Differential Equations for Engineers
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Lecture – 57
Temperature in a Finite Rod

So in the last video we have seen how to solve boundary value problem, initial boundary value problems for the heat equation in infinite domain as well as semi infinite domain. We have shown that solution exists, we have actually constructed the solutions in all these domains and then in this unbounded domain whether fully infinite or semi infinite domains for the one dimensional heat equation.

We also have seen that this solution is unique that is what is shown, so solution is we have a unique solution for all this problems whatever we have construct so far for the heat equation as well as the wave equation that whatever we have considered in the earlier videos. So a problem, a boundary value problem whether is a initial boundary value problem or boundary value problem that is well defined if the solution exists and it is unique and also there is a condition that is called if you have whatever the initial data and boundary data you (()) (01:20) little bit.

So this is what is the data you are given, you are given a data that how do you find that, that is normally practically you can get the data at time T is equal to 0 or on the boundary you can calculate, you measure it and try to see it try to get that data, that data when you try to get it that may not be exact data so it may be having some small error very little so you may have error maybe very very small but it is not suddenly the exact data, exact boundary data. So even if it is not exactly correct boundary data, even if you (())(01:57) little bit so that means you take boundary data with little error.

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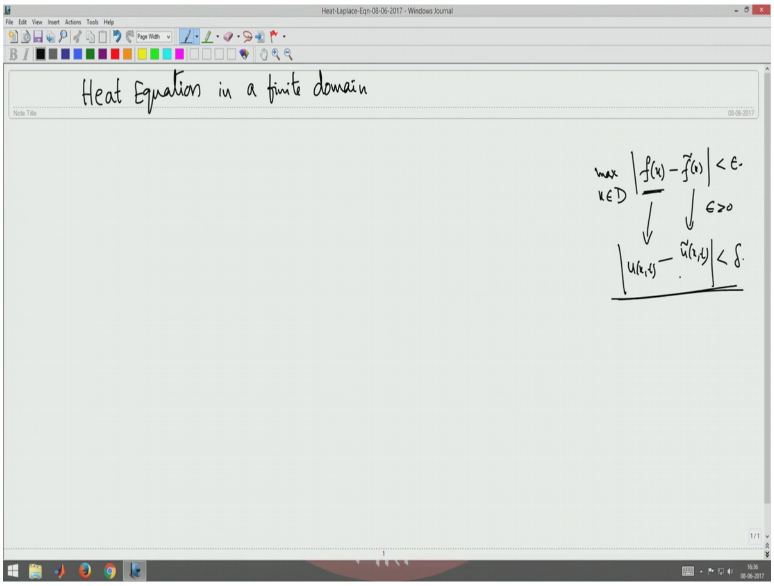

Suppose fx is the boundary, now initial data you take some f tilde very close, let us say this distance, this maximum of this wherever this X belongs to the domain, okay so let us say if this is very very small epsilon, epsilon is positive, such a thing instead of fx, exactly fx, fx you have the solution, okay.

Correspondingly with this initial value fx you have a solution u of x t, now if you calculate, if your initial data is f tilde of x that is very close to f but it is not exactly f, in such a case you have solution let us say you know how to find the solution that is say u tilde of x of t. So you change the initial data there is very close to the actual data so then you have a different solution, then problem is well defined if the whatever you found this solutions, these solutions are also very close, they are also very close to each other.

So let us say some delta, okay so the delta depending on epsilon, so okay, so you choose your data very close to f then correspondingly the solutions also will be closer to each other, so this is called continuous dependence on the initial data or the boundary data, so once this condition is also satisfied then you say that the boundary, initial boundary value problem are, initial value problem is well defined.

So for all those problems whatever we have done this can be done this whatever you change the initial data, you (())(03:42) up the initial data and still you can see that corresponding solutions will be close to each other, that analysis we do not do in this course so you can assume that the actual that is also true, so all this boundary value problems are initial boundary value problems are initial value problems, these are all well defined in that sense.

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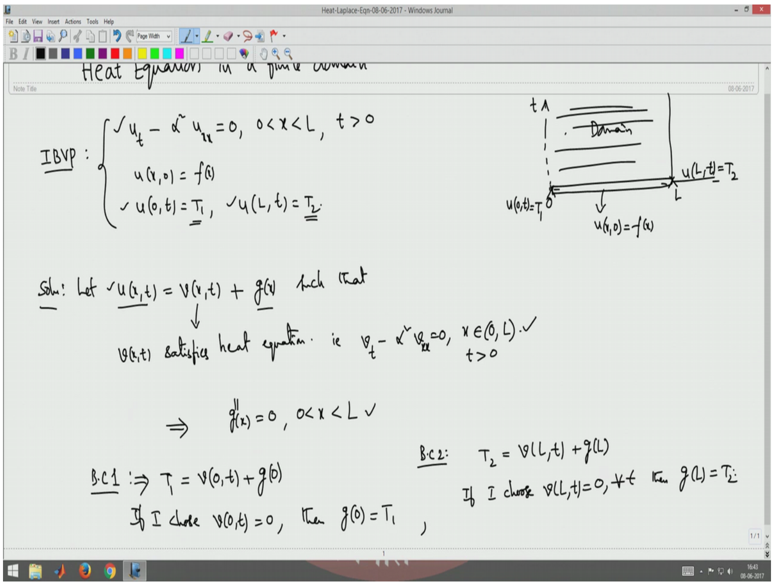

So that is what we have seen in the last few videos, today we will have a heat equation, we will have a solutions for the heat equation in a finite domain so one dimensional heat equation we can consider and we consider over a finite domain so that is associated with this finite rod.

So you take the finite rod that is heated initially, insulated latterly and it is heated initially you have a temperature u at x 0 equal to some fx. So initial data you have, now we have the boundary here so at this points you have a boundary, you can provide the temperature so you fix this boundary data saying that at x equal to 0 and this is L, 0 to L is the distance so this is the length of the rod, L is the length of the rod so you have 0 for all time, you maintain some temperature let us say T1, here also at this end you also maintain u at LT for all times, you maintain a temperature T2.

Then we can find the solution for all times in this rod, how it diffuses when you have, when you maintain the temperature T1 here T2 here, initially rod is at the temperature fx. So let us see let us write this initial boundary value problem, so let us write this first, so this is a heat equation, satisfies the heat equation so you have temperature satisfies the heat equation.

Use a function of x and t, ut minus alpha square, alpha square is a constant, then uxx equal to 0, now x belongs to 0 to L and t is positive. So in this domain, so that basically if you say x and t, this is your t, x is this, x is 0 to L, so you are in the actual in the kind of strip, you are in this, you are, this is your domain so this is your domain so t is positive and this is between 0 to L so this is your domain, so this domain, this heat equation is satisfied and you have the initial data, initial data is u at x 0 is fx and the boundary data is u at 0 t equal to T1 and u at L t equal to T2 so how do we solve this one.

So this is what this what we do is we try to break this problem into this initial boundary value problem into two problems, how do we do this, solution let us see, so basically if I can do, see if I want to extract Sturm-Liouville problem you need the boundary conditions to be homogeneous that means I need these boundary conditions to be 0.

So in order to make it 0, I break this solution u as let us say let u is I want u of x t as some v of x t plus let us say some g of x, only x okay, so this is a trick so you just try to break this into two problems so in this once you break this, this satisfies the heat equation okay so if you impose that so let u be this such that v satisfies heat equation, v of x t satisfies heat equation.

When that is vt minus alpha square, vxx equal to 0, x belongs to 0 to L okay and t positive so this is what is the domain and then what happens so if since u also satisfies the heat equation v is satisfying the heat equation then g should also satisfy the heat equation. So g is the function of x when you pull it in the heat equation it becomes gt, so time derivative of g will be 0 so you have minus alpha square gxx is also 0, okay.

So that means this is simply function of x so is g double dash of x equal to 0, alpha square is constant so this cannot be 0, okay so this implies you have this g double dash if 0 say again the domain is x belongs to 0 to L, so this is what happens for v this is equation for g this is the equation. Now you look at what happens u at x 0 is fx okay and now let us first apply this boundary conditions, if you apply this boundary conditions, if you put x equals to 0, if you put x equals to 0 here, u at 0 t is T1 which is equal to v at 0 T plus g at 0 okay.

So I choose now again I choose here I choose v 0 t as 0 okay, so choose v of 0 t equal to 0, so once you see this immediately then once you choose this if I choose v of 0 t, 0. Then g of 0 is T1, okay, boundary condition one gives you that, gives me this one. Now you do the same thing for boundary condition two that is u at L T that is T2 here okay so T2 equal to v at L t plus g at L, okay. So now again I choose, if I choose v at L t as 0 for every t then g at L has to be T2.

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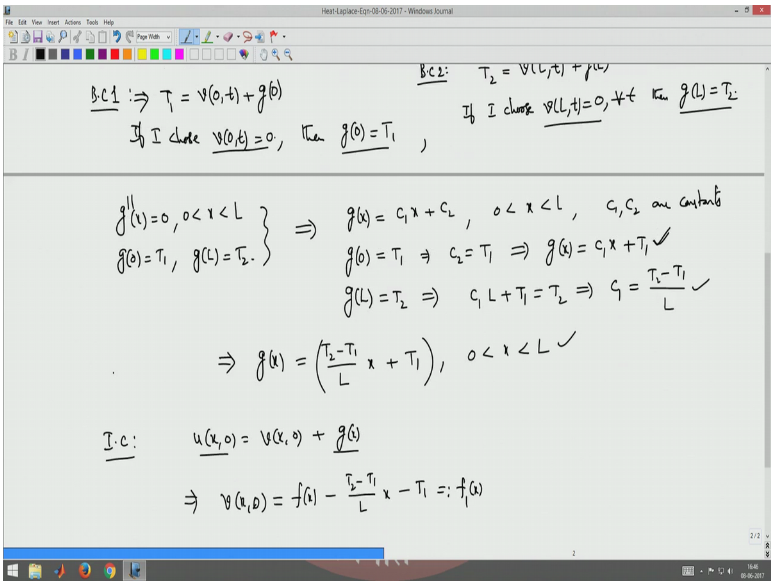

Now just combine what you have. So what you achieved as actually, now you write the problem for v where v satisfies heat equation, v satisfies this boundary condition, this boundary condition and what happens to the initial condition, initial condition you can easily see.

So before you give the look find the initial condition for v first, so v is not completed okay we know only heat equation and some boundary conditions like this, these are the boundary conditions what you have okay, so I have a boundary conditions here this is the one and this is the one so have two boundary conditions and heat equation what we have for v but for g, I have this ODE two derivatives is 0 and this is one condition and this is another condition so that solves okay.

So write first g, g dash of x equal to 0 if x is between 0 to L and g0 is T1, g at L equal to T2. So this how do I solve, so we can solve this so you what is the g double dash of x so if you simply integrate g of x will be some arbitrary constants into x plus C2, C1 C2 are arbitrary constants, x is between 0 to L, C1 C2 are constants, constants you can find. So apply this boundary data so g at 0 equal to T1 so this will give me C2 equal to T1 and now, now what happens so immediately what you get is g of x is C1 x now C2 is T1, so that is what you have.

Now you write, now apply the other boundary conditions so g at L equal to T2 so this will give me, now this is your general solution, g at L that is C1 at L plus T1 equal to T2 so this gives me what is my C1. C1 is T2 minus T1 divided by L. So now you this you go on substituting to this to get so this immediately give me what is my gx, gx is, C1 is T2 minus T1 by L x plus T1, this is what you have so this is why gx, x is between 0 to L satisfying this boundary conditions okay now you apply your initial data your initial data u at x 0, u at x 0 is fx so if you apply initial data here okay so u at x 0 is fx.

fx equal to initial condition will give me u at x 0 equal to v at x 0 plus gx, so this I know already now okay, this is what is gx. So this implies now I have v at x 0 is, u at x 0 is fx minus this gx is this one okay minus gx, gx you can write if you want so T2 minus T1 this is given known into x minus T1 so this you call some f1 of x just by definition okay. This you can define as a definition so this is your (())(13:20)

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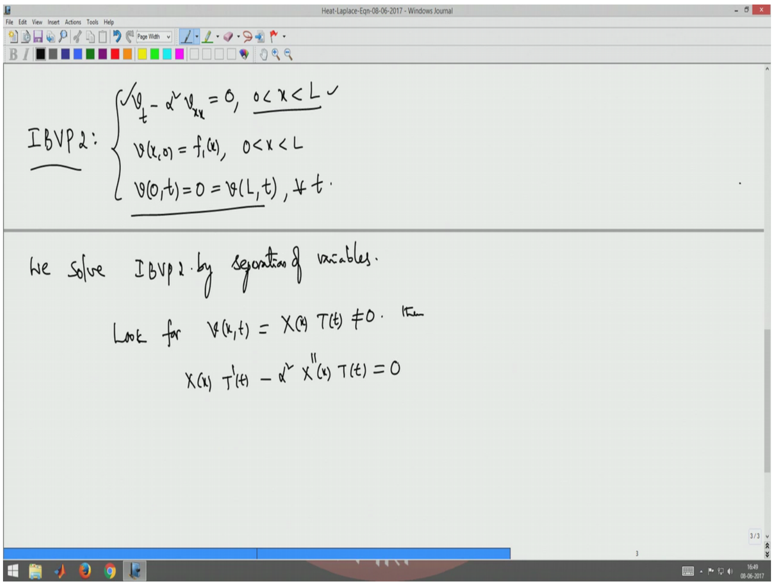

So now I know what is my g, now I know what is completely g, v I do not know, v is actually now satisfying this heat equation this boundary conditions now this initial condition, okay so let me put this problem so what you have is vt minus alpha square vxx equal to 0, x is between 0 to L. v at x 0 equal to f1 of x, f1 is known now because f phi is given, T1 T2 are given so this is this function I know.

L is known, L is given so this is the length of the rod, so you have f1 x so between 0 to, x is between 0 to L and then boundary conditions are v at 0 t equal to 0 now okay these are 0 this is our chosen equal to v at L t. So this is what it has become initial boundary value problem two okay.

If you call this as one, this is one now the initial value, initial boundary value problem one, you want to solve that you reduce to initial boundary value problem two, so that is easy to handle because now I have the boundary data which is 0, 0 homogeneous boundary data okay, this is true or every t. So this is what we solve by the separation of variable method.

So we will solve this, we solve, we solve for v of x t, we solve rather I, initial boundary value problem two by separation of variables, because the boundary is finite okay, so domain is finite so you can easily say so look for solution like this form so v of x t as X of x and T of t, as a non zero solution, because I am only now looking for the solution of the equation so because it is a homogeneous equation you can always look for 0 is the solution so but I am looking or a non zero solution okay.

So you look for a non zero solution then you simply substitute vt in this form, X of x, T of t into the equation so you get X of x, T dash of t minus alpha square X double dash of x, T of t equal to 0 okay.

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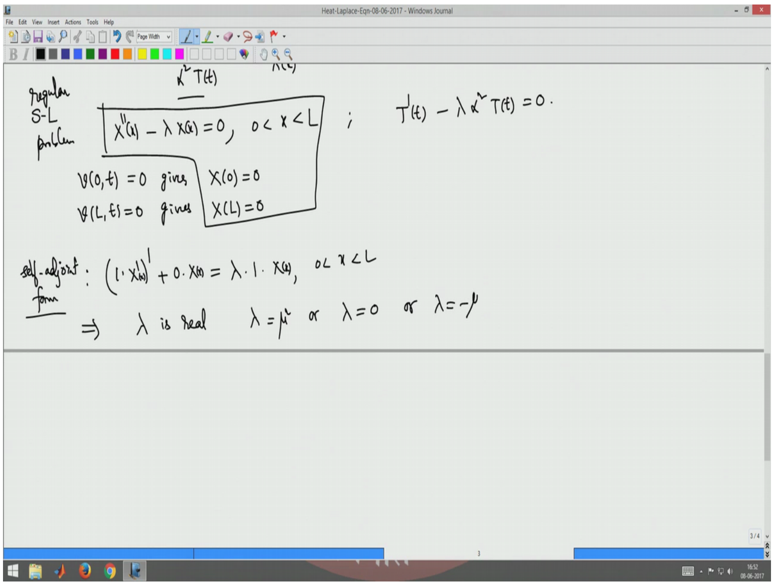

Now because this is non zero I can divide both sides with, I can divide both sides with v of x t that is X of x into T of t so that will give you T dash of t by T of t. Now I bring this alpha square here so alpha square equal to now if you this one will be X double dash of x, T of t will go, so X of x will remain here, now left hand side is function of T, this is the function of X okay, so that means it has to be constant so that is what you have, so now the partial differential equation that the heat equation becomes 2 ordinary differential equations one is X double dash of x minus lambda X of x equal to 0, x is, domain is again 0 to L this is the finite domain, this is where, this is our extracted the Sturm-Liouville problems, Sturm-Liouville type okay and what is other one, other one is this one so this will give me T dash of t minus lambda alpha square T of t equal to 0 okay.

So apply the boundary condition now, if you apply the boundary conditions so if you apply boundary condition v of 0 t equal to 0 gives X of 0 is 0, v at L t equal to 0 gives X at L equal to 0, so this is what is the problem now this problem is your Sturm-Liouville problem, Sturm-Liouville regular, Sturm-Liouville problem so this is how you extract it okay.

So this is your regular Sturm-Liouville problem this already in this self adjoint form so this word we know. This is this, this is X dash P is 1, this is dash that is X double dash, Q is 0 okay plus 0 times X equal to lambda times 1, W is 1 and here X, X of x, X means X of x so this what X of x, you can write X of x, X dash of x, this is actually x belongs to 0 to L so this already in the self adjoint form.

Once it is a self adjoint form, you know that it is the eigenvalues are real so this already self adjoint form or right, Hermitian form, say Hermitian form right, self adjoint form. So implies lambda is real, lambda is real means immediately lambda is either mew square or lambda equal to 0 or lambda equal to minus mew square so you look at these 3 cases.

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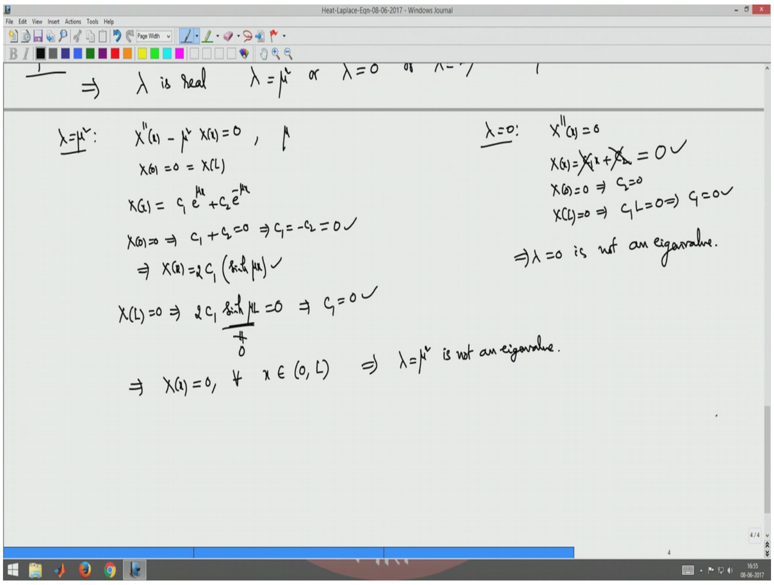

So first start with the lambda equal to mew square, lambda is mew square so what you have is this equation so X, X of x, equation is X double dash of x minus mew square X of x equal to 0 and then with the boundary condition X of 0, X of L equal to 0 so this one how do you find the general solution of this is c1 e power mew x plus c2 e power minus mew x.

Now we apply X of 0 equal to 0 will give me c1 plus c2 equal to 0 so this will give c1 equal to minus c2 so this implies X of x is c1, c1 is minus c2 so you have c1 take it out so e power mew x minus e power minus mew x so you can say this is 2 times that is sine hyperbolic mew x okay.

Now you apply X at L equal to 0 will give me for this, now the general solution is this now you have a 2 c1 sine hyperbolic mew L equal to 0 because sine hyperbolic function is 0 only at 0, mew L equal to 0. Mew cannot be 0 since mew is positive okay mew square is positive so this is with mew positive so mew is always positive so it cannot be 0, this is non zero so implies c1 is 0.

So c1 is 0, c2 is, when c1 is 0, c2 is also 0, okay. Immediately what you have is the general solution becomes completely 0 for every x, x belongs to 0 to L implies lambda equal to mew square is not an eigenvalue, okay. So you can all do you can also do for lambda equal to 0, let me test lambda equal to 0 in this case the equation becomes, so this equation becomes X double dash of x equal to 0 so you have Y X, general solution of this is c1 x plus c2 and then you apply X of 0 is 0 will give me c2 is 0 so this is gone and then X at L equal to 0 will give me again c1 L equal to 0.

This gives me c1 is 0. So that means this is also 0, so implies completely 0, X of x is 0 implies 0, lambda equal to 0 is not an eigenvalue, okay. So that means there is no non zero solution for this problem with this boundary condition, so with this lambda.

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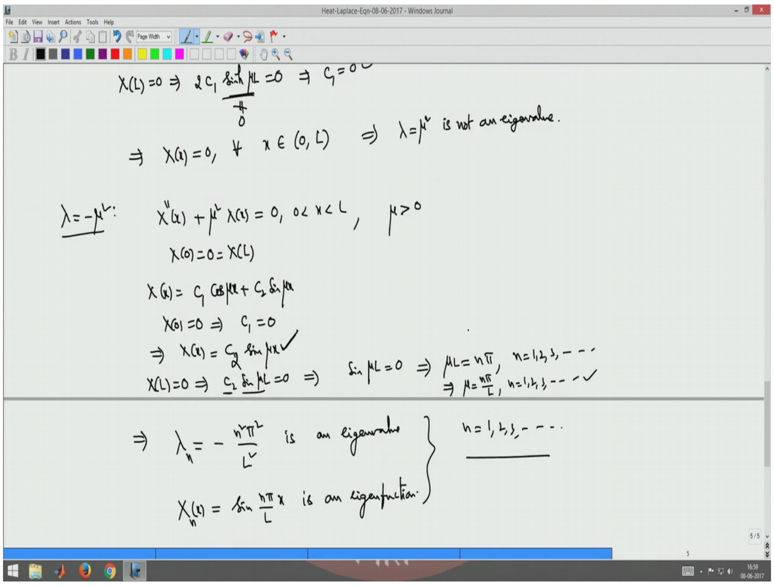

Now we have left with only lambda equal to minus mew square so this will give me, this eigenvalue so you have X double dash of x, now you have a plus mew square X of x equal to 0, x is between 0 to L, X of 0 equal to 0 which is except L so what is the general solution here because this is minus mew square, if you look for e power IK, e power K X kind of solution it will become, K becomes K is plus or minus I mew so that you have e power I mew X and e power minus I mew X (())(22:15) independent solution so similarly if you add and subtract you will see their sine mew X and cos mew X are solutions. So you have general solution is c1 cos mew x plus c2 sine mew x, now you apply this boundary conditions, X 0 is 0 will give me so apply the boundary condition this gives me c1, c2 into sine that is 0 okay so C1 equal to 0 okay.

So this immediately X of x is nothing but c1 cos, c1 is 0 so what you are left with is c2 sine mew x, so this is what is the general solution, now you apply other boundary condition so if you apply the other boundary condition that is X at L equal to 0 will give me c2 times sine mew L equal to 0. So sine function can be 0 for certain mew values, so here mew is positive and so c2 for those mew values for which sine mew L equal to 0, c2 can be arbitrary that means sine mew x such that mew is satisfying sine mew L equal to 0 will be eigenfunction, okay.

So you can see that sine mew L can be 0 implies either c2 is 0 or sine mew L is 0, c2 is 0 means completely 0, X of x is completely 0, it is not an eigenvalue, okay. But there are certain mew values for which sine mew L is 0 so sine mew L is means mew L equal to n pi, mew is always positive so n is running from 1, 2, 3 onwards. So this implies mew is n pi by L, n is from 1, 2, 3 and so on.

By the way I missed something here so when you as soon as you write this in the, in this form immediately you can see the dot product of this functions here, okay, solutions, dot products of the solutions of this let us see some phi and the psi dot product I can define as dot product is defined as integral 0 to L, W is 1 so there is no weight function and you have a phi x, psi x bar dx, because these are all real so eigenfunctions are also will be real value function so bar does not matter. So this is what is this.

So now you have this, these are for these values you have sine mew L is 0, c2 can be arbitrary implies lambda equal to minus mew square that is n square pi square by L square is an eigenvalue for n is from 1, 2, 3 and so on and the corresponding so because it depends on n and let me call this lambda n and corresponding solutions, let us call this phi n, okay phi n or X n, X n of x these are the eigenvalues, that is when you have for the when you put M equal to n pi by L sine mew x so sine mew is n pi by L x, sine mew x okay.

This depends on n so I am putting X n, X of x as X n is eigenfunction, is an eigenfunction. Eigenfunction corresponding to this lambda n so for n is, so in both the cases n is running from 1, 2, 3 onwards. So now we find this eigenvalues and eigenfunctions so I know what is my X, for these two problems okay I have one problem here, I have another problem here so for this I know the eigenvalues and eigenfunctions.

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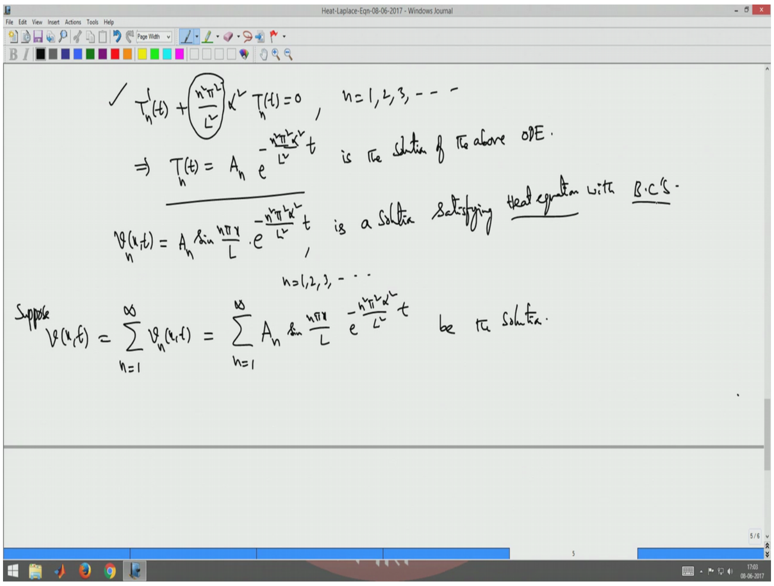

So I have X n that satisfies so that same lambda you can put it here so what happens T dash, T dash of t minus lambda is minus mew square, so plus mew square so mew is n square pi square by L square into alpha square okay so lambda into alpha square so you replace, put this lambda into T of t equal to 0 so this is first order ODE, linear ODE.

So I can solve it directly so the general solution of this is, so this see because it depends on for each n, n is running from, replace this lambda with lambda n so I call this the ODE for each n I have a different each ODE okay corresponding to n I have one ODE satisfying this, I have a ODE so I can say that is T n of t that is some arbitrary constant A n depends on n okay, fix n so A n is fixed. So A n times so A n is corresponding to that ODE arbitrary constant A n times e power minus n square pi square by L square alpha square t, does it satisfy?

Differentiate n square pi square alpha square by L square okay and this will be this minus of that will be there so this will be plus so 0, okay. So this is your solution, so this is the solution, is the solution of the ODE. So now what you have so for each n my v of x t for each n depending on n so I am calling this vn as X n of x into that is sine n pi x by L into this T n of t that is A n times e power minus n square pi square by alpha square by L square into t so this is what you have now I take this sum so this is a solution, this is a solution satisfying heat equation with boundary conditions, clearly right.

So far this is each of this is satisfying the heat equation with the boundary condition so I superpose all this solutions or each n okay, n is running from 1, 2, 3 onwards so if I superpose this solutions this sum is also I can assume that is a solution that you call it V of x t, V of x t as a superposition for all this solutions n is running from 1 to infinity that is n is from 1 to infinity A n sine n pi x by L e power minus n square pi square alpha square by L square t, okay.

As superposition be the solution now this solution because it is a superposition we will assume that suppose this is a solution superposition we can, we can actually show once you get this constants one can show that is actually converse this uniformly so that means actually you can assume superposition principle if you apply this is the solution so you can assume that this is the solution that satisfies the heat equation with the boundary condition.

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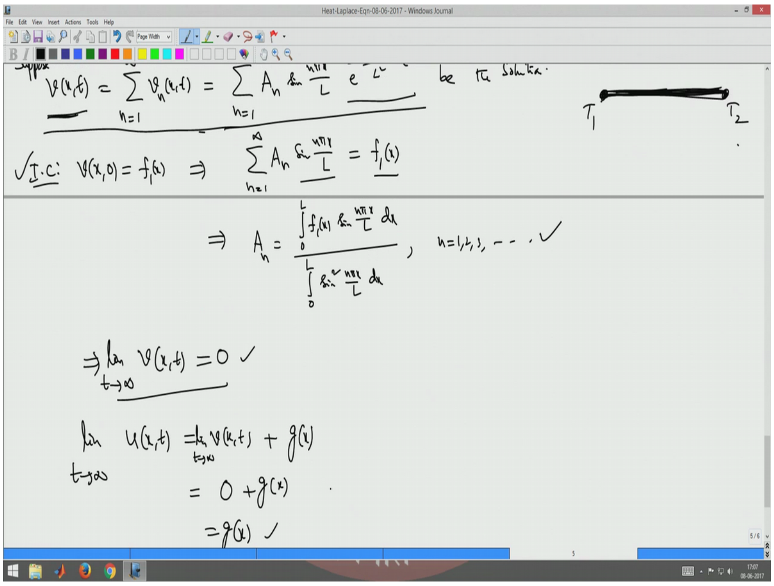

Now we apply the initial data that is initial condition is, so far we have not used so v of x0 equal to f1 of x, f1 of x is fx minus g of x that you have evaluated earlier, so implies v of x0 is this sum n is from 1 to infinity A n sine n pi x by L this when you put equal to 0 that is 1, so this is equal to f1 of x.

Now sine n pi x these are all eigenfunctions these are Xn of X so these are all this, this are orthogonal, these are complete orthogonal set. That is what from the Sturm-Liouville theory so this implies I can get my A n simply multiply the same eigenfunction and integrate so that means you take the dot product with the eigenfunctions both sides so you can see that integral so dot products with f is f1 of x right hand side, f1 of x into sine n pi x by L dx divided by integral 0 to L sine square n pi x by L dx okay this is what remain now n is from 1, 2, 3 onwards.

So this solution vx this is the solution where with A n are this is actually now satisfying heat equation boundary condition and the initial condition okay this is the required solution, so this is what you need, this is how it behaves you can see physically as t goes to infinity suppose you have a rod that is initially some temperature and you maintain constantly at T1 and T2 okay so what happens eventually even though you maintain a temperature T1 and T2, what happens to V, V solution is, by the way V solution is boundary terms are 0, boundary you maintain at temperature T0, 0 0 here and as T goes to infinite after some time and whole temperature of the rod becomes, is cool down.

So once the heated rod and both the ends are everything is insulated even the, you maintain both the ends of the rod is at the temperature 0 and what happens slowly it diffuses, heat diffuses and finally temperature will become 0 that is what you see as T goes to 0, as T goes to infinity that means for larger times this is becoming, this is going to 0.

So that means finally V of x t is zero for all times okay, so we can see that remark is V of x t as limit t goes to infinity is actually 0 so rod becomes cool okay it becomes at 0 temperature this is what is V so actual rod is what you have is this is a T1 and T2, this is a T2 so what happens V is 0 what is your solution is V of x t that is this solution and place g of x that is what we have.

So as limit t goes to infinity and this limit t goes to infinity and this is nothing but this is 0 plus g x so this is equal to g x so what is g x? g x is simply, so this reaches the steady state so T1 is here and T2 here, in between you simply have a steady state solution so T1 so linearly it varies so this is the linear function, g is a linear function that is, that is what we have seen.

So what is the linear function you can see if you look at the g, g is when you put x equal to 0 it is a T1 at x equal to L it is actually T2 so its linearly varies, this linearly varies, one is at T1 other one is at T2 so that is what is your g x function so once you have this the eventually as t goes to infinity if you maintain the rod at temperature T1 and T2 eventually as time goes to infinity you see that as the temperature simply rod will have a linearly, a temperature maintains linearly okay,

This is what would you get out of this, so it has two problems one is temperature of a rod, finite rod we want to find out for all times when you maintain at the two different temperatures one end at T1 other end is at T2 okay. So this you convert into a problem with both the ends you maintain at zero temperature, okay.

So zero temperature if you work out so you the worked out so this is what you actually solve and that actually finally solves rod which you maintains at the temperature T1 and T2 okay, so you have at both the problems are combine so if you are given at 2 problems either of the problem one is problem with both the ends are at zero temperature or you are given a rod with two different temperature at the both the ends so both things you can solve okay if you are given two different things you convert into 0 0 type by this technique, convert boundary value, initial boundary value problem two and that you solve that is what is at 0 degree okay, 0 degree at both the end points so this is how you can solve this now what else we can do with this rod.

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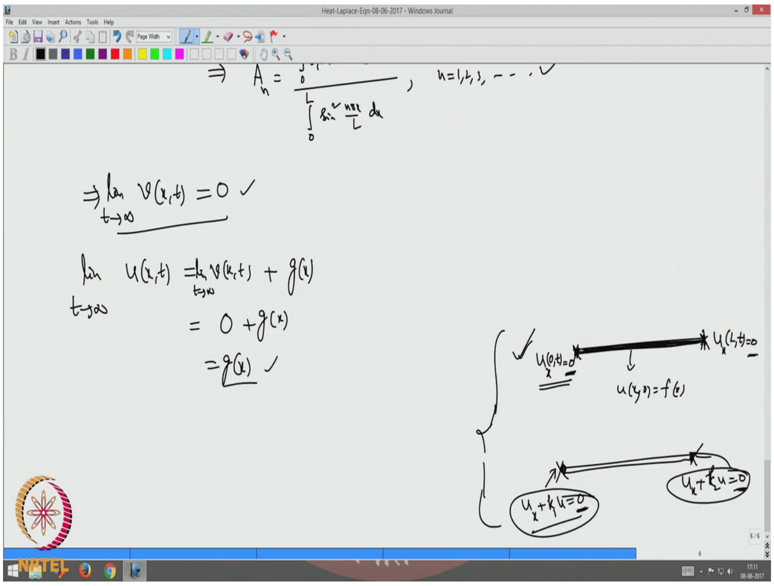

Rod is, if you are given a rod this is insulated and you have a boundary condition finite rod and you have initial data u at x0 equal to fx and you can also insulate, what you can do is you can insulate one end and you can allow it to be say some you can maintain some temperature u at L t, L t is say some t, some temperature let us see, some T1 okay.

Here you say this insulation means you are not allowing the heat to go out that means the flux that temperature normal derivative of the temperature that is ux at x0 for all times you maintain 0 so if you insulate it this is the condition you have okay or you can simply interchange or you both can have so both can have, so both side if you insulate you do not allow the heat initially the rod is at some temperature you are not allowing everything is, so all heat will be inside and you have ux this side also if you insulate ux, what do you expect ? You expect whatever the distribution of this temperature here that will be after sometime it will reach the steady state, okay, it will reach the steady state that is what is should be the solution so this is you solve we can solve this in the next video okay.

And also this is what we will see in the next video and we can also make one more problem so take the same rod you allow the heat exchange, you allowed the allow if you allow the heat exchange at the boundary that means the flux is proportional to the temperature that means ux plus some constant k times u equal to 0 so this is the boundary condition mixture of u and ux will be the boundary condition here now similarly you can also make here ux plus k some constant k2 here k1, k u equal to 0.

So if you keep this boundary condition you can also work out the same way so only thing is boundary conditions are all having 0 0 boundary condition okay so you have a homogenous boundary condition that is what is required for the Sturm-Liouville problem which you want to extract okay if they are not you have to use the technique that I have, we have seen in this video initially so you try to write the problem u equal to v of x t plus some g of x okay.

For g you give this non zero boundary conditions now to g and zero boundary condition to v but initial condition, so once you solve this you can, initial condition for v will be different that you can solve it, so these are the three things, so two different kinds of problems one can look at it.

So you will try to see one of this maybe you will do this one so both the ends are insulated are you maintain one at the insulated other one is maintain at some temperature all this things you can work out okay, I will try to, we will try to see the next video when the temperature of the rod initially at some temperature along the rod and both the ends are insulated what is the temperature for all times this is what we will see in the next video. Thank you very much.