains you can solve, okay. So that means some circular domains means, this is, in this suppose you are given different things, okay. Suppose this is nonzero, some A, let us say some C1, I maintain some nonzero temperature C2. What you have to do is, you try to use some transformation, something like, so you extract, you basically write U as V plus, U1 plus U2 or V plus some psi, psi is a function of X alone. And this is, function of X, Y.

If you do this, you may, so you may be able to find this, if you can find such a V that satisfies these non-homogeneous boundary conditions so that V satisfies homogeneous boundary conditions here and this may be little different, little different but this is something else, okay. Like that, the idea is, if you have a nonzero condition, if we can reduce by some transformation, by some way to the problem with a homogeneous conditions, that is 0, 0. Then you know its solution, so that you can solve even for the non-homogeneous conditions, okay, non-homogeneous boundary conditions.

So this way, this is a variable, separation of variables method which you can apply to solve certain Laplace equation in certain domains either it is rectangular domain. We have seen that it is a finite rectangle domain, so it is the rectangle domain or semi-strip domain, okay. So I have seen the semi-strip, semi-infinite strip domain. And also for the curved domain, circle inside the circle, exterior of the circle, annulus region also here in this sector, in the sector or in the semicircular domain or sector in the quarter plane, okay. So like this you can solve Laplace equation.

That is actually a steady-state heat equation, once you reach that, once you, once the plate, okay the two-dimensional plate, once it reaches the steady-state, okay, so by keeping the temperature at the boundary, fixing the temperatures at the boundary so you can solve this problem. So what else we can, so far for this Laplace equation we have only provided for this curved, curved domains, what we have provided, only temperature at the boundary, we will provide the dirichlet condition. We can also provide the normal data, so what is a normal data here? You can provide UX, it is something like U Theta, okay.

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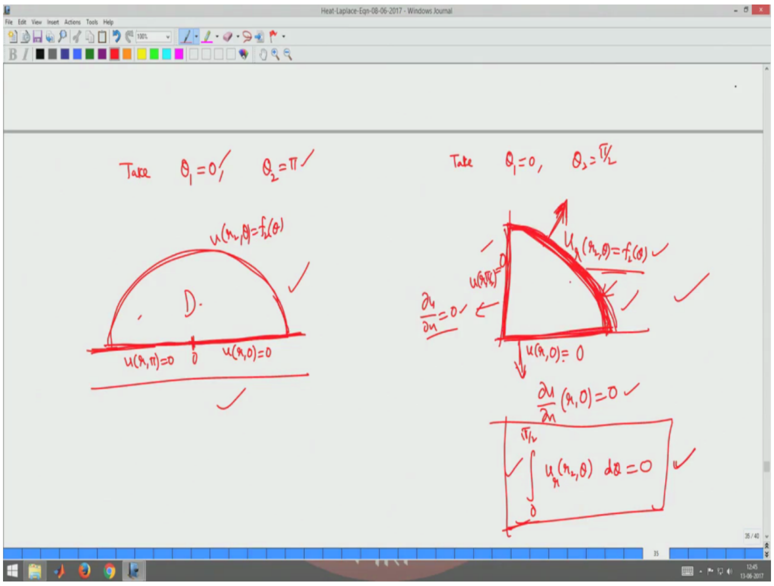

U Theta is your normal, so here, so, suppose this is the boundary and this is the boundary and this is the boundary, so whatever you extracted, so Sturm Louisville problem when you extracted, what you have is U U provided. Here you can provide doe U by doe n equal to 0 at r, Theta equal to 0 is 0. So what is that thing, that normal thing you have to find out, okay. So what is the normal here, this normal in polar coordinates you have to provide that has a data here. Okay similarly you can provide this doe U by doe n here 0, okay. So that means you insulate these 2 ends, still you can extract Sturm Louisville problem because these are homogeneous boundary conditions. Okay.

And only thing is that you have to have, finally, you can do this, here also suppose you want to do this, this normal is only R, so you have U r, clearly I can see that UR, this is a normal, normal is this R, right. This normal vector is R is simply, so doe U by doe R is the normal here for this circular domain. So if you give this one together, so it should have, from UR of r, Theta, r2, Theta, Theta is from 0 to pie by 2 has to be 0, this has to be 0. This is a necessary condition for you to have, so you insulated, so you have not insulated here, so the insulation here, you are giving some external source.

So whatever comes in should go out, if you are providing heat inside some part of it, some part of it should be able to go out so that eventually that effectively net flux through this surface, through this boundary, the net flux of the temperature should be 0. Okay. So anyway here this is 0, flux is 0, insulated, inserted here, the only boundary through which you can either allow the heat to inside or to go out. So effectively that flux, heat flux should be 0.

So that is a necessary condition, so you can provide these insulating boundary conditions, but the only thing is you have to maintain 0 flux 0 net flux of the heat, so that you can solve these boundary value problems, okay, otherwise you cannot. So this is how we, we have solve this Laplace equation in certain domains, okay. So this is how you solve Laplace equation in curved domains, thank you very much.