Differential Equations for Engineers

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Lecture – 60

Laplace equation over a Rectangle with flux boundary conditions

So in the last video I have seen how to solve laplace equation in a rectangular domain with boundary conditions being dirichlet conditions that means you provide as a model if you look at the plate problem so heated plate initially at some temperature and if you have a boundary of the plate is kept at certain temperature so what is steady state temperature in the plate can be calculated by the as a solution of this boundary value problem so that is what we have seen.

Today we try to see the same thing steady state solution we will try to look at the solutions for the laplace equation with boundary conditions being Neumann type that means you give flux conditions okay normal derivative is provided instead of function itself instead of the data is being the function is normal derivative is provided.

So physically for the plate if initially at some temperature sorry, once for the heated plate once it reaches the steady state and if the boundary conditions, if the boundary of the plates are insulated then you say that it is a well posed problem, okay so you can find the steady state distribution of the temperature inside the plate you can find out so by just by solving this problem, okay.

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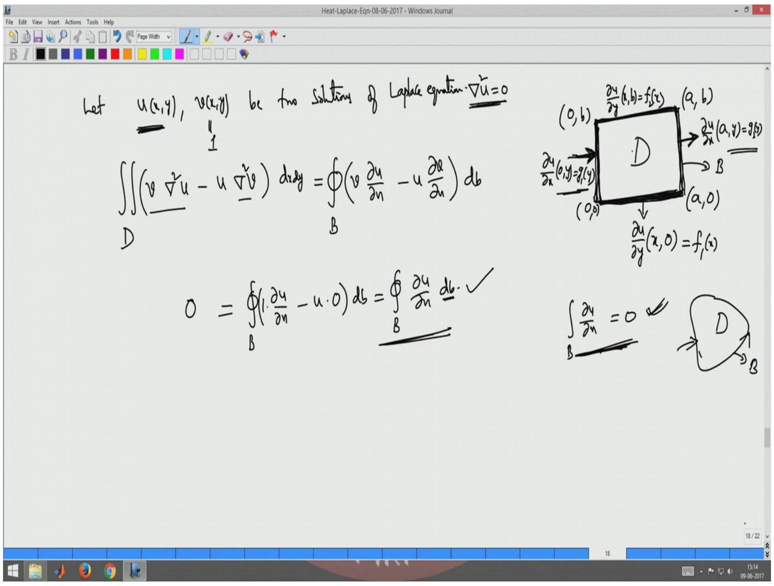

So to do this it should be well defined problem that means suppose you consider a plate, you consider again a plate so what we are providing is here, this is the normal derivative so this is, the data is, this is 0 0, 0 a and 0 b and a to b so this is the plate you have and here the normal derivative is dou u by dou y, okay dou u by dou y at x, so this is a 0 at x and y 0, okay.

So this is here, normal derivative is, so this is what is provided as f1 of x so here the normal derivative is dou u by dou x at x is a y equal to g2 of y, here dou u by dou y at x b is f2 of x and here dou u by dou x at 0 y is g1 of y so this is how you provide data instead of as a solution.

Suppose f1 f2 and g1 g2 are 0 that means for the plate is being insulated both ends all along the boundary the plate is insulated that means the flux is neither going neither so it is basically is insulated no heat is going out of this surface okay, out of this boundary.

So if you really want a steady state of heated rod, okay. So what you need is you should not have sources, so no heat should go out so effectively, effective heat should not be so net heat either going outside or coming inside should be 0, okay. So no exchange between, no exchange out through this boundary heat should not go out or enter this plate in that case you can find, you can reach the steady state otherwise suppose you say some heat is constantly coming inside so net flux is non-zero.

So that means flux let us say which is coming inside it will never reach the steady state so if you want steady state whatever comes in should go out okay then it can be at some steady state, for example g1 is the entering through this surface you want g2 be minus g1 y so that so the integral value so integrals allow here and here it should be compensated now same way, so physically that is what is the meaning.

So in a plate these two can be insulated these two boundaries but these two can be non-zero but you cannot give any data here so any flux data, only thing whatever comes in should go out then only this steady state will be maintain so if you want this g1 to be here, you need g2 to be minus g1 of y okay that way you can obtain the steady state and get the solution for this problem.

So why we make insulated these ends f1 f2 or 0 because if you want to solve so you can have this you can also have f1 and f2 as non-zero solution, non-zero flux here but again here also f2 has to be minus f1 or f1 has to be minus f2 okay flux should be whatever comes in should go out so something like that okay, heat whatever it is entering if it is positive it is entering whatever so okay so it is whatever goes through this (())(05:30) through these two boundary should be 0.

Similarly these two should be 0 physically that is what is the meaning actually throughout if you would say throughout as 4 boundaries if you say all this four boundaries integral if you call this as a boundary, integral over b whatever your u, not u so dou u by dou n this is a normal derivative should be 0 so this one can show if you want to have a solution it is necessary that this has to be zero okay.

So we will first show that this necessary conditions is always true okay physically that is what you can easily see you want a plate to be to have a steady state no heat should enter or heat should not go out of this okay, effective heat should not go out that means whatever if at all there is a source of heat coming from outside it should be, it should go out by some other means, some other boundary but if it comes through this boundary should go out that boundary okay so we will see how we will prove this one so for this let us consider let u of x y and v of x y be two solutions of laplace equation.

Laplace equation is Uxx plus Uyy is 0 okay, then you can write the you know the Green's formula from the calculus so this is your D if this is your D and boundary is B so I am not saying I am not taken actually the plate general domain D this you call it D the boundary is B so if you take this over this you take this v and then you have laplace equation you can also write gradient square of u equal to 0 okay.

So gradient square u minus u gradient square v this is what is a Green's function so this is dx dy this is a double integral because this is a plane area, this is equal to now you have a line integral that should be a closed curve over B v dou u by dou n minus u dou v by dou n oriented curve okay some kind of orientation chosen.

You have a curve orientation always considered your domain is the D is always in the left hand side then this is true otherwise you may have to take with minus sign so let B be the solutions we can see that v if I take it as 1 constant 1, if v is 1 it satisfies the laplace equation derivatives are 0 so v is 0 let u be the solution as usual okay.

So if you put v equal to 1 what you get is so del square u since u is satisfies this equation this is 0 and v is 1 so del square v is 1 is 0 so the left hand side is 0, right hand side is so over B v is 1 dou u by dou n 1 here minus u into dou u by dou n the derivative of constant term is 0.

This is DB let us say okay this is like a line integral okay so this our DB is 0 so this is nothing but (())(09:04) over B dou u by dou n into db this is B okay b is the parameter of this curve so if you want to have a solution for this problem with the Neumann data on it so the Neumann has to be, the Neumann all along the boundary if you are given a Neumann data that is the flux should be 0, effective flux should be 0 that is what it says.

Now we will consider this triangular plate problem okay what we consider is physically feasible 1. Ones are both sides two sides we insulate it and one side also three sides we insulate other side we allow it to be we allow it to have something okay, allow it to have some flux, but that flux has to be but that flux in such a way that the integral has to be 0 okay.

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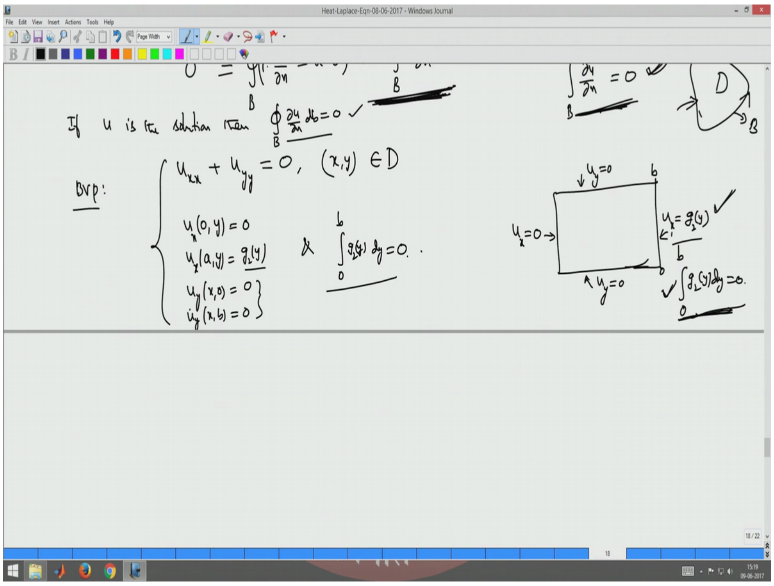

So we will see, we will pose that problem so the laplace, let boundary value problem definition goes like this Uxx plus Uyy equal to 0, x y belongs to the rectangular domain D you have this okay, now what you have so I make uy, uy is 0 here and I have a ux equal to some function of y, ux at let us say g2 y and here u2 ux is 0 here so let us make like this problem okay.

Now like this you by changing so you make this three 0 and here you keep one okay, you can also like earlier you can break this problem into 4 problems okay only thing is f1 and f2 g1 and gy or this rectangle should be, that is underlying assumption this is a necessary condition okay so if u is the solution then necessarily you should have this boundary dou u by dou n d b has to be 0.

So this is the necessary condition that means whatever the data f1 f2 g1 g2 that integral has to be 0 okay so with those f1 g's for which this integral is 0 if you are given then if you are, they are all non-zero then you break them into 4 problems like this by three of them you take it as 0 you call this u1 okay here this if I call u1 problem now u2 will be like earlier so remaining you take this 0 and other so keep 1 as non-zero okay like that you make u1 u2 u3 u4 problems and then combine it.

Once you solve one of them remaining three are similar so then you combine them to get the solution for this so you do one so that is why we choose only one so one such thing where ux is this boundary you have this j2 but it is not, but you cannot actually break it okay you can only break, see if you want to solve this problem so this has to be 0 to b, y is between 0 to b, g2 of y dy has to be zero, from the necessary condition because other places anyways 0, so this has to be true so if you are given f1 f2 g1 gy such that each of them integral 0 for each f1 g1 f2 g2 let us see each of that integral say 0 to a f1 of x dx is 0, 0 to b g2 of y dy is 0, 0 to a f2 of x dx is 0 g1 of y 0 to b dy 0.

So if each of this are true then I can break this into 4 problems okay because necessary condition is anyways satisfied even after breaking the necessary condition still satisfied okay let us anyway choose this problem if you have otherwise also you need two ends you need the insulation that is what is required okay.

So you can see that one so otherwise you will not be able to solve, you will not be able to break it into few problems even though it is a linear problem you cannot do so after breaking each problem should satisfy the necessary condition okay so such a way you have to break it but once you break it you should have, two opposite and should have zero boundary conditions.

So zero boundary conditions if you have then only you can extract the Sturm-Liouville problem you can solve by separation of variables that is idea okay so we will just take boundary condition like this, this is laplace equation boundary conditions are ux at 0 y is 0 ux at a y equal to g2 y, u y at x 0 is 0, uy at x b is also 0, so I have two opposite zero boundary condition so this problem will solve okay.

So g2 in such that integral 0 to b g2 of x dx no actually y okay this anyway dummy variable you can choose anything you want so dy equal to 0 such that, g2 is such that so this has to be true then only you have the steady state you can find. So you will have a solution that is physically feasible only if this is true.

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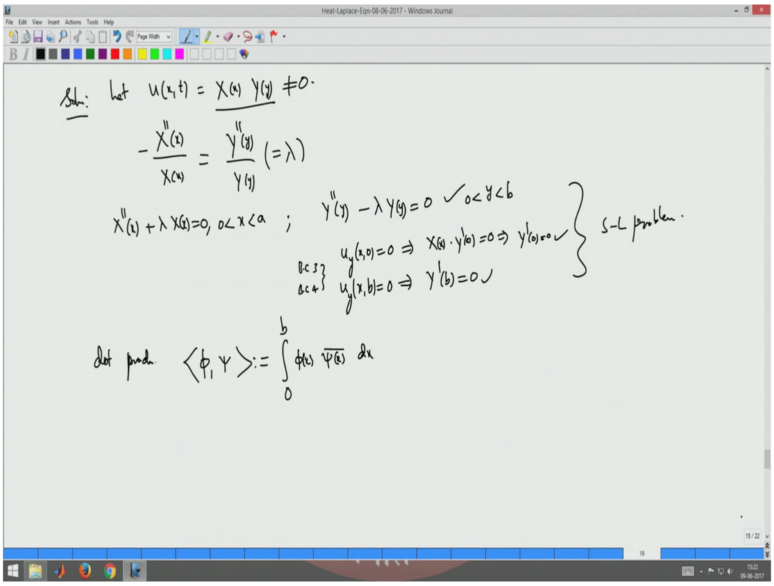

So the solution we can now work out by separation of variables that is the standard technique so you have, let solution be X of x, Y of y which is non-zero you substitute into the equation you break it so that you have X by x by X of x like earlier okay and divide it okay you substitute this and you divide with this u.

So both sides you will get this one, this is equal to so you put minus bring it here so you have Y double dash of y by Y of y so this is the form both functions of x here, functions of y here this cannot be same unless they are constant so this is a constant so now you have this laplace equation becomes 2 ordinary differential equation like in the earlier video so you have this plus lambda X of x equal to 0, x is between 0 to a and you do not have boundary data boundary things or that is the first two boundary conditions okay.

So now for the other equation you have Y double dash of y minus lambda Y of y equal to 0 now the boundary condition 3 4 boundary condition 3, boundary condition 4 will give uy at x 0 equal to 0 will give me what you get uy is X x into Y dash of y , y at 0 equal to 0 so this implies Y dash of 0 is 0, similarly uy at x b right, uy at x b equal to 0 will give me Y dash of b equal to 0.

So you have the derivatives are 0 for this Sturm-Liouville problem, this is the Sturm-Liouville problem as usual again and you have this is already in the self adjoint form simpler form so you can define the dot product as phi psi because W is 1 because 0 to b, same 0 to b domain is 0 to b, y is between 0 to b, phi x psi x bar dx so this is the definition for the dot product okay.

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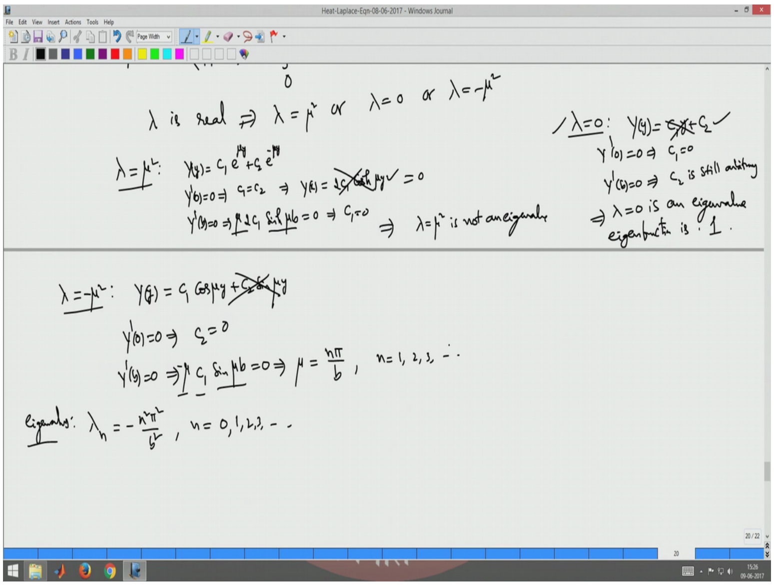

So once you have this find the so it is already in self adjoint form so you can write, you can see that lambda is real, so lambda is mew square or lambda equal to 0 or lambda is minus mew square so lambda is mew square cannot be an eigenvalue so we can see this lambda equal to mew square we will see this one so let us quickly we will just find out so lambda equal to mew square means Y so the solution of this ordinary differential equation is general solution is C1, e power mew y plus C2 e power minus mew y so if you apply Y dash of 0 is 0 will give me C1 equal to C2 so implies Y of x is C1 equal to C2 means so you get 2 C1 cos hyperbolic mew y.

Now you apply Y dash at b equal to 0 for this if you do this 2 C1 sine hyperbolic mew b, you have a mew comes out because of the derivative this is equal to 0 so sine hyperbolic mew b cannot be 0 because mew is positive b is positive 2 mew cannot be positive so implies C1 is 0 so it makes it this is also 0 so this implies lambda equal to mew square is not an eigenvalue.

But lambda equal to 0 is an eigenvalue, so this you can see by finding the general solution by putting lambda equal to 0 this becomes two derivatives of y0, the general solution is C1 y plus C2 so Y dash of 0 equal to 0 will give me C1 is 0, so this is gone and Y dash of b is equal to 0 satisfied by 0 implies C2, C2 is nothing actually satisfied okay when you do this C2 is still arbitrary implies you have a non-zero solution so implies lambda equal to 0 is an eigenvalue.

Eigenfunction is, corresponding eigenfunction is 1 okay call this 1 so this simply takes c2 as 1 constant that is 1 so you also look at lambda equal to minus mew square this part will give me general solution is, so general solution should be function of y okay Y of y so I cannot write this x, Y of y is C1 cos mew y plus C2 sine mew y, so Y dash at 0 equal to 0 will give me C2 is 0, okay you can see that and then so that means this is gone so Y dash be equal to 0 will give me C1 sine mew b, mew comes out okay because minus has to be 0 that is the derivative and put y equal to b.

So mew cannot be 0 if you want non-zero solution this has to be 0, so this is possible certain mew positive value so mew equals to n pi by b, mew is positive n is from 1, 2, 3 onwards okay now if you put n equal to 0 mew is 0 so that is lambda is 0 so the lambda is 0 is an eigenvalue so you can combine them so anyway you can write lambda n minus n square pi square by b square or eigenvalues, eigenvalues are this n is from, I can also include 0 so from here so 1, 2, 3 onwards from here okay.

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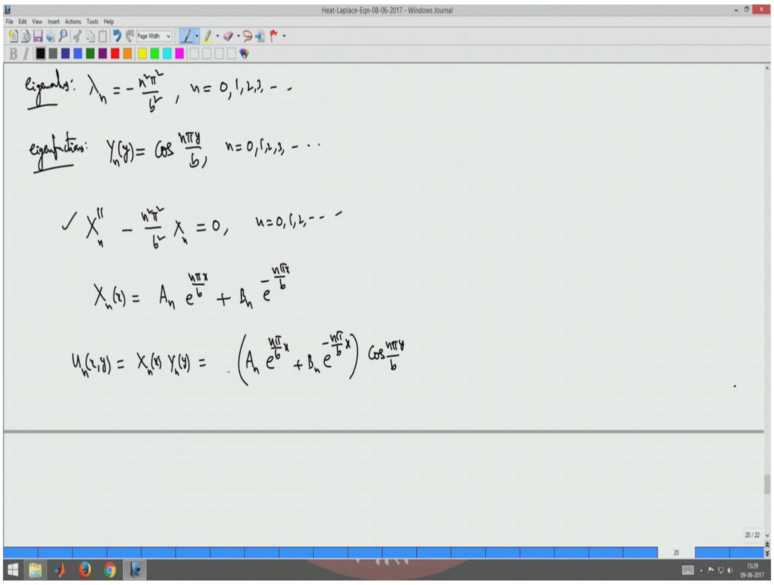

So that is lambda n square and eigenfunctions or Yn of y which is equal to, so what are the solutions what is left with this arbitrary C1 you can take it as 1 so you get cos mew is now n pi y by b, n pi by b into y so this is our for n is from 0, 1, 2, 3 onwards so I got eigenvalues and eigenfunctions now you can look at a problem for Xn okay so x just look at the x problem, x so you do not have to do the same ways so what is the now we look at the x problem because you have one zero boundary condition you can remove one arbitrary constant here itself so how do we do this X double dash of x plus lambda X of x you consider only X double dash of x plus lambda X of x.

Earlier we replace with this lambda with lambda n and then do it, you need not do this because you have a third boundary condition is zero boundary condition okay you can see that third boundary is rather first boundary we use the boundary three four, first boundary condition is 0 so that makes it one constant you can nullify so the general solution of this X of x is C1 e power i root lambda plus C2, this is tricky so let us use the same way what we follow okay.

So let us not do this one so given this lambda n so for each n you have lambda so X double dash plus lambda, lambda is minus n square pi square by b square x, so x is depending upon now I am replacing lambda by lambda n so this, label them as Xn, Xn double dash equal to 0 for each n is from 0, 1, 2, 3 onwards okay.

This case what is your Xn of x is simply A n e power n pi by b, n pi x by b plus B n e power minus n pi x by b so that is a general solution of this equation now what is your Un of x y which is X n of x into Y n of y this is simply A n e power n pi by b x plus B n e power minus n pi by b x that is my X n of x, Y n is cos n pi y by b.

So cos here y n I can also here I can include some other constant but that constant Cn if I write Cn into cos n pi y by b if I take C n A n will be one arbitrary constant C n B n will be another arbitrary constant anyway two arbitrary constant will be you can come, so product of two arbitrary constant is still one arbitrary constant only, so that way you still you can consider only one here so this is the solution or each n.

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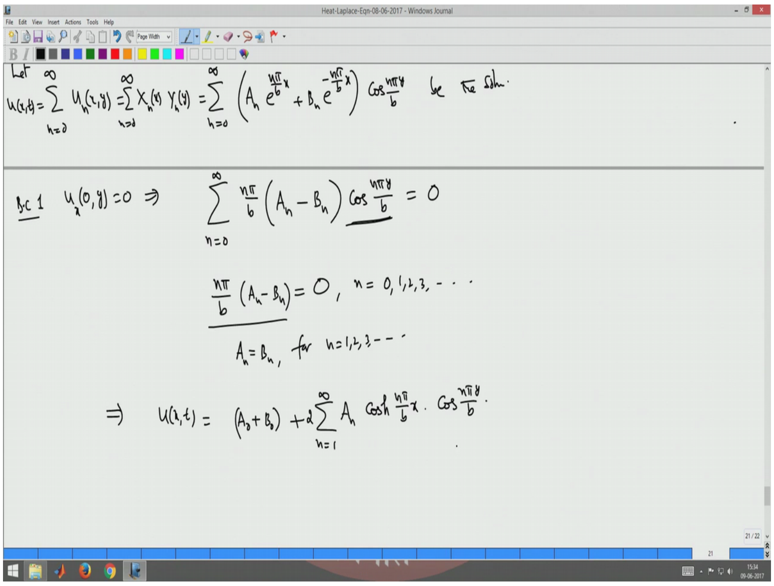

So now you can superpose the solution all these solutions as your solution u of x t, n is running from 0 to infinity okay, so if you do this let be the solution if this is the solution that satisfies, is already satisfying laplace equation and the boundary conditions 3 and 4 okay now we apply 1 and 2 to find this An and Nn.

So apply first ux at, let us see what is that ux 0 y is 0, ux 0 y is equal to 0 will give me this is the boundary condition 1, ux 0 y is 0 will give me so now you can both sides you can differentiate Uxx with respect to x you get n pi by b comes out you have A n e power and you are putting n equal to this n equal to so putting x equal to 0 after differentiation so it will be 1.

So you have only A n and this will be minus B n okay cos n pi y by b equal to zero now both sides you apply this eiginfunction take a dot product with those eigenfunctions to see that n pi by b times A n minus B n will be 0 because right hand side is 0 just multiply with the same cos n pi by b and integral from 0 to b so that the constant has to be 0 okay because that integral value is it is something it is b by 2 so you have, this has to be 0.

So this is true for every n 0, 1, 2, 3 onwards so what happens when n a equal to 0, n equal to 0 what you have is A n minus B n can be arbitrary A n minus B n so A0 minus B0 need not be 0 okay right, for n equal to 0 when you put n equal to 0 so this A0 B0, A0 minus B0 so you cannot do this so what is your solution, so this implies An equal to Bn for n is from 1, 2, 3 onwards because when you put n equal to 0 this can be zero, this is actually 0 right.

So you cannot say that this whole thing is 0 because n is here when you put n equal to 0 it is becoming 0 so A0 minus B0 you cannot have anything okay so only An equal to Bn you find, so implies what you get is u of x t is simply n equal to 0 you can write it separately so A n plus so A not plus B not so you call this some constant and you put n equal to zero plus this is sum n is from 1 to infinity, now An equal to Bn so what you get is 2 times A n because I am replacing B n as A n so you get cos hyperbolic n pi by b x into cos n pi y by b, okay, this is what you have, so this is what you have.

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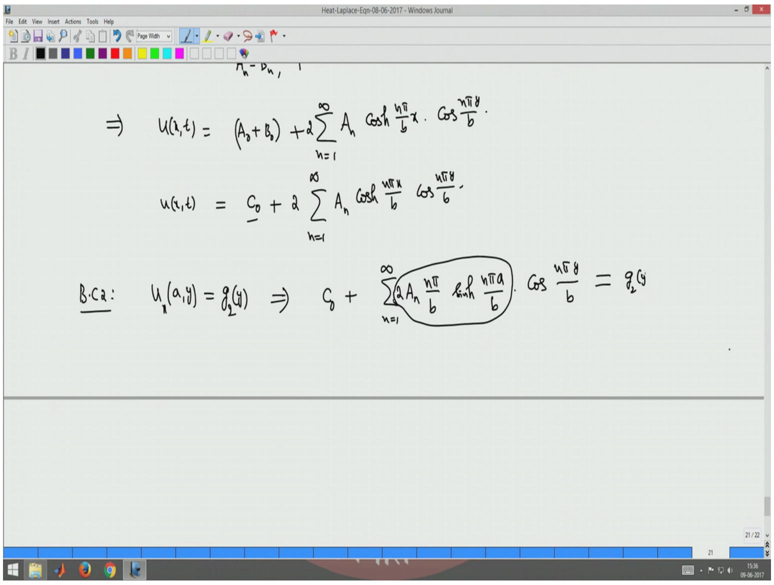

So what is this, now you have this is your arbitrary constant so now you have an equation now your solution becomes with boundary condition 1 and 3, 4 laplace equation the solution is so this is the solution that satisfies the laplace equation and the boundary condition 1 and 3, 4 so I am writing here so this you call it some constant C not plus this is A not B not is still arbitrary so A not B not so that you call this C not so and as a sum together it is 1 arbitrary constant so you have c not plus two times this n is from 1 to infinity An is an arbitrary constant cos hyperbolic n pi x by b into cos n pi y by b, okay.

Now you apply the boundary condition 2 that is ux at a y equal to f, f or g, ux at a y is g2 of y so g2 of y this is g2 of y so this will give me c0 plus 2 times c minus from 1 to infinity A n cos hyperbolic n pi so you are differentiating this cosine function it will become so n pi b comes out and you have this becomes sine hyperbolic n pi when you are putting x equal to a so n pi a by b so n pi a divided by b into cos n pi y by b so 2 you bring in so that the whole this is the whole constant this is the constant you will get, this equal to g2 of y.

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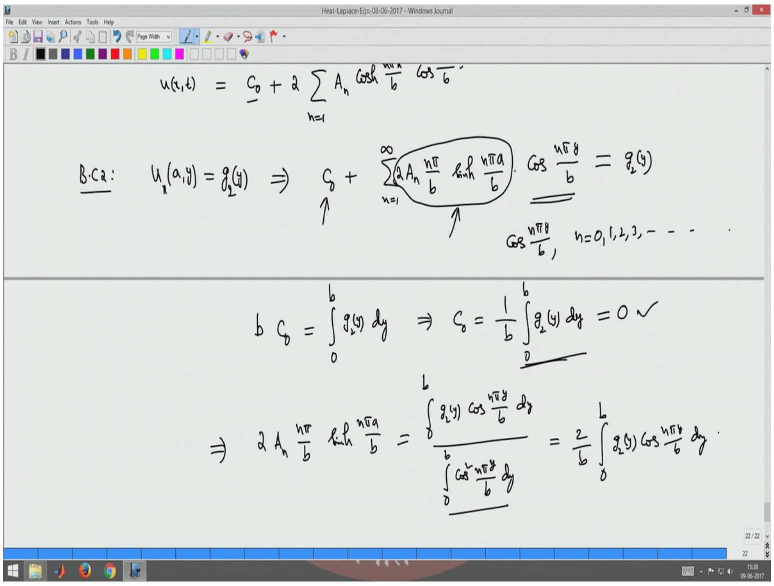

Now this constant and this constants can be gotten from this eigenfunction so what are the eigen functions n pi y by b n is now running from 0, 1, 2, 3 onwards okay simply multiply 1 and integrate from 0 to b okay what you get is this sum would not contribute anything only c not, so c not multiply, so c not into b is actually equal to integral 0 to b g2 of y dy, I simply multiplied 1 n equal to 0 cos n pi y by b is 1.

So I multiplied that both sides 1 and integrate from 0 to b so this will be 0 and this will be simply that is what you have so implies C0 is 1 by b, b is non-zero so 0 to b g2 of y dy but for in order to have the steady state we know that this condition is already satisfied so implies this is 0 plus this integral is a necessary condition to get the steady state okay this is a necessary condition to have a solution for the steady state for the plate if you want.

That means that is same as saying to have a solution that is the unique solution if you see this is non-zero what happens phi 0 is so you cannot have physically you cannot have this is the necessary condition we have seen mathematically also okay I have shown that this has to be 0 contribution Neumann condition on the boundary has to be 0 so this is 0.

So this implies what happens to other boundary condition other if you apply cos n pi, n is from 1, 2, 3 onwards you get 2 An n pi by b sine hyperbolic n pi a by b equal to integral 0 to b, g2 of y cos n pi y by b dy divided by cos square so that is also will give me integral 0 to b, cos square n pi b by y, n pi y by b dy so this is because this value is b by 2, so you have 2 by b integral 0 to b g2 of y cos n pi y by b dy.

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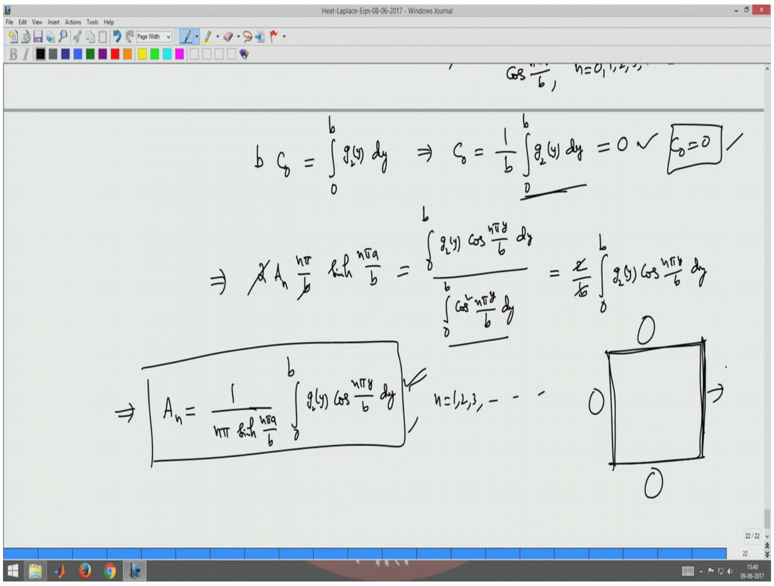

So this gives me A n as you can remove this thing so you can cancel b b what you get is 1 over n pi sin hyperbolic n pi a by b times integral 0 to b g2 of y cos n pi y by b dy so this is what you have for n is running from 1, 2, 3 onwards now C0 is 0 so this implies C0 is 0 so this together this, this if you put it into this this is the required solution.

So this is the required solution that satisfies all the boundary condition with c0 is 0 so you have 0 and you have this A n with a and b this one, okay. This is how you can solve Neumann data and the boundary for the laplace equation okay. So as a mathematical problem for the laplace equation if you are given a Neumann data so what you need to see is that necessary condition it has you have to verify first of all, the boundary data Neumann data should be that integral has to be 0 then you will have a solution otherwise you may not be able to work out the solution okay.

So as a physical problem that we have seen as a physical, the heated plate when it reaches the steady state and is whatever, see physically what it means is another things is you have a plate 0 here 0 so all the three places three boundaries you insulated only here is not insulated flux is you allowed the flux that means some part of the boundary, so heat is coming in some part of the boundary heat is going out so that is the meaning of that integral 0 to b g2 y dy 0.

So the flux is, net flux is 0 that means whatever comes through that same boundary whatever heat comes in net amount of heat comes should be same as net amount of heat that goes out of the same boundary okay so this is how you can see that then only it will reach the steady state then the steady state temperature of the plate is the solution of this problem so as separation of variable method you can apply and get it like this okay.

So this is how we can solve the laplace equation in two variables and we will see in there so what we have seen is for rectangle Cartesian co-ordinates this on the simpler domain like rectangular domain use this Cartesian co-ordinates and work out separation of variables we can also work out certain other domains such as circular domains, circle, interior circle, exterior circular domain. We will see in the next video how to find the solutions for the laplace equation in a curved domain such as circular domains, circular basically circular we will use polar co-ordinates okay. Thank you very much.