Differential Equations for Engineers
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Lecture – 58
Temperature in a Finite Rod with Insulated Ends

Welcome back in the last video I have seen how to find the solution of the heat equation in the finite rod when the two ends of the rod are maintained at two different temperatures. So today we will see how to find the temperature in the rod when the both the ends are insulated.

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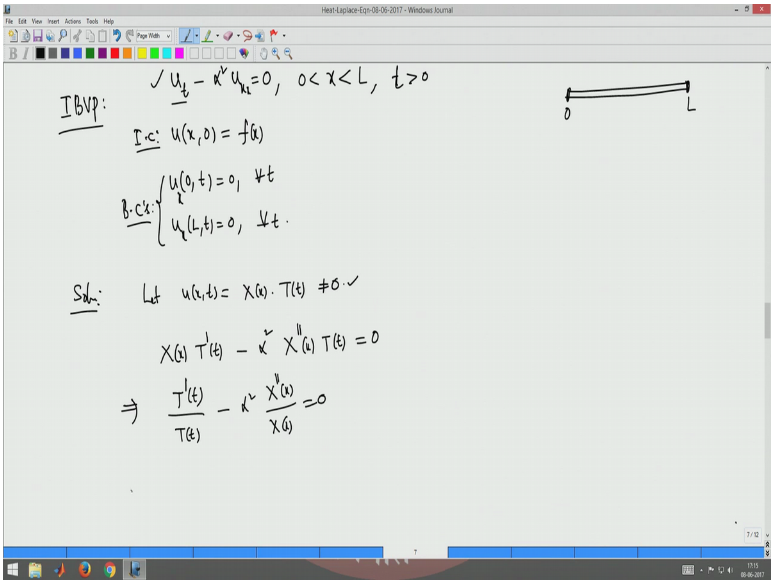

Okay so we will just write the problem as our initial boundary value problem, as satisfies along the rod ut minus alpha square uxx equal to 0, so alpha square is the normal diffusivity constant depending on the material of the rod, so this you have what you have is this 0 and L is the length of the rod so you have spatial variable is x that is L and t is for all times, 0 to t and the initial condition is this is the given initial condition t is equal to 0, rod is at some temperature fx and boundary condition is ux, the flux will be 0 so it is at 0 t, 0 for every t, this is 1, this is what is a boundary condition, okay this is the initial condition, ux at 0 not other end L t is also 0 for every t, okay. So these are the boundary conditions.

So overall this is the problem we are going to solve so this is straight forward this just because you see that the boundary conditions are having homogeneous terms so homogeneous boundary conditions that means ux or u plus ux that is what is a combination of them, right side is zero.

So right side if they are constant some other things then you have to worry you have to use the technique that I have explained in the last video so because these are homogeneous conditions so we can just take forward the working out we can work out method and the straight forward so let us see we will solve this by separation of variables so let u of x t be X of x into T of t okay, into T of t and you want a non zero solution so let us say it is a non zero solution.

Look for solution in this fashion so substitute into the equation you look for solution of the heat equation okay, so as a non zero solution, so because you, zero solution is always there so look for non zero solution substitute into the equation what you get is X of x, T dash of t that is for ut minus alpha square uxx will be X double dash of x into T of t equal to 0, okay. So you can divide it because it is non zero so you what you get is T dash of t by T of t minus alpha square X double dash of x by X of x equal to 0.

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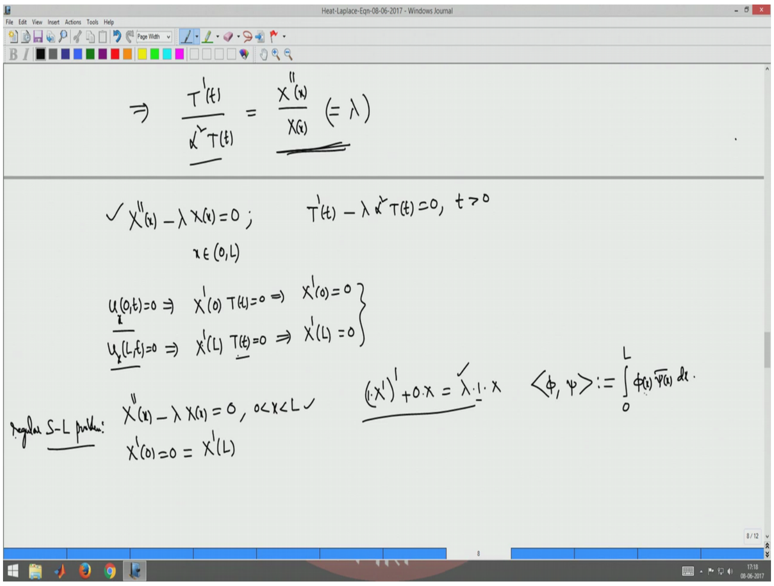

So one more step you can write T dash of t by alpha square you bring it here alpha square T of t equal to X double dash by X of x so left hand side is function of T, right hand side function of X.

So that is possible only if they are constant so should be some constant lambda, arbitrary constant so the partial differential equation that is heat equation becomes two ODEs. So this is what a special domain that is where you have the boundary conditions that will give you the Sturm-Liouville problem, so you extract the Sturm-Liouville problem for this X.

X double dash of x minus lambda X of x equal to 0 that is one, okay. Rather one is T dash of t minus lambda alpha square T of t equal to 0 this for t positive this is for x belongs to 0 to L, now if you apply the boundary conditions here so ux at 0 t equal to 0 will give me X dash of 0 into T of t that is what is this equal to 0 okay, T of t equal to 0 this will give me X dash of 0 equal to 0 similarly ux at L t equal to 0 both ends are insulated so the other ends is insulated so what you get is X dash of L, T of t equal to 0.

T of t cannot be say function that cannot be function, T cannot be 0, so what you have is X dash of L equal to 0. So this is what is this boundary conditions with this equation this is ordinary difference, this is the Sturm-Liouville problem, this together will give you the Sturm-Liouville problem, so let us write together so you have X double dash of x minus lambda X of x equal to 0, X dash of 0 equal to 0 which is also same as X dash at L so this is the Sturm-Liouville problem which is already in the self adjoint form, so this is the Sturm-Liouville problem, regular of course, regular Sturm-Liouville problem, okay.

So in your working in this Cartesian co-ordinates you can expect always regular Sturm-Liouville problem, is what by the experience okay. So you can see that this is already in the self adjoint form, this equation so x is the domain is between 0 to L this is already self adjoint form that is X dash whole dash P is 1 plus Q is 0 into x equal to lambda into 1 into X, okay so W is this immediately solutions of this Sturm-Liouville problem so you call this phi and psi the dot product will be from 0 to L that is the domain W is 1 so there is no weight so you have phi x psi bar of x dx, okay.

So bar does not matter finally because this is already in the self adjont form so implies lambda will be real so you can make, eigenfunctions are also real so these solutions will be what we use as a eigenfunction will be real so that is how we use it.

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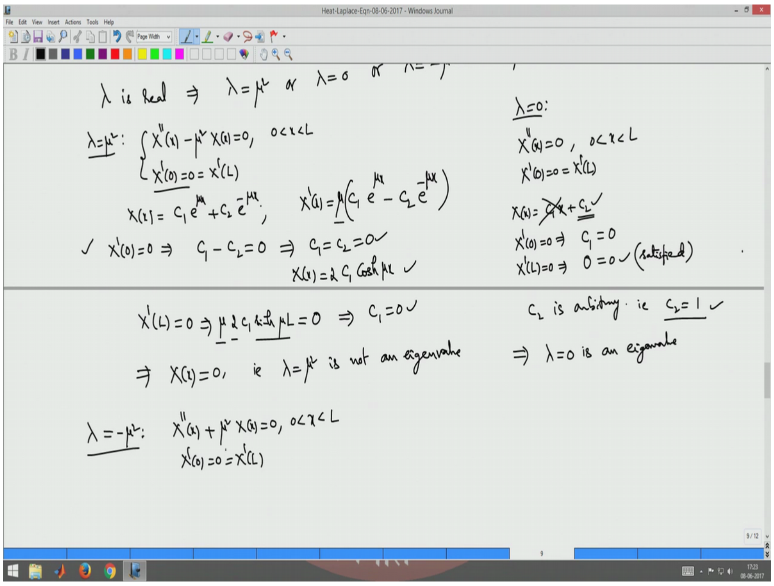

So let us see let us try to find the solutions here so because this is already self adjoint form so lambda is real so lambda is real implies lambda is mew square or lambda equal to 0 or lambda equal to minus mew square with mew positive okay.

So lambda equal to mew square if you see X double dash of x minus mew square X of x equal to 0, x is between 0 to L, X dash of 0 is 0, X dash of L, so this one what is the general solution of this, first you can write the general solution C1 e power mew x plus C2 e power minus mew x, now apply this first boundary condition, X dash of, so for that you need X dash of x, that is C1 mew e power mew x minus C2 e power minus mew X, mew is also, mew take it out.

So that is how it is e X dash of mew cannot be mew is positive so mew cannot be 0, so X dash of 0 equal to 0 implies mew times this one is 0 so you have C1 minus C2 equal to 0 so this will give me C1 equals to C2 so once after applying this boundary condition the general solution becomes X of x becomes C1 equals to C2 so that is C1, C1 take it out so you have C1 e power mew x and C2 is this is going to be cos hyperbolic mew x if you multiply with 2 so this is what it becomes, this is general solution now.

Now for this you apply other boundary condition X dash of L equal to 0 if you do this you get 2 C1 sine hyperbolic mew x into mew comes out as a derivative equal to 0, of course when you put X equals to L, mew L equal to 0, now this implies mew cannot be 0 mew is positive 2 cannot be 0, sine mew L, sine hyperbolic L cannot be 0 so implies C1 is 0. Once C1 is zero C2 is also 0, so this means X of x is completely 0, okay.

So implies X of x is identically 0 that is lambda equal to mew square is not an eigenvalue. Now you can see this lambda A equal to 0 here so for this X double dash of x equal to 0, x is between 0 to L that is what is the equation becomes and now the boundary condition x0 is 0 is x dash of L, now what is the general solution here is C1 plus C2, C1 x plus C2, now you apply the boundary condition X dash at 0, this 0 will give me X dash of x that is C1 equal to 0.

So that is what is 0 so C1 is 0 so this is gone now we apply the other boundary condition X dash at L equal to 0 will give me, so anyways 0 so C2 if you differentiate at X dash and put X equal to L that is 0 equal to 0 so satisfied it is satisfied okay, satisfied so the boundary condition is satisfied with any C2 so C2 is arbitrary C2 is arbitrary that means that is C2 can be taken as 1, okay if you take 1 non zero solution, 1 is the solution corresponding to L lambda equal to 0 implies lambda equal to 0 is an eigenvalue and so you can say that lambda 0, lambda 0 you can represent okay as a eigenvalue correspondingly X0 you can write is as 1 okay,

So that is what we will do at the end so now you see that at now look at the other cases say lambda equal to minus mew square in this case X double dash of x plus mew square X of x equal to 0, x is between 0 to L, X dash of 0 equal to 0 equal to X dash at L so this boundary value problem for the ordinary differential equation.

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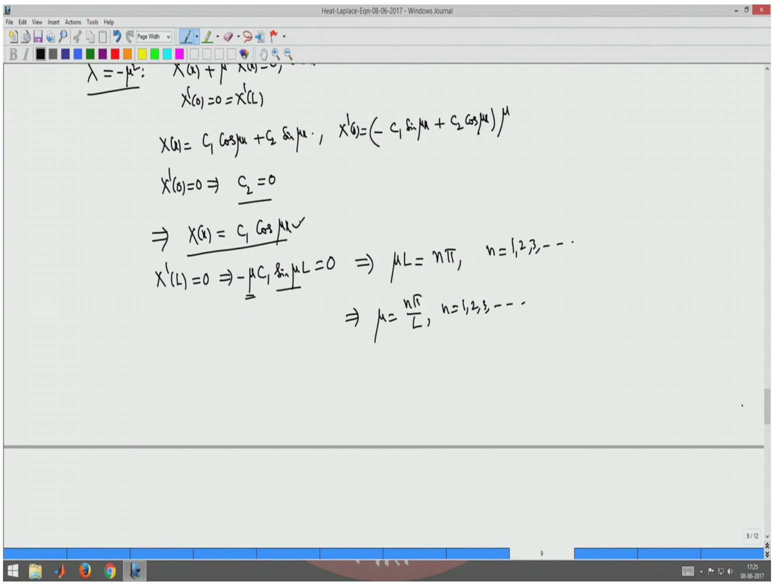

You can find the first general solution, this is C1 cos mew x plus C2 sin mew x now we apply the boundary condition X0 equal to 0 will give me so what is X dash of x is C1, minus C1 sine mew x plus C2 cos mew x into mew. Okay that is what is the X dash of x in this put X equal to 0 what you get is C2 is 0 when you put X equals to, cos mew 0, cos 0 is 1 so C2 times mew cannot be 0 because mew is always positive so C2 is 0 so this gives me X of x will be C1 cos mew x.

Now on this you apply the other boundary condition X dash at L equal to 0 will give me C1 so minus sine mew x into mew, mew comes out so you have this now when you put X equals to L mew L equal to 0 so this implies anyway mew is always positive but this quantity can be 0 for some mew, positive mew values that is mew L equal to n pi, n is running from 1, 2, 3 and so on because mew is always positive either so I am choosing n is running from 1, 2 onwards.

So this implies mew is n pi by L okay, n pi by L, n is running from 1, 2, 3 onwards these are for these values, your solution, this can be a solution cos mew x is a solution because C1 can be arbitrary.

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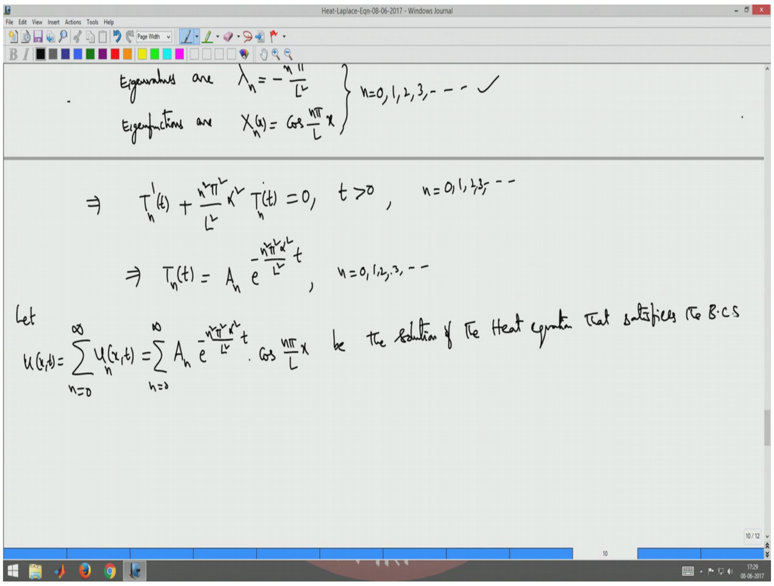

So eigenvalues are minus mew square lambda n so because it depends on n you may write lambda n minus mew square is n square pi square by L square and eigenfunctions are let us call this X n, X n of x they are cos mew x, cos mew is n pi by L x, now in both the case n is running from 1, 2, 3 and so on.

Now these are the eigenvalues and eigenfunctions in this case we have already seen that lambda equal to 0 is an eigenvalue, that means if you put n equal to 0 lambda becomes 0 lambda n is 0 so I can actually include 0 here that earlier one lambda equal to 0 by just by writing n equal to 0, 1, 2 so include 0 lambda n is 0, lambda 0 is 0 that is what we have seen lambda is an eigenvalue.

Corresponding function when you put n equal to 0 cos n equal to 0 pi by L x is actually 0 that is cos 0 that is 1, so that is what is the eigenfunction so I include these are the final eigenvalues, okay, these are final eigenvalues and eigenfunctions.

So this means I have un of x t equal to X n of, no once you have X so you solve this X problem now you can solve for P problem so this P problem by putting lambda equals to eigenvalues so T dash of t minus lambda what happens this T dash of t minus lambda is n square so for each n I have this lambda n so I am calling this T n dash of T minus lambda n that is plus n square pi square by L square into alpha square T of t, T n depending on n so I am putting T n equal to 0 for t positive, okay.

So this is running from n is again 0, 1, 2, 3 onwards so for this we can see that T n of t will become simply for each n you have an arbitrary constant An e power minus n square pi square alpha square by L square t okay and is running from 0, 1, 2, 3 onwards. Now you can write this un of x t so for each n, un of x t is X n of x into T n of t that is A n e power minus n square pi square alpha square by L square t.

This is T n of t into X n of x is cos n pi by L x and you take the super position of, so this one is the solution that satisfies the boundary condition so far okay each is the solution for each n equal to 0 to 0, 1, 2, 3 onwards for each n this is the solution that satisfies the boundary condition.

You superpose all these solutions that is running from 0 to infinity, n is from 0 to infinity so this is also solution assume that this is solution actually you cannot say this is a solution only if this this series is uniformly convergent one can actually show by applying the initial condition so one can actually show that this series is uniformly convergent, okay.

Once you know this apply the initial condition formally and get what is A n and then you can actually show that the series is uniformly convergent because it is uniformly convergent now you are saying that it is you can differentiate term by term and then put it into the equation it solves because each term is satisfying the heat equation on the boundary conditions okay that is why this superposition principle works.

So super position of all this solutions will call this u of x t let this u of x t be the solution of the heat equation that satisfies the boundary condition, okay so far we have not used the initial conditions.

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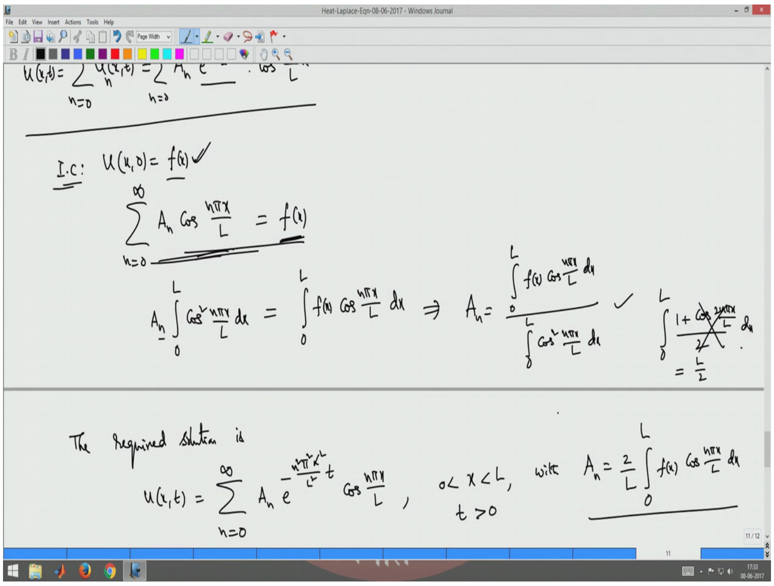

So we can now apply the initial condition. Initial condition is what is the initial condition how, u at x 0 is equal to fx, so from this we can see that n is running from 0 to infinity A n, e equal to makes it 1 cos n pi x by L equal to x.

Now you can use the dot product of the solutions of the Sturm-Liouville problem, these are eigenfunction so make it dot product with the eigen function both sides cos n pi X by L both sides and make a dot product so the right hand side becomes integral 0 to L fx cos, these are real values functions so bar does not matter, so you have cos n pi x by L dx okay right hand side.

The left hand side again it will become A n integral 0 to L cause square n pi x by L dx so this is what it becomes because all other n that is not actually this n will become 0 because cos n pi x by L and cos n pi x by L dx will be 0 because integral 0 to L that is 0 because they are complete orthogonal eigenfunctions so completeness, actually the completeness means when you apply this that is what is actually a completeness so fx any fx I may able to write terms of this okay so they form complete (())(18:39) so we have not use really completeness actually naturally this initial condition is taken care okay.

That means fx I am able to any fx I am able to write in terms of these functions so the natural tells you that these eigenfunctions are complete, complete means any function I am able to write in terms of them as a series so what we use only once we get this form, that is coming from directly naturally from the initial condition you apply the dot product and get your A n so this gives me A n as simple this 0 to L this calculations you can do so f is given, f that can be anything so this you can calculate n pi x by L dx with the denominator you can calculate left hand side, integral 0 to L this cos square n pi x by L this you can do it okay so this is with this A n the solution, the solution is this.

This is your, the this solution u of x t satisfies the heat equation and the boundary condition and the initial condition so this solves the problem so the required solution is u of x t is simply this sum n is from 0 to infinity this is A n e power minus n square pi square alpha square by L square t times cos n pi x by L so this is what this x is between 0 to L and t is positive okay so this is your domain so in this domain you have this, with the A n is as above so A n you can simplify this, this is like 1 plus cos 2 n pi x by L okay so you can write divided by 2 okay this you are integrating from 0 to L dx sin thing will become 0 so this part will, this contribution will not be there so what you get is, is simply L by 2.

So you have what you get is 2 by L, 0 to L fx cos n pi x by L dx, this is actually cosine series, what you get is the cosine series, fourier cosine series we are actually using here. So in the earlier problem when the both the ends are maintained at 0 temperature that is where you are using the sine series okay, fourier sine series so this is how you solve this insulated problem.

Now you can also work out the same way if you allow the heat exchange at the both the, at the boundary points at 0 and L there you will not get explicitly your eigenvalues and eigenfunctions. Eigenvalues satisfy some dispersion relation, some relation okay so if you have some transcendental equation so that satisfy.

So whatever you can actually some transcendental equation that has the roots, those roots are eigenvalues so you cannot explicitly find them then numerical you can, one can find so if you label them as lambda 1, lambda 2, lambda N, they are eigenvalues correspondingly your solutions you call them so the corresponding solutions you will be some either in terms of cos or sine that lambda N so you do not write lambda, you do not replace so like here cos lambda X is the, cos mew X, mew you replace because mew explicitly found here there you will not be able to find mew X explicitly you are calling lambda N itself.

So in that case, cos or sine whatever form comes their mew replace with lambda n so those are your eigenfunctions so corresponding, with, by replacing lambda with that lambda 1, lambda 2, lambda N, you can find T n of T and make a product and superposition and finally apply the initial condition with the dot product, whatever defined okay, there you can have dot product from the Sturm-Liouville, you see what is the dot product when you extract the Sturm-Liouville problem.

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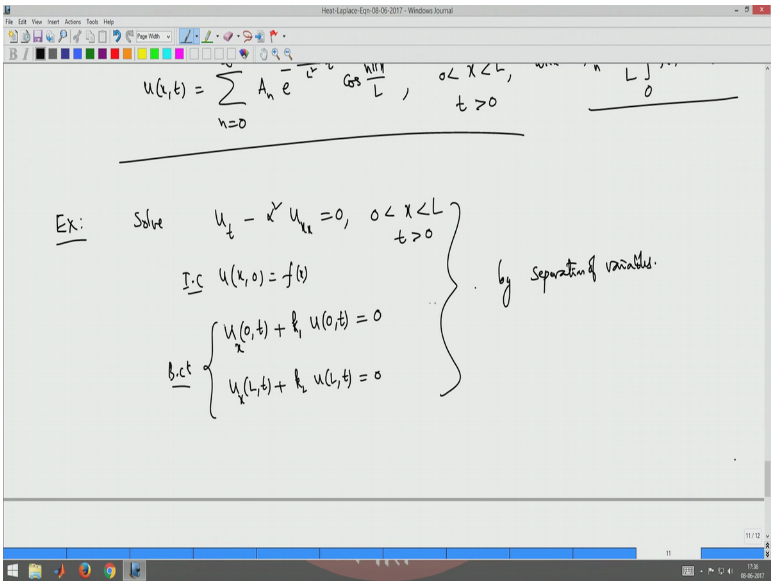

So then find the solution for the problem with the heat exchange problem so you can work out I will just write it as a problem so x, I will write it as an exercise, solve this problem ut, this initial boundary value problem ut minus alpha square, uxx equal to 0, x is between 0 to L, t is positive and initial value is u at x 0 equal to rod is at the temperature fx, boundary conditions is now ux at 0 t plus some k okay some exchange with this constant k1 so this proportional at 0 to t so heat is going out or coming inside some exchange is happening at the boundary at x equals to 0 this is the one boundary condition, other boundary condition is similar one, L t plus some other constant k2 u at L t equal to 0 so these are your boundary conditions, initial conditions here so that is your heat equation.

So this together you can solve this by separation of variables okay because the domain is, special domain is finite so this is how you can solve this heat equation okay, so in the next video we will try to, so far we have dealt with wave and heat equations in unbounded domains and finite domains okay and.

If you consider so far we have actually considered 2 dimensional wave equation and we solve in a circular domain okay as a drum problem we have solved just to demonstrate you that how do we use the Sturm-Liouville, how to extract the Sturm-Liouville problem that is the (())(24:29) type of Sturm-Liouville, similar Sturm-Liouville type of system you can extract and try to solve it okay for the wave equation.

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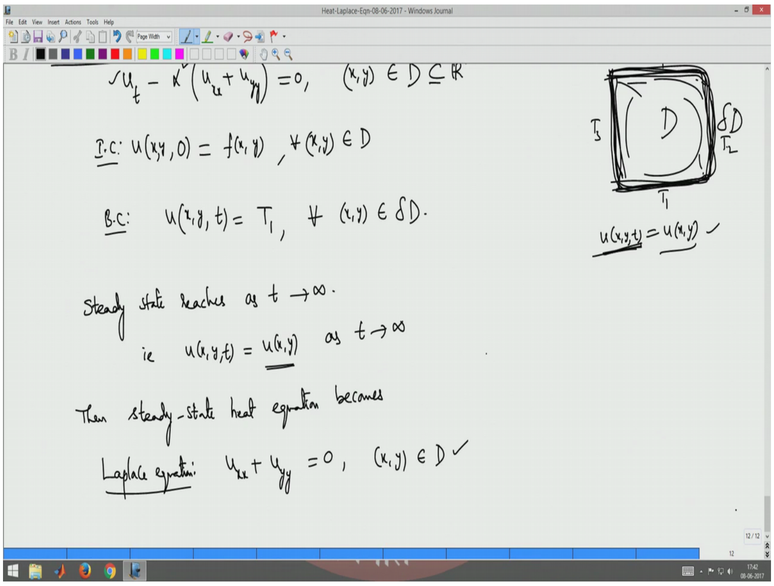

Heat equation we have so far only seen the 1 dimensional heat equations so if you actually see the 2 dimensional heat equation that is like, what is that heat equation so if you see if you instead of considering the rod if you consider a plate so let us say 2 dimensional plate okay so let us say 2 dimensional plate if you consider so what you get is everything is same except ut minus alpha square instead of uxx what you get is the uxx plus uyy.

So this is the 2 dimensional heat equation, equal to 0, x y belongs to the plane let us R2 so if you have infinite plane that is what you have or you say some domain D that is part of plane okay it can be even same as plane itself so if you have this like this and you need to provide plate is at the temperature, initial temperature x0, xy. So now it is a function of xy right and t equal to 0, u is now is a, U is a function of x y t okay, u is like this so u satisfies this, this equation and this initial data x y t equal to 0 that is now you can give some f x y, f depends on this and then so where for every x y belongs to D this satisfies this.

This is your initial condition and boundary condition is you can provide, depends on the boundary. So boundary if you call this is your D, delta D if you call the boundary you can say some you can give any boundary condition you can maintain the plate at some temperature at the boundary so that if you say on the boundary xy for all T equal to 0, if you maintain some temperature let us say T1 for every xy belongs to this boundary delta D okay so this is the 2 dimensional heat equation so this is the initial boundary value problem for the 2 dimensional heat equation.

So this also one can solve using, so let us not do this one as a heat equation we will not solve because it is involved now what we do is when you have a plate like this initially at some temperature and you maintain some boundary, boundary keeping, maintaining the some temperature at the boundary.

So what happens after some time it reaches the steady state right, so that is what we have seen as t goes to infinity, maintain keep maintaining at 0 temperature initially at some heated plate eventually it will become 0 that is the steady state condition so eventually when you as t goes to infinity u of x t, x y t becomes simply 0 okay that this right hand side this whatever the function as T infinity for larger T, this is simply function of xy okay.

There is no time, it is not changing this simply stagnant, reaching the steady state that means u of x y t is actually u of x y itself okay so that means it does not depend on T steady state reaches the steady state if you maintain the temperature 1, this is at T1, this side is T2, this is T3, T4. So you will have some kind of steady state combination of T1, T2, T4 so that is so some linear layer whatever you may get all, as T goes to infinity that is what you see okay if you maintain at the temperature T1, T2, T3, T4 for the boundary.

So in any case you reach the steady state okay, steady state reaches as t goes to infinity. So what is the steady state means that is u of x y t the temperature which depends on t becomes function of x y as t goes to infinity this that does not depend on t okay so if that is the case what is ut, ut becomes then the equation becomes steady state then steady state heat equation is, heat equation becomes what happens ut, ut is actually now function of, it does not depend on t that is 0, minus alpha square now uxx plus uyy equal to 0.

Okay alpha square this cannot be 0 so this plate parameter so you have this is 0 this is nothing but your laplace equation, so laplace, is the laplace equation for the temperature. So steady state temperature satisfies laplace equation okay so u is only depending on xy so for every xy belongs to this D okay. So this is the laplace equation, this is what we can solve okay.

So we can solve so though you are actually as you are if you are next in the next few videos we will solve methods to solve boundary value problems for the laplace equation because there is no time we do not give any initial condition so only boundary will be there you have to provide the boundary data.

So when you are solving the laplace equation actually dealing with 2 dimensional heat equation which is at the steady state so in that sense we are actually dealing 2 dimensional heat equation okay so this is what we will see in the next video so we will try to solve methods to find solutions of the laplace equation with the boundary data okay and both Cartesian co-ordinates, you start with the Cartesian co-ordinates and then we will move on to circular domain or where different kinds of forms okay different kinds of circular domains what are all possible domains we can solve we can solve by this separation of variable technique okay in which we can actually the main idea is to extract Sturm-Liouville problem and then make use of eigenvalues and eigenfunctions for the superposition of these solutions okay get the solutions and its superposition finally get the solution in a separable form this is what we will see in the next video. Thank you very much.