Differential Equations for Engineers
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Lecture – 59
Laplace equation over a Rectangle

Welcome back the last video we have seen that a steady state heat equation 2 dimensional steady state heat equation becomes laplace equation, so laplace equation is actually a steady state 2 dimensional or 3 dimensional, basically steady state heat equation is a laplace equation, so that is what we have seen so we will try to define problems for the laplace equation.

So as an application for this heat equation, so you can see once if you consider plate that is heated plate so initially at some temperature and at the boundary, 2 dimensional so the boundary terms, at the boundary you maintain certain temperature so if you maintain certain temperature then actually you can find the temperature of the plate for all times so that we can solve for all terms.

So once it reaches the steady state so we have to find the solution that means so you want to see the steady state solution of how the temperature distributes through the plate with these boundary conditions that is to solve laplace equation with this boundary conditions we will see how do we pose this problem as a boundary value problem.

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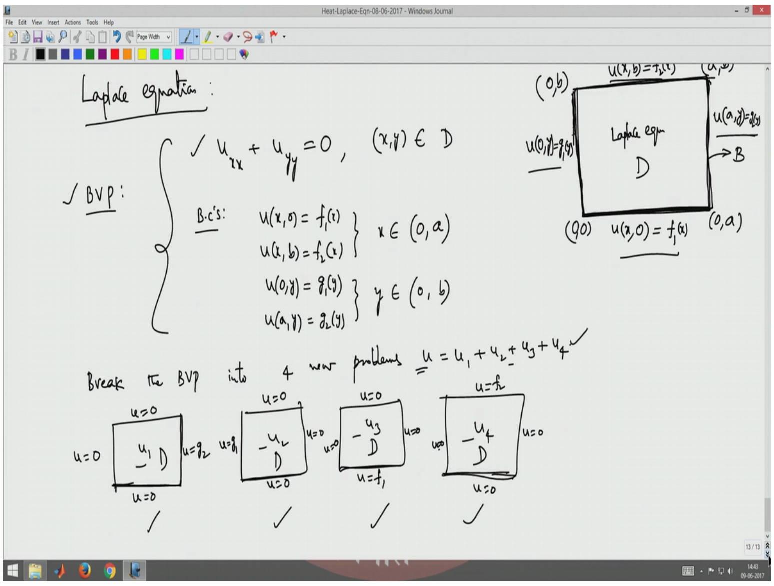

So for laplace equation we solve the laplace equation so we can say first 2 dimensional, only 2 dimensional we will solve so Uxx plus Uyy equal to 0, so this is x y belongs to some let us consider some domain D. See for not all the domains we can solve this boundary value problems for the laplace equation so we use separation of variables technique when you want if you want to use the separation of variables technique basically you want to extract Sturm-Liouville problem so you should have boundary domain in one direction and boundary, you should have a homogeneous boundary condition at those n points.

So that means basically what domain should be finite or semi-infinite something like this one so if you, so what we do is we consider this laplace equation in a boundary domain for example rectangular plate okay if you consider this so you consider the rectangular, D is the rectangular plate okay so if you consider this as a D, it is a rectangle so you say this point is 0 0, so 0 0 and this point is 0 to a and this point is 0 to b.

So obviously this will be a to b, so this is what is the rectangle plate you have and here you have this laplace equation satisfied here inside this domain and you have to provide there is no time so you have to provide the boundary data on this so you have to provide the boundary data on this okay on this boundary, this is your boundary so if this is your domain D, the boundary is this, boundary let us call this boundary B.

So on this if you want to give this, so you can actually you can provide data everywhere so all along the boundary you can provide so here so if I say this is a dirichlet problem size, I provide the boundary value problem so I will give you the boundary value problem as, so what I provide is I can provide first of all the temperature, so this is, this models has steady state heat equations so the plate is, heated plate in a steady state.

So you want to see how the temperature distribution throughout this plate inside okay once you know the boundary however if you give the boundary how the temperature varies along the inside the domain so that is what you want to find so there is no time so the boundary value problem is this so what I do is I try to maintain this boundary at certain temperature fix temperature for example u at this is what x equal to, x is varying x is between 0 to u so x is actually varying and y is 0 so you have a 0, u x y equal to 0, okay.

Let us say I do not gave equal to, so these let us say f1 of x, okay and then here u of x b is actually f2 of x and here u at now here on this boundary y is varying, so you have x is 0 and y this you call it g1 of y and here you provide data that is u at the temperature, x is a and y is varying so that you give some g2 of y okay if you give like this, so the boundary conditions are basically so you can write what it is.

So u at x 0 is f1 x, U at x b is f2 of x. u at 0 y equal to g1 of y, u at 0, not 0 this is a y is g2 of y, so this is what you have so where x belongs to both the cases x belongs to 0 to a, both the cases x belongs to, y belongs to 0 to b, these are the (())(06:05) so if you how do we solve this boundary value problem so the best way is so, many ideas we try to solve by separation of (())(06:13) as we said to do that we need to have opposite, two opposite boundary should have zero boundary conditions, okay.

So you need zero boundary conditions that means you want this say either these twos are u at here 0, here 0 or here 0 and here 0, anyone of them, okay and then and here you can have both of them non zero, so if you want to reduce the calculations involve what we do is in the here we try to make it 0 both the u at, and these boundary we make it 0 and on this boundary we can provide the data but in that I provide one 0 okay I make this one side 0, other side is function of y, okay.

So if you do that because it is a linear problem, linear equation and you can actually break this problem into 4 types okay, 4 problems, so u1 so break the boundary value problem, above boundary value problem into 4 new problems, how do we do this so I will just graphical I show what it is so you have u1 so call this u equal to u1 plus u2 plus u3 plus u4, okay.

And then u1 is satisfying the laplace equation, u2 is satisfying the laplace equation, u3 is satisfying the laplace equation and D this is actually D the domain same domain what we have considered, u4 is also satisfying this domain D only the boundary you have to be careful now, so now here I get f1 here so I make f1 0 so u is 0 here, u is 0 here okay and here I make u is 0 and here I make u equal to g2 so I make all this three, this three will be 0 and only this one will be non 0 in the first case.

Now I what I do is I make this one non zero okay, this side is non zero and here make it 0 so that is our problem number 2 so here u0 same u0 0 u0, here I make u equal to g1 and here u equal to 0 so u at a y is 0, u at 0 y is g1 and now similarly here, so for the problem for problems u1 and u2 1, 2 what you consider is you have the same homogeneous conditions in the opposite side so that is at y equal to 0 and y equal to b and here now you change this here you make u0 here and u0 here and here you vary so now here I make this f1 and here I make this 0.

Similarly I make this 0 here I make this here as f2 and here obviously u0 0. So this is how you break then what happens u1 plus u2 plus u3 plus u4 at this boundary, at this boundary 0, u1 0, u2 is 0, u3 is f1, u4 is 0 so together it is f1 so that is what here okay similarly here so and this boundary here it is g2, g2 plus 0 plus 0 plus 0 so that is 2 what we have okay, similarly here u3 is so if you apply here so 0 0 0 and f2 so this is what we will give you similarly here.

So this side you can see that 0 g1 0 0 on this, so this is how all the boundary conditions are now if you combine them all these problems together this satisfies, still satisfy the laplace equation because it is a linear equation and okay, so you break this boundary value problem into 4 new problems even you call them u1 u2 u3 u4 and such that they satisfy the laplace equation u1 satisfies the laplace equation here u2 satisfies la[lace equation, u3 satisfies laplace equation, u4 also satisfying the laplace equation and the boundary data you just split it you just distribute it in some sense.

So that, so the boundary is now for u, the boundary is as it is but if you combine all the data boundary data for u1 u2 u3 u4 will give you the boundary data for u and since u1 u2 u3 u4 are satisfying laplace equation sum is also satisfying laplace that means you got this back u3, okay.

So if you can solve one of this problems then one problem you solve and similarly you can work out other problems so that if you are given any problem like this boundary value problem with 4 sides are non-zero data then you can by solving this 4 problems, you can give this and then sum it up. You add all of the 4 solution that will give you the solution of this boundary value problem so we will see one, so let us start with u1 so we will take this up, for u1 okay solution method for u1.

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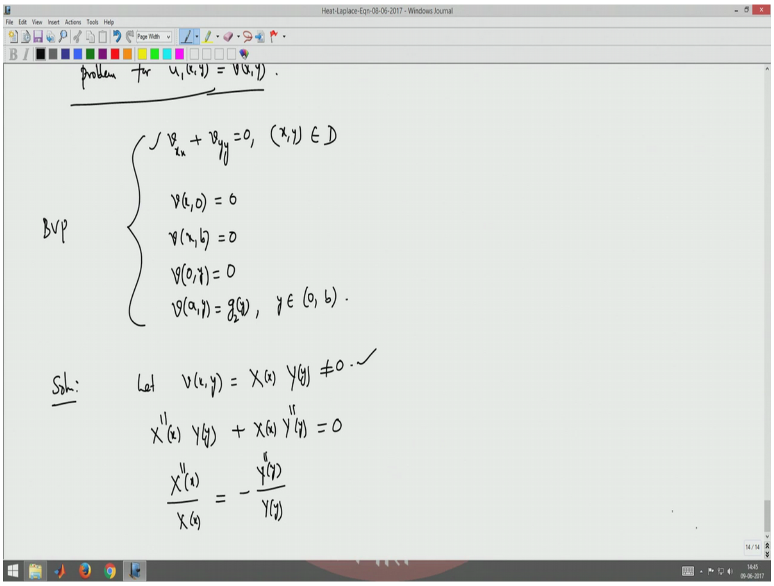

So we will write this problem for, problem for u1 of x y. So let us call this I do not want to work with u1 let us call this v of x y, call u1 is as v so Vxx plus Vyy equal to 0, x y belongs to the domain that is the rectangular domain and v at x0 is 0, v at x b is also 0 and v at 0 y is 0, v at a y equal to, a y is g2, g2 of y, y belongs to 0 to b. So this problem will work out so we will try to solve this boundary value problem okay.

So this problem how do we solve this, so give the solution by separation of variable so let the solution be v of x y, X of x, Y of y as a non-zero solution if you look for non-zero solution for the laplace equation then if you substitute this v into the laplace equation you will get X double dash of x, Y of y, the place of Vxx plus X of x times Y double dash of y equal to 0.

Again the same procedure because this is non-zero you can divide with a both side so you get X double dash of x by X of x equal to minus Y double dash of y divided by Y of y now you can see that both sides, one side is function of X other side is function of Y so it should be a constant, so this should be a constant parameters you call this lambda.

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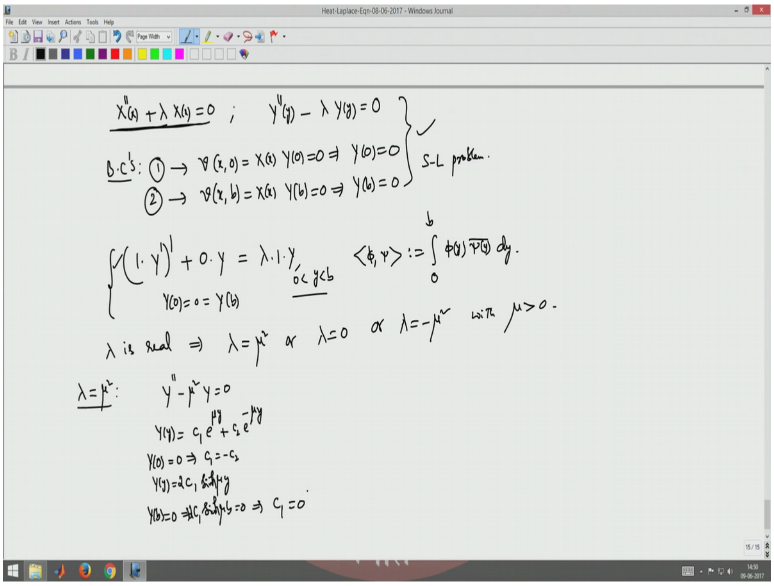

And so the laplace equation becomes two ordinary differential equations that is X double dash of x minus lambda X of x equal to 0 this is one and other one is Y double dash of y and then plus lambda Y of y equal to 0 so this is has become okay.

Now if you apply this zero boundary conditions wherever this one, okay. v x 0 is 0, v boundary condition first two, one and two, one will give me v at x0 is X of x and Y of 0 is 0 implies y of 0 is 0 because X of x cannot be 0 is a function if it is 0 then whole thing is 0, v of v 0 which is against our assumption that it is non-zero solution so boundary condition two will give me v at x b, v at x b is 0 that is again X of x, y of b equal to 0 that will give me y at b equal to 0 so this is the problem so now what you get is boundary condition, zero boundary condition will give you Sturm-Liouville system for Y of y.

Once you get this eigenvalues and eigenfunctions here you substitute for eigenvalues for lambda here and try to solve, try to find the general solution for X n, X n and Y n you combine and put it as vn and then take a super position of vn that satisfies the laplace equation with this two boundary conditions one and two.

Apply the boundary conditions three and four as a initial like we have applied for initial condition for wave and heat equations this method you can get those constants okay that is the procedure so let us find, so let us see what is the Sturm-Liouville, so this is the Sturm-Liouville problem obviously regular because in the Cartesian co-ordinates.

So you have y double dash, y dash 1 into y dash dash plus q, q is 0 into y equal to minus lambda okay so what we have is minus lambda so if you do not want minus here so you can make minus here so plus here, so here this is, this way also you can make, okay. Before you after, by dividing both sides with this you can write this way also, by that you can see the this will be plus and this will be minus so that, so that you are Sturm-Liouville problem is also in the exactly in that form in the self adjoint form and as a eigenvalue problem, y okay lambda into 1 into y so lambda into (())(16:50) as 1 y.

Obviously y 0 is 0 which is equal to yb okay so this one is already self adjoint form w is 1 so the phi size dot product is for the solution you can define it as, this is what is the domain between, this domain is y is between 0 to b so the domain is between 0 to b and you have a phi x because the function of L is say phi y psi y bar dy this is the dot product.

So the Sturm-Liouville so immediately self adjoint form so lambda is real implies lambda is either mew square or lambda is 0 or lambda is minus mew square with mew is always positive okay and look at this all this three cases lambda equal to mew square so this will give me.

If you consider y double dash minus lambda so you have a y double dash minus mew square y equal to 0 so the general solution of this is y of y is C1 e power mew y plus C2 e power minus mew y, I think earlier also we solved we see that this is not satisfying so this lambda is not an eigenvalue okay anyway we can repeat the same so Y 0, Y 0 will give me C1 equal to minus C2 so your general solution is C1 comes out 2 C1, sine mew y now if you apply y at b equal to 0 will give me sine mew v has to be 0. So this will give me sine mew sorry this is sine hyperbolic mew, sine hyperbolic mew b into 2 C1 has to be zero that implies C1 has to be 0.

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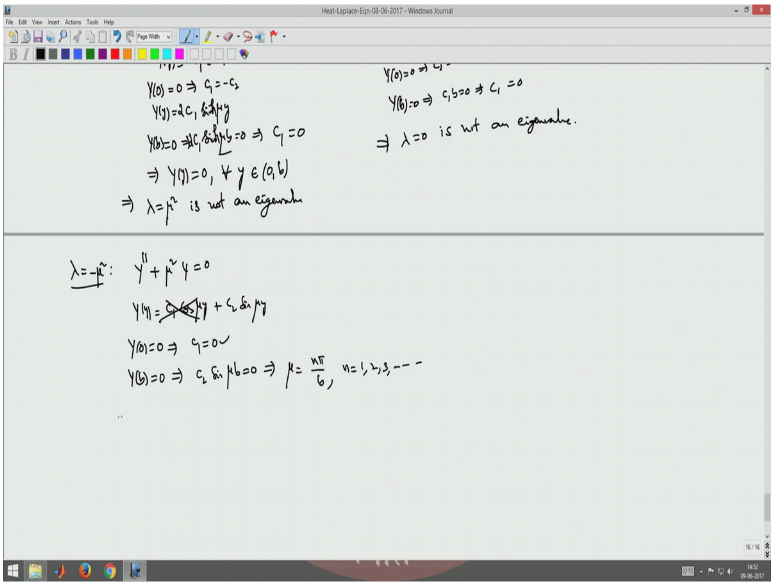

Because sine hyperbolic cannot be 0 for mew is positive, b is positive so C1 is 0, so implies Y of y is completely 0 for everyone belongs to 0, so this implies lambda equal to mew square is not an eigenvalue.

Now same way you can see that lambda is 0 is also not an eigenvalue so Y double dash lambda v0 so you have this one so y of y is actually C1 y plus C2 the general solution, y of 0 is 0 will give me C2 is 0 so this has gone, y at b equal to 0 will give me now C1 b0 that gives me C1 has to be there so that means this is also gone so completely 0 so in place lambda equal to 0 is not an eigenvalue.

So what you are left with the third case that is lambda is minus mew square, so if you do this Y double dash plus mew square y equal to 0, Y of y is C1 cos mew y plus C2 sin mew y now apply Y of 0 is 0 it will give me C1 is 0 so this has gone now Y at b equal to 0 will give me C2 sin mew b equal to 0 and so you want a non-zero solution C1 is anyways 0, C2 is you want to be non-zero so that sin mew b 0 that gives me mew as n pi by b, so n is running from 1, 2 onwards because mew is always positive.

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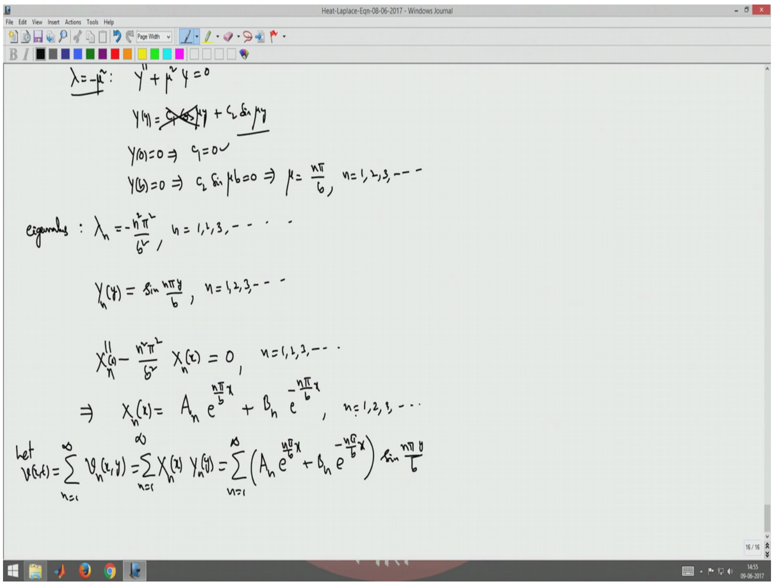

So your eigenvalues are lambda n eigenvalues, lambda n equal to n minus n square pi square by b square n is from 1 2 onwards okay and then eigenfunctions are this solution sine eigenfunctions call this Y n of depends on n so just labelling as Y n as sine, it can take C2 as 1 sin mew is n pi y by b again this is from 1, 2, 3 onwards, so these are eigenfunctions.

Now you consider X double dash minus, so now you have a plus, plus lambda is, lambda is this one so you have a minus n square pi square by b square so because they are depending on n lambda n replacing with lambda n so I am calling this X n, X n double dash plus lambda X of x so this is what you have so X double dash plus lambda is this now it becomes minus so you have minus X n of x for this function this is equal to 0.

Now what is the solution, general solution of this X n of x is, so you can call this A n so two arbitrary constants involved for each n, okay. n is from 1, 2, 3 onwards for each n you have a differential equation like this whose solution is on arbitrary constant for each n, A n, B n are arbitrary constant A n e power n pi by b x plus B n e power minus n pi by b x, okay, this is true for every n, 1, 2, 3 onwards.

So what is your Vn of x y the solution combining which is the product of for each n you can multiply Xn x with Yn y so if you do that you will see that A n e power n pi by b x plus Bn e power minus n pi by b x into Yn is sine n pi y by b.

So this is the solution that satisfies the boundary conditions, two boundary conditions, one and two so you can take a superposition let this v of x t be this superposition of all this n is running from 1 to infinity so now if you do this, now you apply other two boundary conditions to get what is your An's and Bn's.

So other boundary conditions are what is the other boundary conditions what you have is just go back and see what is your first, first problem so the problem for other boundary. V0 y 0 and V a y is g2 y, V0 y is 0.

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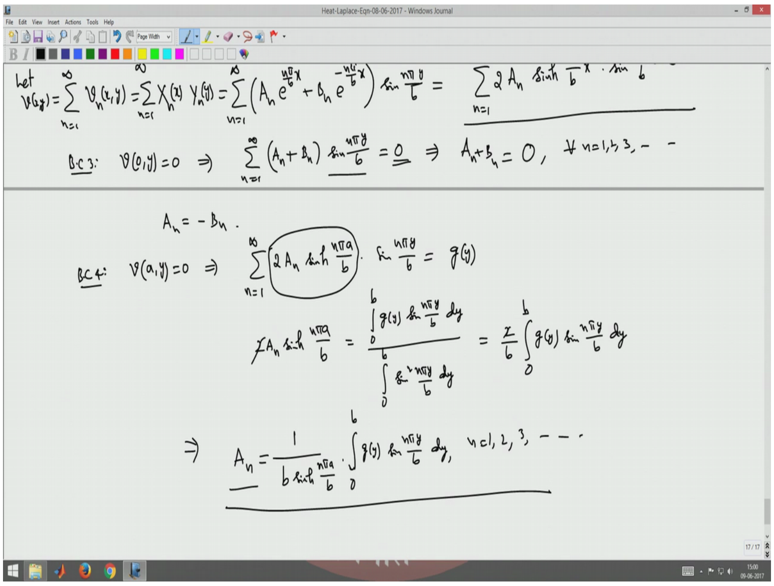

Now you apply the boundary condition three, v0 y equal to 0 so this will give me v its xy so v 0 y, 0 means the left hand side n is from 1 to infinity An when you put x equal to 0 this simply An plus Bn times sine n pi y by b equal to 0 so immediately you will see that An's how do we get this An plus Bn, An plus Bn is actually if you take the dot product with the eigenfunction sine n pi y by b both sides this side, right hand side will be integral 0 to b this is a function is 0 so 0 into sine n pi by y by b divided by integral 0 so integral 0 to b sine square n pi y by b so that dy okay so that one if you do this becomes 0.

This is true for every n 1 2 3 onwards so it means An equal to minus Bn so go back and see so if you substitute here so what happens once you know n equal to Bn you can combine it, n is from 1 to infinity An equal to Bn or Bn equal to An so you can combine An, Bn as minus so 2 An what you get is sine hyperbolic n pi b by x into sine n pi y by b so these are your this is now general solution.

Now you apply the other boundary condition, boundary condition four that is what you have so V at a y equal to 0 now for this now your general, this solution becomes this you apply, n is from 1 to infinity 2 An sine hyperbolic n pi a by b into sine n pi y by b equal to g of y. Now again so this is a constant this constant you can get it by applying dot products both sides with the eigenfunction.

So sine hyperbolic n pi a by b is actually equal to integral 0 to b, g of y sine n pi y by b dy divided by integral 0 to b, sine square n pi y by b dy so you can see this will be 1 minus cos 2 n pi y by b if you apply we will see that you can see that it is going to be b by 2 so you can say that this is actually 2 by b integral 0 to b, g of y the denominator evaluated as b by 2 so that becomes 2 by b, g of y sine n pi y by b dy okay.

So this is how you can get your this (conso) what you need is An so An are nothing but 2 2 goes both sides so what you get is 1 divided by b sine hyperbolic n pi a by b into integral 0 to b, g of y sine n pi y by b dy this is from n is 1 2 3 onwards so with this An you put it into here that is the solution that satisfies now all the boundary conditions so implies so you have a this is the required solution with An being this okay that is the required solution.

So this is how you can solve laplace equation with this boundary data so you need two opposite, one opposite, two opposite ends at least one or two boundary condition should be homogeneous, zero boundary conditions you need okay, so this is how if you solve so what you have done is you solved only a problem you want.

So in the same fashion you can solve for u2 apply the same technique u1 u2 u3 u4 you combine them to get the solution for u, this is how you can solve this rectangle problem with Dirichlet boundary condition, these kind of boundary when you provide u a temperature at the boundary of the plate is called the Dirichlet boundary conditions okay, Dirichlet boundary conditions.

If you provide the, if you insulate them that is kind of Neumann if you provide the flux conditions Ux, the insulating condition at the boundary so you get Ux or Uy depending on the boundary you say that, that is Neumann boundary condition so that is some for example normal derivative to the boundary is Neumann.

You provide simply u at the boundary this is dirichlet if you actually combine them that is robin type, robin conditions are combination of them so that is dou u by dou n plus u okay so if you give this dirichlet boundary condition we have solved we will try to see how we can see with this Neumann boundary condition we will not deal with this we will not look into the problems with robin condition okay.

So let us see let us fix so maybe we can see, so in the next video we will try to see we try to solve the boundary value problems for the laplace equation in a rectangular domain with Neumann data on it okay so we will see that in the next video. Thank you very much..