Differential Equations for Engineers.
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Lecture-6.
Methods for First-order ODE's - Reducible to Exact Equation (Continued).

So far we have solved exact equations and some non-exact equations with integrating factor that is only function of x or function of y alone. So we will continue to reduce non-exact equation into an exact equation, but functions of x or functions of y alone, integrating factors may not work. Integrating factors that are functions of x or functions of alone may not work when, when they do not work, you may have to look for some function of x, y as an indicating factor, so this is called the general method.

So inside, if you if you want the general function, generally integrating factor as a function of x and y, so what are the expressions to check, given certain form of indicating factor, you may have to check, check certain expression beforehand based on your M and N, M of xy and N of xy from the differential equation. We will see the general procedure in this video and, we will see, not all the equations you can get such integrating factors, so whenever it is possible, so you, so you will give an expressions that are to be checked whether you will get integrating factor of function of xy is exact form or not. Okay.

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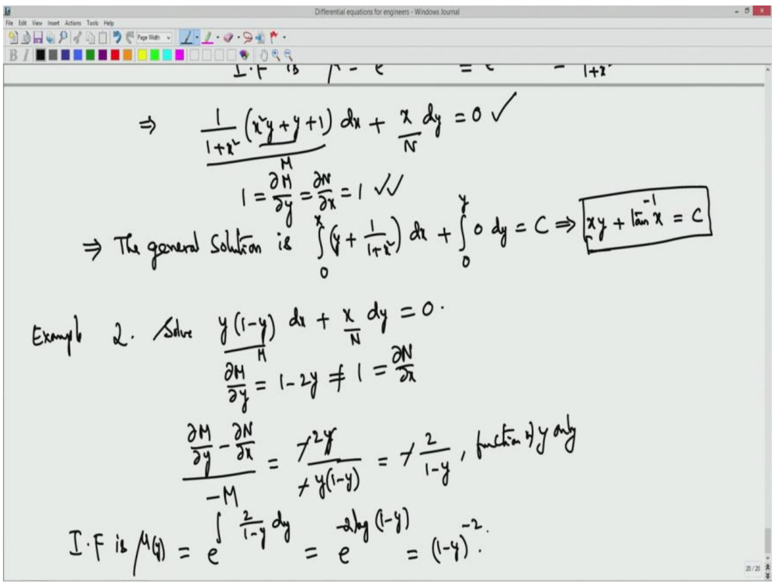

So let us take one more example where you may get, we will see some more examples. For example 2, so I will take simple example. Solve y into 1 minus y dx plus x dy equal to 0. So how do I solve this? Clearly M is this, N is this, dow M by dow y, dow M by dow y which is equal to 1 minus 2y is not same as 1 which is actually dow N by dow x, it is not exact.

So, but I try to make it an exact equation by, so I, my check is dow M by dow y minus dow N by dow x, what is this value, this is equal to 1 minus 2y minus1, so minus 2y, so if I divide this with N, it should be function of x. If I divide this with minus M, then it should be function of y. So this is my M, this is my N. If I divide with N, it is not function of x, right, divided by N for example. So if I do it with x, it is not function of x. Okay. This is not function of x, so this will not work.

So what I do is, I divide by minus M, that is the condition you have, if you want for the function of y. So what is my M, y into 1 minus y. So this is equal to, y, y goes, so you have -2 divided by 1 minus y which is function of y. Once you know this function of y only, what is my integrating factor? Integrating factor is mu equal to mu which is a function of y as e power integral this function whatever you got minus 2 divided by 1 minus y dy which is equal to, what is its integration? e power log, 2 log, 2 takes out, 2 log 1 minus y, okay, plus some constant will be there, so that e power this is e power that constants, okay.

So that will come here, that we can take it as 1. So only at the end, I want only some function of y with multiply, multiply any constant is also an integrating factor. So you have infinitely many integrating factors, simply by multiplying a constant. 1, 2, 3, anything, any number you can multiply with this, you can say that there is an integrating factor. There is this log 1 minus y whose derivative is 1 by 1 minus y minus, so this is equal to, you simply have a 1 minus y whole square.

I made a mistake here, so I have minus M, minus M is minus, minus M is, M is minus, so M is y into minus y, so minus minus goes, so this is plus, so this is plus. That is why should I have minus here, so that you have -2. So this is your indicating factor, okay. Once you know the integrating factor, you multiply this integrating factor to the equation. So what is your equations, y into 1 minus y and multiply this with 1 by 1 minus y whole square dx plus x diverted by 1 minus y whole square dy equal to 0.

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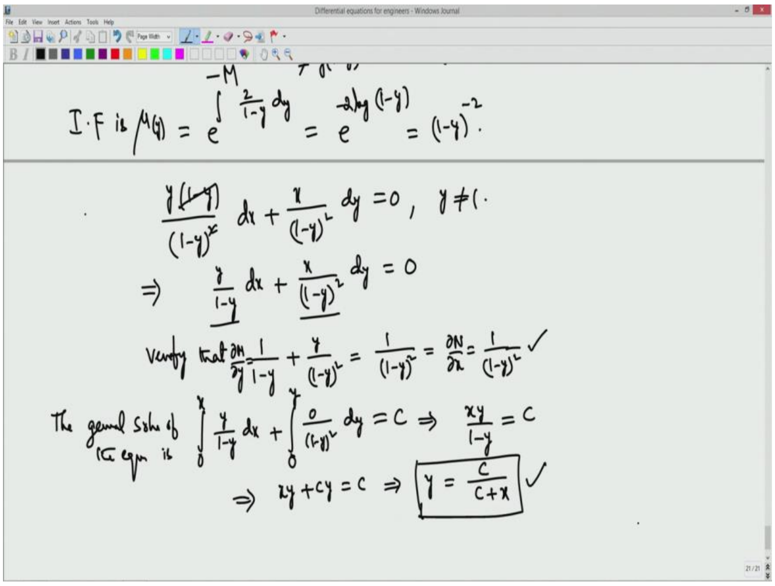

See, you have got this equation where x belongs to full R, I do not have issues, there is nothing like one by x, x is defined everywhere, y is also defined everywhere. Now you are multiplying this equation like this. You are multiplying this equation with 1, 1 by 1 minus y whole square. That means at y equal to1, this equation is not defined. you multiply an integrating factor, after multiplying, you have in issue at y equal to1, so it is defined, the reduced, what you got is the, after multiplying the integrating factor, the differential equation is defined only at certain domain, it is not defined at y equal to1, okay, y should not be equal to1.

So that means you can consider any domain that does not involve y equal to1, okay. So this my, so you need not worry, whenever you seen the denominator, the denominator is becoming 0, that point it is not defined. So this is your equation, so this will give me, what happens, so this one goes, so you have y by 1 minus y dx plus x by 1 minus y whole square dy equal to 0. So this is now exact, you can easily see dow M by dow y, can you verify now, dow N by dow x is this, dow M by dow y is, so if you can see this is my M, this is my N.

Now you can verify that dow M by dow y, that is 1 by 1 minus y plus y by 1 minus y whole square, so that will give me 1 by 1 minus y whole square which is equal to dow M by dow y equal to this, which is equal to dow N by dow x which is simply 1 by 1 minus y whole square. So this is an exact equation, now I know how to solve it. I fix my x equal to 0, x0 equal to 0 and y0 equal to 0.

So y0 is 0 is there in the domain, so I can simply integrate M with respect to, integrate from 0 to x, y by 1 minus y dx plus integral y0 I take 0 to y, I replace x variable as x0 which is 0, so 0 by 1 minus y whole square, I do with respect to integration constant. So the general solution is this. General solution of the equation is, is this. So what is this exactly? So this becomes y by 1 minus y as a constant, so xy by 1 minus y +0. So this is equal to C.

So this implies, you can get your y, so xy plus Cy equal to C. Implies y equal to C divided by C plus x. This is your general solution of the given equation. So you have 2 examples where we could check dow M by dow y minus dow N by dow x divided by either N or by minus M, verify whether it is a function of x or function of y and we can get accordingly what is my integrating factor by multiplying that we can make the equation exact and then solve the equation.

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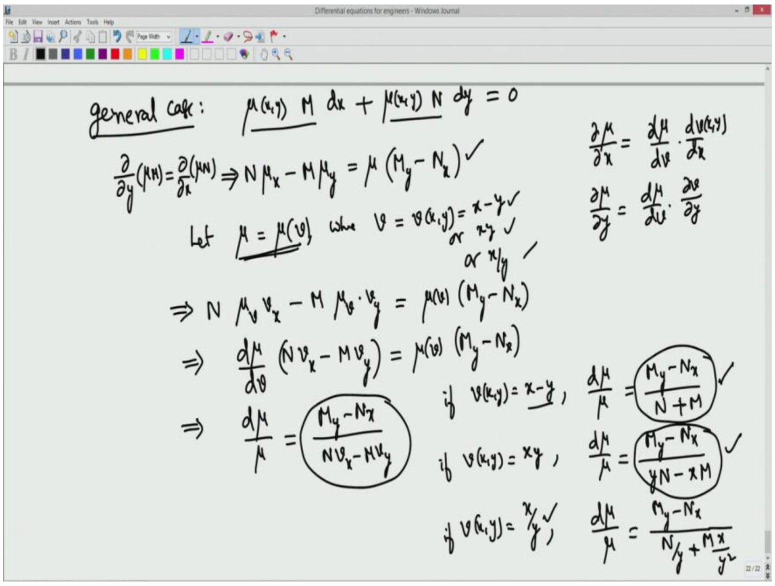

So let us, let us go back to the general case, general case, I do not want, see, it, it may not work. Dow M by dow y minus dow N by dow x divided by N divided by minus M may not be function of x alone or may not be function of y alone, it is function of x, y, then what do you do? Then what I do is, I still multiply, so if you multiply mu of xy, N dx plus, I multiply this, N xy, N dy is equal to 0, this is the given equation which is not exact, M dx plus N dy, I multiply this.

If I multiply, I hope this to be exact, so you have that equation, mu has to satisfy, finally mu has to satisfy N mu x minus M mu y equal to mu times My minus N x. This is exactly what we had earlier. Okay. So if, this is simply, this is just nothing but, this is my M, this is my N, so dow by dow y of mu M equal to dow dow x of mu N will give me this. Okay. Now I look for mu as a function of x y, okay. It is not any general xy.

If I say let mu is function of some v which is where v is, v is a function of x, y, the specific form, this I can choose like, like v as v of x minus y, v of x, y, v of x by y, these kind of patches once I can choose there is function of x, as a function of x, v, okay. So let us choose, let us take like this. If I choose mu as mu of x, y, as mu of v, v is function of x, y but v of xy is actually, in specific form like v of, maybe something like x minus y, x minus function of, this is a of, so I do not write the same.

So maybe some function is let us say G, G of x, y, some G of x, y. So G of x, y is, as, rather so it should be known, it should be v of x, y square v of x, y if some function, let us say G of x minus y. Okay, this or this or this, okay. Something like this. This is how you look for it, then what you should, you just simply assumed the form, mu equal to mu of v subsidiary to this. If you substitute, what we have, N, I read at mu x, mu x is, mu is a function of v which is a function of x, right, you can see this.

Mu is nothing but dow mu by dow x, this is equal to dow, d mu by dv into dv by dx. v is a function of xy, okay, so, right, so if I write like this, mu x is given by mu v into vx minus M mu y, that is dow mu by dow y is, mu is a function of v, so d mu by dv equal to dow v by dow y. Okay. So this will give me mu v into vy, the place of mu y which is equal to mu of v dow M by dow y, y minus Nx. This is what it becomes, okay. That implies, what is this d mu, d mu by dv.

Mu is common but I write d mu by dv into N vx minus M vy equal to mu, mu is a function of v, M by minus Nx. So this implies, I can now, you see this mu is one variable, d mu and other things M vx, minus M vy, so you simply write, rewrite this as d mu by mu which is equal to My minus Nx divided by N vx minus M vy. So you see mu is a function of v. Okay. If I want to integrate this to get my mu, again like earlier, this should be function of v. Okay. So we can verify whether it is a function of v.

So what is your v? So early you choose, you can check if mu is, if v of x, y is some function of, some function of let us say x minus y, okay. If you want this, then it should be, you simply choose G, not function of x, simply v of x, y is simply x minus y, this as xy. Simple form, x by y. Any of these things, okay. Then let us say x minus y some if you choose this, this thing, what happens to this equation?

This equation becomes d mu by mu, what is mu, mu is function of x, so that is mu is, mu of x minus y. Okay, equal to. Now you see this My minus Nx divided by N vx, what is vx, 1. Now N minus M, vy is, vy is minus1, so that make it plus. So this is what you have, so if you want mu as a function of x minus y, which is v, then this quantity you have to verify, if it is after verifying you know M and N from the given equation, you can verify this, if it is function of x minus y, then you can integrate and get your integrating factor, that is the idea. Okay.

So we will see if v of x, y is xy, in this case, this equation becomes d mu by mu equal to, numerator is same dy, My minus Nx divided by, now N into vx is y, vx is y, y into N, A vy is x, so x into M. So now you will verify whether this is a function of your xy, okay. If that is the case, then you can get your integrating factor. So if v of x, y is x by y, so you can choose any form, it can be x plus y the whole square, so anything in general. Okay, I will give you only one example here for any one of these cases, just to demonstrate how it works, okay.

So d mu by mu, you have again My minus N x divided by numerator, what happens, N, what is vx, 1 by y, N divided by y minus M, what is vy, v, x divided by y square. This is what, so you have to see whether this right-hand side functions is function of x by y or not, if that is the case, I can integrate both sides and get my mu, okay. So we will see one example.

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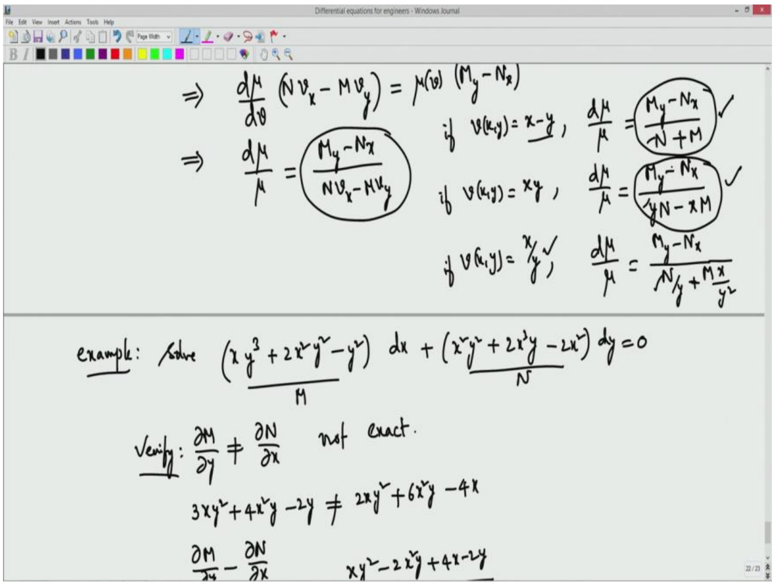

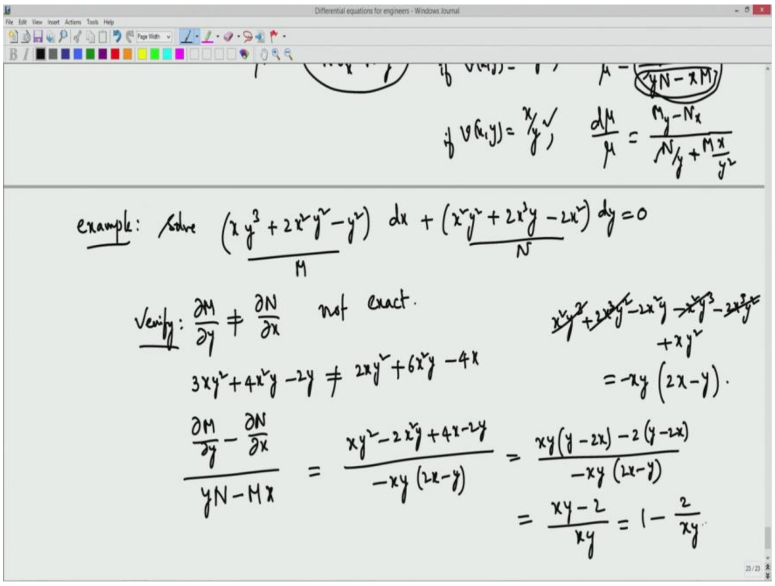

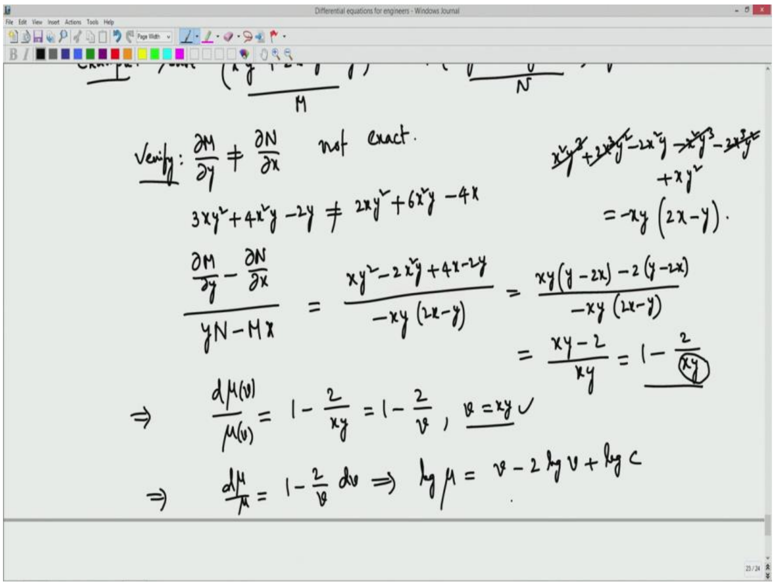

Solve xy cube +2 x square y square minus y square dx plus x square y square plus 2x cube y minus 2x square dy equal to 0. How do we solve this, this is not an exact equation? As you know this is my M, this is my N, you can verify that dow M by dow y is not equal to dow N by dow x, okay, you can verify. This is what is the case, so it is not exact, implies not exact. So how do we make exact, dow M by dow y you calculate, actually it becomes dow M by dow y is 3xy square plus 4x square y minus 2y, this is what is.

Which is not equal to dow N by dow x is to xy square plus 6x square y minus 4x, obviously they are not same, so it is not an exact equation. But if you calculate dow M by dow y minus dow N by dow x, what is this? This, what is the numerator, I am just writing it, this becomes dow M by dow y is 3 xy square - 2 xy square, so we have xy square 4x square y minus this, 6. It is going to be minus 2x square y minus 2y plus 4x, plus 4x minus 2y.

So this is what is your this thing, numerator. So if I divide this, like in the earlier case, now I check in the, N plus M, yN minus x, xN or this, this quantity, either this or this or this in the denominator I choose and see what happens, okay. Because I know this so I will check only this one because this will work. If you do N plus M, you divide it. If I do N plus M here, this may not work. N plus M is again for x minus y, when I do this N plus M, I may not get this whole thing, when I write this N plus M, M plus N is known, this M is known, this whole function need not be, it will not become as a, as x minus y function. So that will not work.

It should be a function of x minus y. So I choose y, yN minus xM. So if I do this one, so what is yN, yN is 2 xy cube plus 6x square y square -4 xy minus M is this, M is, sorry no, I made a mistake. So y into M, so this is my N, y into N is x square y cube plus 2x cube y square minus 2x square y minus Mx, that is x square y cube minus 2x cube y square plus xy square. This is what is my y N minus M x, so what exactly is this?

x square y cube goes, 2x cube y square, 2x cube y square goes, so what you are left with is minus 2x square y, so xy is common, x square y, x + y. So this is what you have, so what I have is -2 xy into x plus y, x plus y or x minus y, x minus y right. 2 xy, so no actually not right, so you have xy is common minus xy you take it out, so you have minus, so 2 x and here xy square, so minus y, okay. So you have minus xy to x minus y, what is this one, this is actually equal to xy, numerator xy you take it out, you have y minus 2x here 2 times y minus -2 times y minus 2x divided by minus xy 2x minus y.

So you can cancel minus 2x minus y in the denominator, what you get is xy -2 divided by xy, so this is exactly equal to 1 - 2 divided by xy. So you can see if I choose this, if I, I just verify, this quantity is, right-hand side of this equation, this quantity is function of xy alone, this is a function of xy, you can see this as a variable. 1 - 2v, v is a function of xy, so this implies equation, once I verify this, I have this equation, from this I have to find my mu. So d mu by mu equal to d mu by mu equal to1 minus 2 by xy. Okay.

So mu is function of xy, mu which function of v, function of v is 1 minus 2 by v, v is xy. That is how we see, v of x, y is xy. Now this implies d mu by mu which is equal to, this I have dv here. Okay. d mu by mu, when I rewrite, I have a dv, so what I have is, so we have dv, dv you have here. Okay. So d mu by mu, I have 1 - 2 by v into dv, these are, variables are separated, which are variables are mu and v. So this I can easily integrate.

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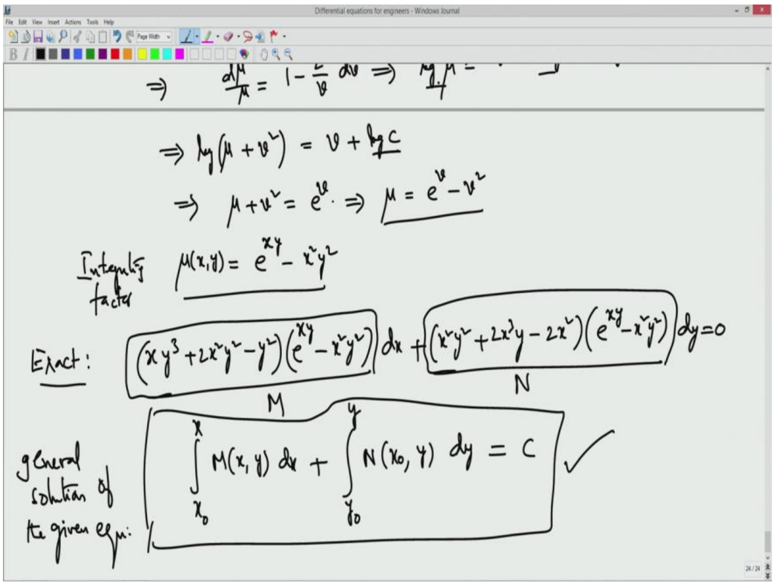

We get log mu equal to, I can integrate, so v - 2 log v, okay plus log C. So what is the integrating factor, mu is the integrating factor, if I calculate mu plus mu plus v square is my log, if I take this term and this term together which is equal to v plus log C. So this gives me mu plus v square equals to e Power v into e power C1, so if you call this C1, there is a constant, so I can take it as 1. So this will give me mu has e power v minus v square. So this is my integrating factor.

So integrating factor is mu x, y is e power v is xy minus x square y square, xy square, v is. So this is the integrating factor, integrating factor. So if I multiply this integrating factor to the equation which is this, we can get, we can make the equation exact, that is how we make the equation exact. If simply write x cube, now I take the equation xy cube plus 2x square y square minus y square into this, this integrating factor e power xy minus x square y square into dx plus x square y square plus 2x cube y minus 2x square into, again integrating factor e power xy minus x square y square dy equal to 0. So this is now exact.

What is the solution? Solution, simply integrate this, this is my M, this is my N, new M and N, now exact, with exact. So simply integrate, so solution is x0 to x, M of x, y dx plus N, this one y0 to y, N of, x you fix it as x0 to y dy equal to constant. So this I leave it as an exercise, so you take this M and N as these things, this is my N, this is my M, you put it here, that will give you the general solution, okay, general solution of the given equation is this. This is the equation for this equation, okay.

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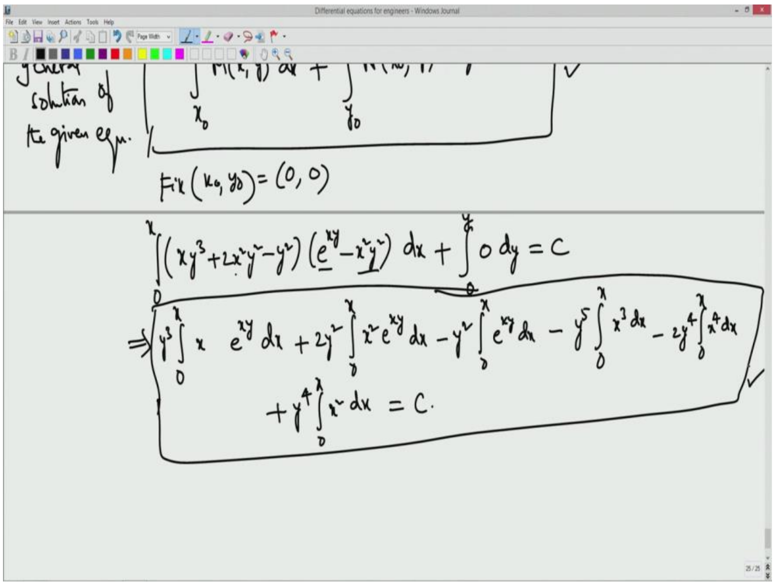

This, this is actually equivalent to the given equation because I could simply cancel both sides this integrating factor. So that gives you original equation. Okay. So you can actually see the calculation now, I am not here, so I may be, I will do the calculation now here. So what is that, I can fix my x0, y0 as 0, 0 because this is part of the domain, there is no issue with 0, 0.

So you have integral 0 to x, M is this one, xy cube plus 2x square y square minus y square into e power xy minus x square y square dx plus y0 is 0 to y, now N, in N you fix x equal to 0, x equal to x0 which is 0, 0 in this becomes 0, this becomes 0, this becomes 0, so x0 into whatever may be 0. So you have 0 dy equal to constant. So you simply have to integrate only this part if you can, okay. So how do we integrate this?

So this is what is the integration, so 1st term is 0 to x xy cube into e power xy dx, so y cube I can take it out because since constant. x into e power xy dx, 2nd term, 2y square is integral 0 to x, you can expand it, okay. x square e power xy dx, now if you multiply this with this, now the last one is minus y square integral 0 to x e power xy dx. This is what happens if you multiply this with each of these 3 terms and integrate.

Minus y power 5 integral 0 to x, x cube dx okay. And this you multiply the 2nd term here with the 2nd term, 3rd terms you get - 2y power 4 integral 0 to xy power 4, sorry, y power 4 we have, so you have x power 4 dx. This is not power, so you have plus y power 4 integral 0 to x, x square dx equal to constant. So these are the integrations which you can do, okay, so it is possible to do. So think of y as constant here, this you can integrate x into e power some constant into x, that you know how to integrate, similarly x square into e power some constant times x.

This integration can do and finally get, this is your, whatever the, after simplification you can get this is your general solution, I leave it here, okay. So we have given a general procedure of finding integrating factor mu of x, y. To do this, we have to check certain expressions, different expressions for different mu of x, y. If mu is a certain form, something like mu xy, function mu is, mu is a function of x, y which is like function of x into y or a function of x plus y or certain mu of G of x, y, G is known.

Then , we have seen what exactly the expressions in terms of M and N to check, so when you check that expression should be function of this G of x, y, known function G of x, y, then mu of G of x, y, you know how to integrate, so you can find that integrating factor. Okay. So we will give, so we have given the procedure how to find this general integrating factor with an example in this video, okay, so far you have seen that. So maybe we will do some more problems later.