Course on Differential Equations for Engineers
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Lecture 55
Temperature in a semi-infinite rod

In the last video we have seen temperature distribution of heated rod for all times that is exactly so if you are given a rod with initial temperature how the heat is diffusing in that rod over a time. So that is what is the meaning of finding a solution for the initial value problem for the heat equation of over a spatial domain that is from minus infinity infinity, okay. In this video we will try to see if you are given a rod infinite rod 1 dimensional rod, okay 1 dimensional with initial temperature distribution so let us say initially you know the temperature of the rod throughout.

And it satisfies the heat equation and the boundary condition at one end you are fixing the temperature either 0 you fix the temperature as 0 or you insulate it the end end point. So that is what is so you have a two initial boundary value problems, okay so for the heat equation so we will just define and try to solve it just based on the solution that we derived in the last video for the infinite rod just occupying minus infinity infinity.

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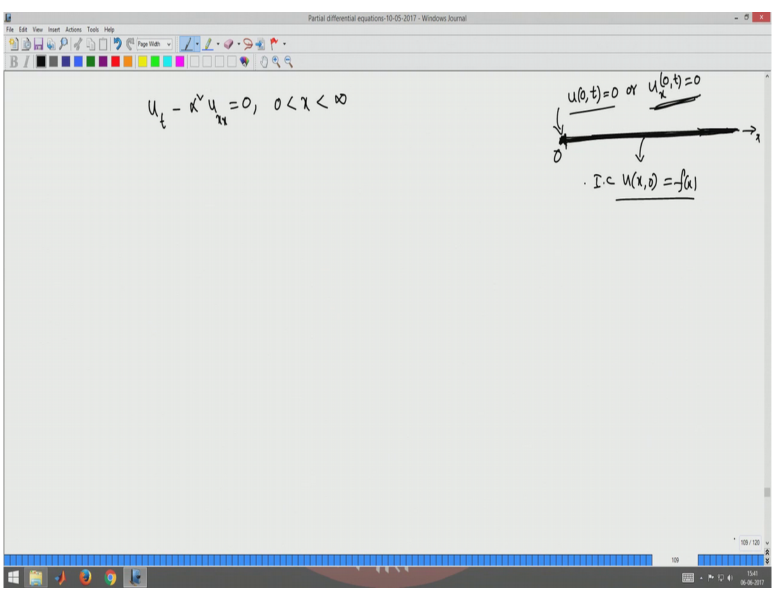

So let us see what are those semi-infinite rod, so if you have a heat equation on semi-infinite domain, so heat equation or let us say initial boundary value problem so let us say heat equation is this ut minus alpha square uxx equal to 0 now x is only positive so that is 0 to infinity so you have infinite rod with an end one end is 0 other end is simply you have a infinite rod so you have a infinite rod other end so with the boundary is only this one.

So initially you know the temperature of this so u at x, 0 is actually given, so that is given so this is initial data and this boundary data at this point you have a boundary data that is u at x is 0 and for all times you maintain either 0 or you insulate this also so you do not allow the heat to pass from this end. So if you do that that is because this is x axis so the normal velocity normal derivative has to be 0 the flux that is the flux going out, okay so that should be so derivative with respect to x at so dough u dough x at 0, t equal to 0 so either this or this, okay.

So we have seen this kind of boundary condition we know how to do it so when you look at when you look back into a wave equations on semi-infinite string so what we have done when you have a when you attach the string with displacement is 0 at the boundary for all times then you know that you have to extend those functions involved as odd function so that if you extend as an odd function this boundary condition is automatically satisfied.

If you attach the string with this condition that means there you allow it to be free ux is 0, okay. So there what happens you extend as an even function and we so that boundary condition is automatically satisfied and we made use of D'alembert solution and then that is over the full real line so we made use of that solution and give you the solution for the semi-infinite string vibrations vibrations for the semi-infinite string.

So we do the same thing here so now instead of the string you have infinite rod with initial heated rod so at time 0 you have some temperature and you want to see how diffuses along this how the heat is diffuses that means how the temperature changes over a time in this rod semi-infinite rod.

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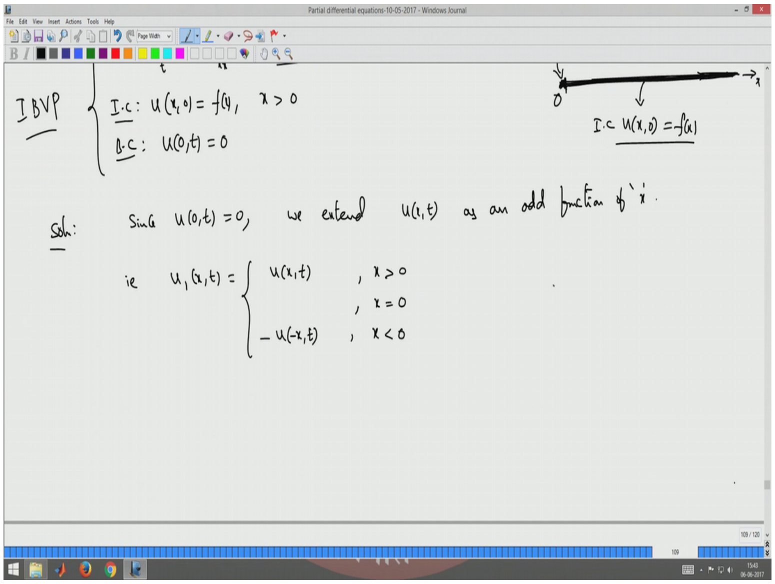

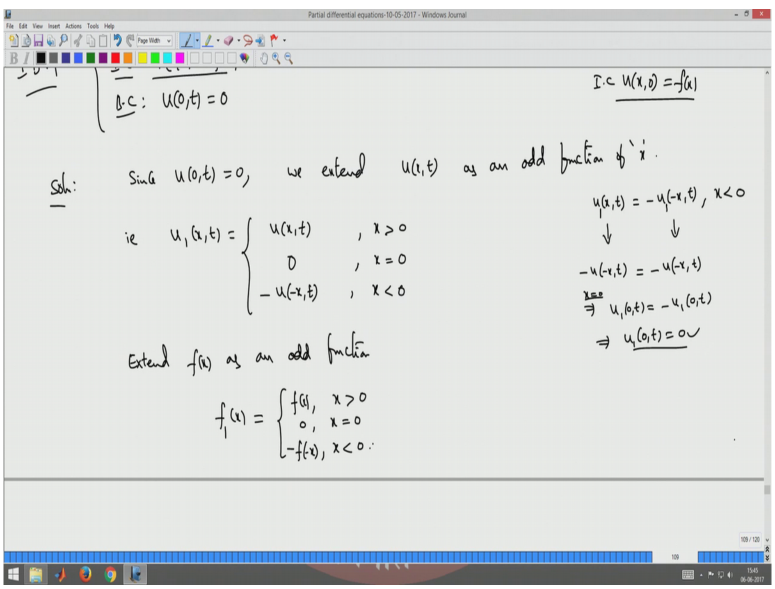

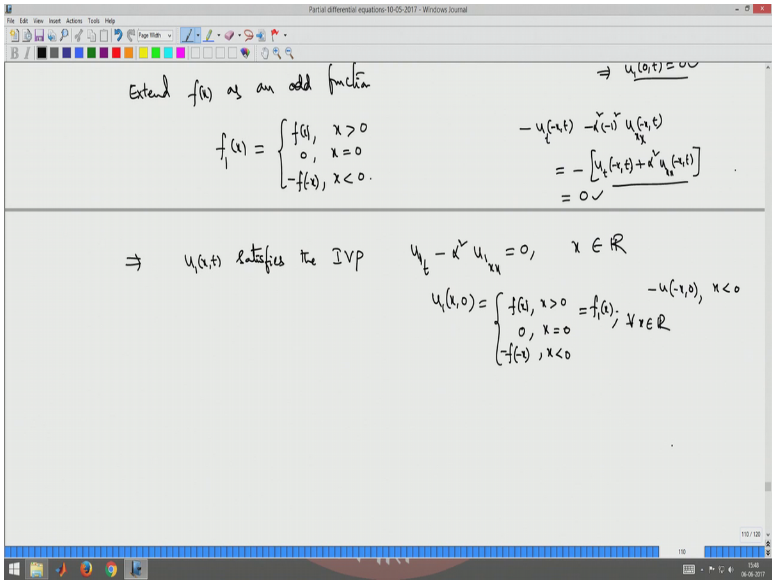

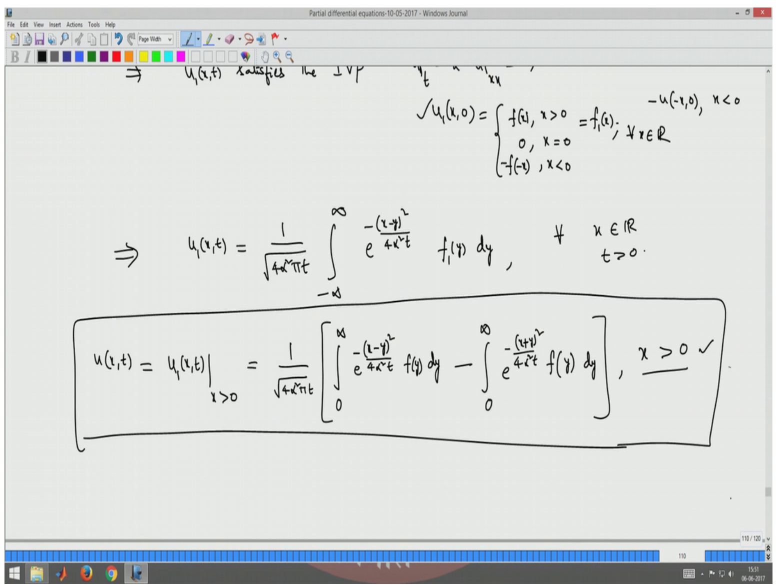

So the initial conditions we write so ux initial condition is this you have x, 0 equal to fx and x is positive and the boundary condition is u at 0, t so let us start with this initial boundary condition so you have this is the initial boundary value problem for the heat equation in the semi-infinite domain x positive and is also t is involved that is the time t positive, okay. So how do we find this solution? Solution we just make use of because of the boundary condition since ux u at 0, t equal to 0 we extend u of x, t as an odd function of x.

So that means I define a new function u1 of x, t as if x positive you take same u of x, t so that is satisfied by the heat equation and otherwise for x negative we just do it as an odd function. So that is u of minus x, t so when x is negative minus x is positive so that is u of positive quantity t that is actually what we have already that is the same. So if you extend it like this, okay so what happens at 0? So at x equal to 0 so what happens u so if you do like this at x equal to 0 it has to be so if it is a odd function that means u at minus x u at x, t equal to minus u of minus x, t for x negative so that is what is the meaning, okay.

Or u1 is this one that is actually equal to u1 of minus x, t, okay. So what is the u1 at x, t x negative is minus u minus x, t, okay in terms of this if x is negative minus x is positive so that is same as minus when minus x is positive u1 when x is positive is u of x, t u of minus x is positive so that is what is the same so both are same, okay. So that means that is 0 so now once you put x goes to 0 here so you put x equal to 0 here so this will put x equal to 0 so you see that u1 at 0, t is same as minus u1 at 0, t.

So that means this is possible only if 0, t equal to 0, okay. So what you have is this is simply 0 at this value at x equal to 0, and similarly u extends so you need initial value even for x negative so you extend to get that you extend this initial temperature also as an odd function so extend fx as an odd function so you get you write this as a odd function is f1 that is fx if x is positive and 0 if at x equal to 0 obviously because it is odd function minus f of minus x if x is negative, okay.

And the way you do it for x positive is when you we are satisfying the heat equation for x negative you if you do twice if you do so u1 x negative what is a t derivative is minus ut minus x, t, okay and then plus if you do twice x derivatives of this two x derivatives of minus u of minus x is again minus 2 derivatives so one derivative will get 1 minus minus one times ux of minus x, t if you do one more derivative with respect to x that will be one more square so you get this is simply minus is common this will be plus so ut of minus x, t plus you have alpha square, okay just try to add, okay put it in this if you multiply with alpha square, alpha square uxx minus x, t this is x minus is common.

So this is for x negative so x negative means minus x is positive and minus x is positive this already satisfied the heat equation that is what is given problem so this is equal to 0. So you see that u1 x is satisfying this implies u1 of x, t satisfies the initial value problem, okay. So that is u1t minus alpha square u1 xx equal to 0 now this is defined for everywhere so this is this one and u1 at x0 is nothing but here fx this is fx and minus u of minus x, okay what is that one so at x equal to 0.

So this is exactly you can see that this will be u1 at x0 is u of x, 0 that is fx when x is positive, okay and then what happens when x is negative, so when x is negative and this becomes minus u of minus x, 0. So when x is negative, so when x is negative minus x is positive this is nothing but f of minus f of minus x, okay because minus as it is minus u of minus x, 0 when x is negative minus x is positive so that we already know that is f of minus x so this is what you have.

So this is nothing but now it is a odd function so that makes it x is at x equal to 0 this has to be 0. So this is nothing but your f1 of x, okay. So for every x belongs to r this is true, so this is the initial value which we solved in the last video, so you can now write your u1 of x, t as the solution so which we know what exactly is that solution you have here so 1 by square root of 4 alpha square pi t, okay and then integral minus infinity infinity e power minus x minus y whole square divided by 4 alpha square t.

Now here f1 of t f1 of y dy, so this is your solution this is true for every x belongs to r and t positive this is from the solution over the full real line for the heat equation with this initial data, okay. So but you need u u1 x, t when you restrict x only positive side that is exactly your uxt, right? So to do this so when x is positive you have 1 by square root of 4 alpha square pi t and this is from so this is exactly what you have so we have 0 to infinity you break this integral to 0 to infinity so what you get is e power minus x is only positive y square by 4 alpha square t and f1 of y, y is now between 0 to infinity so that is f of y dy.

Now plus now minus infinity to 0 what you have is this is also this is same this is x minus y square by 4 alpha square t. Now f1 of y when y is negative minus infinity to 0 y is negative that is minus f of minus f of minus y dy so this is what is the solution. Now here if you change the variable y as minus y if you replace y by minus y then you have a f of y and dy becomes minus that is going to be plus y and you have minus y you are replacing with y so you have minus y so you are replacing with so x plus y so this is what it becomes after change of variables.

So now y equal to minus y earlier y is minus infinity t and you have, so the limits will change to so you have infinity to 0, okay. So you have minus minus plus so you get infinity to 0 so let us say earlier what you have is minus y equal to y1 let us say. So if you put minus infinity so (()) (14:23) minus infinity y is minus infinity minus minus plus infinity so you have plus infinity to 0 this is what it becomes.

So if you now change this limits to 0 to infinity you have a minus you have to replace with minus so this is what you have, now this is only for x positive so this is your solution of the required initial boundary value problem, okay. So we will do the same thing if you replace now you consider this problem so the same initial value take the heat equation with the same initial data and now you consider this boundary condition.

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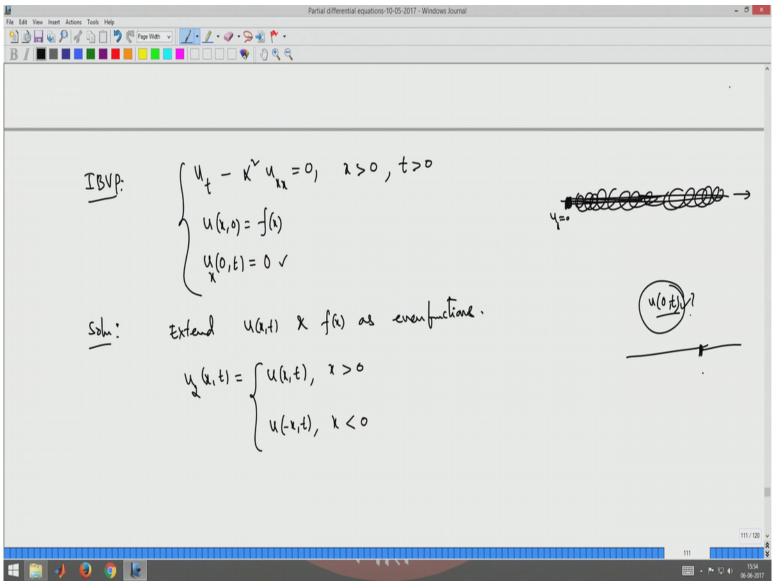

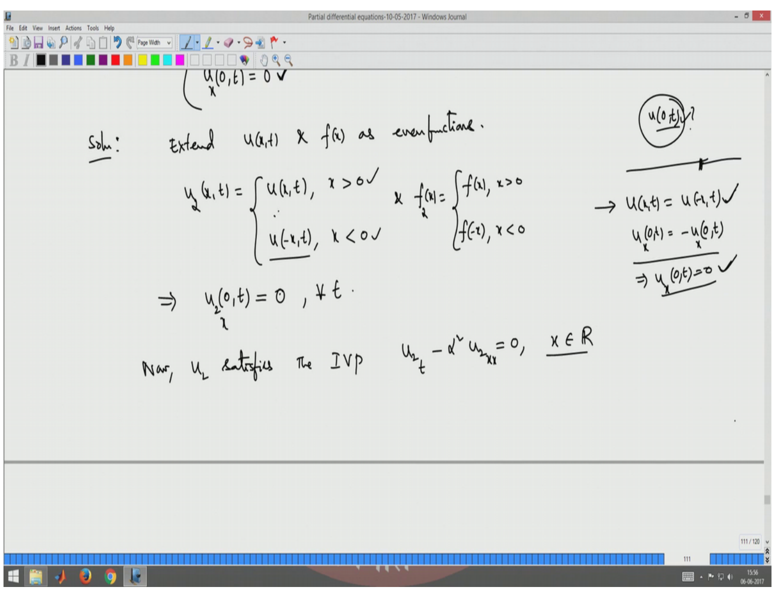

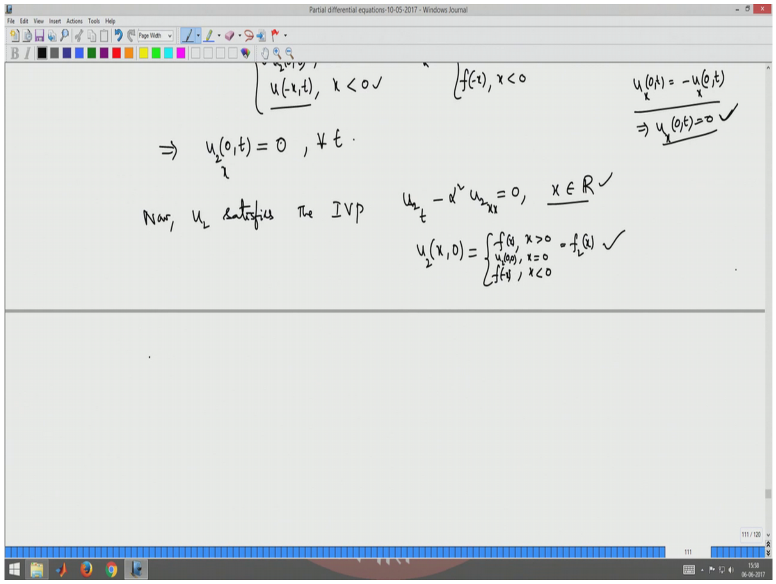

So what you do is you similarly what we do is in this case when you have so let us do that one so if you have the problem initial boundary value problem with ut minus alpha square uxx equal to 0 x is positive and t is positive and u at x, 0 is fx and u at 0, t this one ux 0, t is 0, okay is like you insulate that one end of the rod if you insulate at this point is also insulated all along this we have insulated lateral (())(15:47) so that heat dissipation only in one direction so here also you insulate this end so that it will not heat it will not go out so that is what is the ux equal to 0. So you have semi-infinite rod again so you have this is the initial value problem. So in this case again so when you have problem like this when boundary condition is like this so what you get is extend the functions extend u of x, t and fx as an even functions, how do we do this? So start with u1 of x, t as this is extended function even extended function as calling it let us call it u2, u2 because we used for odd extension as you want this is even extension that is u2 if I call this u2 so what you have is uxt if x is positive if it is even what you should have is u of minus x, t is same as this one, okay.

So in this case you do not know exactly what happens at x equal to 0, okay. So this we do not know so this is clear so once you extend this at that point you are not doing it, okay at one point you are not extending, okay so this is still unknown so you call this you do not know at x is equal to 0. So earlier then you extend it you know that if we extend as an odd function you know the value at x equal to 0 but here it is not clear so nothing is known, okay.

But you know that because this is a even function so u x, t equal to u of minus x, t at x equal to 0 if you differentiate ux of put x equal to 0 is same as minus u x at minus x is minus g that is 0 t. So this will give ux at 0, t equal to 0, so this condition boundary condition is automatically satisfied now, okay you differentiate this equation because this is a even function this is same you both sides you differentiate with respect to x what you get is this and put x equal to 0.

Once you put x equal to 0 you are evaluating ux at x equal to 0 so you have this one so this will give me this boundary condition automatically satisfied. So u2 satisfies so this implies u2 of u2 x at 0, t is automatically satisfied as 0, okay for every t and you can also extend once you do this and fx is also you can extend as even function so you have fx x positive f of minus x as x negative, okay.

So once you have this one so you call this f2 my even extension I am calling f2. So what you get is you can see this this satisfies the heat equation already and this is also satisfying a two x derivatives will not give any minus, minus minus plus so this will also satisfy the heat equation. So clearly u2 satisfies now u2 satisfies the initial value problem u2t minus alpha square u2xx equal to 0 now x belongs to full real line, okay.

So you do not know exactly at x equal to 0 so just one point so that is why is full real line so, okay you still call this you call this itself you do not know exactly what it is, okay. So you call this u2 at 0, t you still do not know at x equal to 0, so that is how we extend your extension is like that. So still this is just a point is just an fixed number so that satisfies the derivatives when you put in this two derivatives is 0 is satisfied n at that point so included that point so x belongs to r.

And what is the initial data so at 2x2 x, 0 is now, what is x2 at 0? So x0 u x, 0 is fx x positive and x negative what you get is when t u of minus x, 0 that is u of minus x means f of minus x so you have f of minus x, so this is nothing but your f2 of x, okay. And again here so still some constant f at 0 you still do not know exactly what it is, okay and you do not know this is some number. So you cannot say it is f2 so you call this u2 of x, 0, okay u2 of 0, 0 at x equal to 0 it is simply u2 of 0, 0 that is simply constant.

So this is exactly your f2 so if you want you can write here also here also you can say u2 of 0, 0 at x equal to 0 that is the number you do not know that is still arbitrary number.

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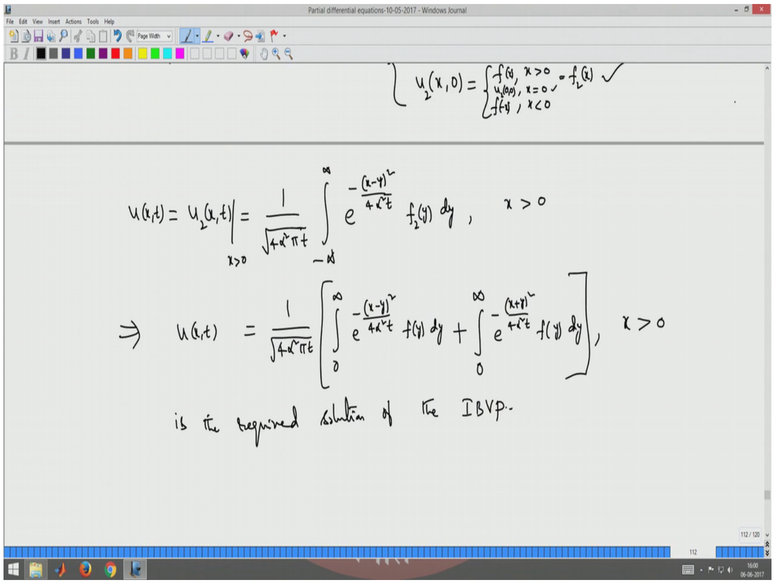

So this is my f2, so this will give me so immediately so we know the solution of this u2 because in the last video we have solved for this initial value problem we know the solution. So you can write u2 of x, t as a solution what is that solution what you have so like this u2 so u2 is exactly 1 divided by 4 alpha square pi t under root, okay and then you minus infinity infinity e power minus x minus y whole square by 4 alpha square t into f2 of y dy because this is the integral that point x equal to 0 does not matter you are going to evaluate at f2 at 0 does not matter so if you break it (())(22:04).

So integration at one point is always 0, okay so you need not consider certain points finite number of points you can always remove and in integral integral value is same. So this if you restrict only for x positive that is exactly your u of x, t what is required required function over t this is exactly is this, okay. So this is for x positive so in this integral you have to consider only x positive so this will give me 1 by 4 alpha square pi t under root and now this is from 0 to infinity e power minus x minus y, x is positive y is now between 0, y is also positive 4 alpha square t.

Now f2 of y and (())(22:45) and y is positive is f of y dy, now plus now you do from minus infinity to 0 this is cannel is same that is x minus y whole square by 4 alpha square t. Now we have 2 of y when y is between minus 1 y is between minus infinity 0 that means when y is negative this is f of minus y dy. Now you do the change of variable here so what you get is x plus y this is fy and you have a dy is minus dy that is minus it is going to be infinity to 0.

So when you write this infinity to 0 as 0 to infinity this will become again plus, so this is what is your solution required solution so is the implies u is the required solution of the initial boundary value problem, okay. So that is here so this is what you have so this is the initial boundary value problem so when you consider this rod with insulated end what is a diffusion process for this initially heated rod semi-infinite rod, okay so this is how you find the solutions, okay.

We then how many minutes are there, okay 24 (())(24:32) other 10 minutes. We will just so this is how we solve some semi-infinite domain heat equation with initial data and boundary conditions, okay.

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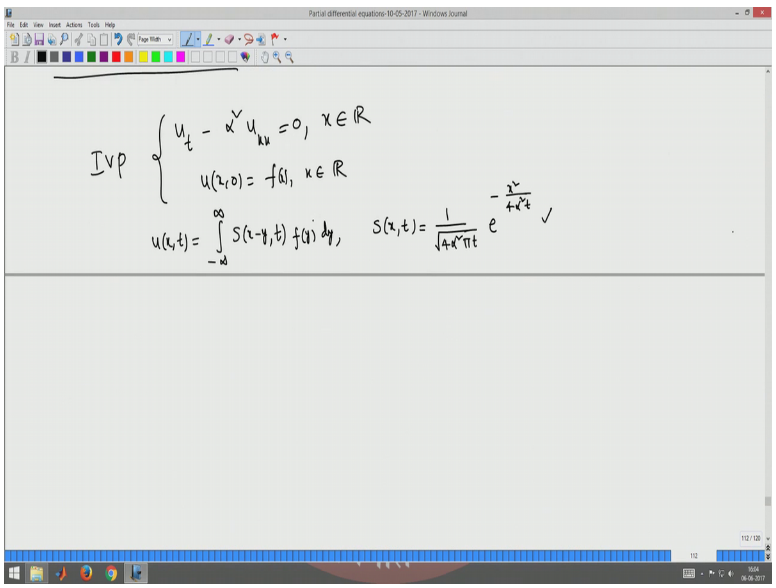


Partial differential equations-10-05-2017 - Windows Journal

$u(x,0) = f(x), \quad x \in \mathbb{R}$

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\, f(y)\, dy, \qquad S(x,t) = \frac{1}{\sqrt{4\kappa^2 \pi t}}\, e^{-\frac{x^2}{4\kappa^2 t}} \checkmark$$

Height of $S(x,t) \rightarrow \dfrac{1}{\sqrt{4\kappa^2\pi t}}$

Approximate curve width of $S(x,t) \longrightarrow \sqrt{2}\,\kappa\,\sqrt{t}$

As $t \to 0^+$, $S(x,0) \sim$

Normal-distribution

$$g(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad \mu\text{-mean},\ \sigma\text{-standard deviation}$$

Height of Bell $\rightarrow \dfrac{1}{\sqrt{2\pi\sigma^2}}$

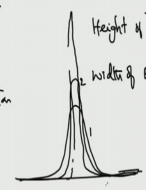
width of Bell curve $\approx 2\sigma$.

Partial differential equations-10-05-2017 - Windows Journal

As $t \to 0^+$, $S(x,0) \sim$

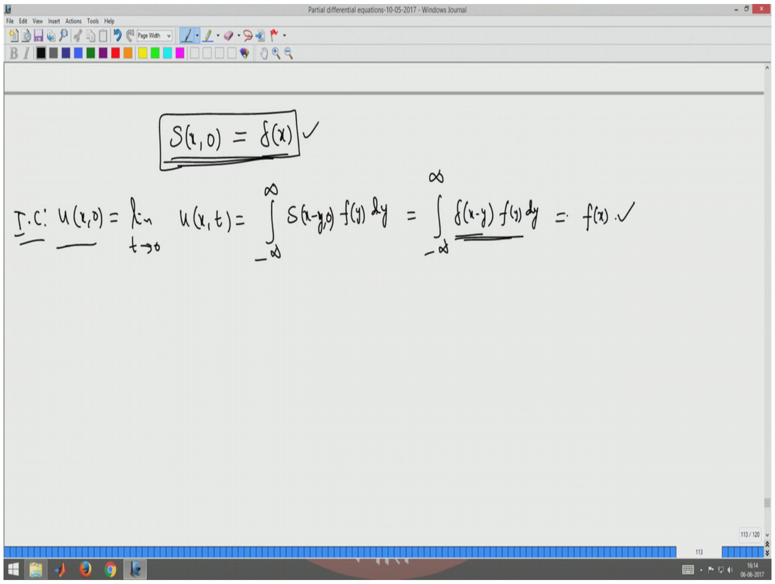
$$\delta(x) = \begin{cases} 0, & x \neq 0 \\ \infty, & x = 0 \end{cases} \quad \text{generalized function}$$

$\delta(x) =$ limit of functions $=$ lim



$$\int_{-\infty}^{\infty} f_n(x)\, dx = 1, \quad n = 1, 2, 3, \ldots$$

$$f(x) = \lim_{n\to\infty} f_n(x)$$



So we will just give you briefly what is the look at the solutions what that we derived, okay so for the infinite things so let us say what is a solution what you have is earlier if you just look back when you look back your solution is this is one so that is let me write this as so let us go back to this problem so ut minus u alpha square uxx equal to 0 x is full real line is infinite rod with the initial data as fx x belongs to r so this initial value problem has a solution this has a solution u x, t as what is the solution you have got derive the solution as integral minus infinity to infinity and this is my S of x minus y phi y dy, okay so where x is S of x and this is, right? So this is S of x minus y comma t and t and fx f of y dy where S of x, t is actually equal to 1 by 4 alpha square that is square root of 4 alpha square by t so this is into exponential minus x minus y so you are replacing x by x minus y so what you get is x square divided by 4 alpha square t so this is what is my S of x, t when you have S of x, t S of x minus y, t is simply what you have is the solution that we wrote earlier, okay.

So that is the solution we derived, if you actually see this one so in the if you know the probability theory so there you can write you have this kind of in the probability theory if you have this two pi alpha square rather sigma square e power minus x minus mu by mu square by 2 alpha square. So if you know this distribution this is the normal distribution this is the curve for normal distribution so that is my f of x is the this normal distribution (())(28:12) let us call this some g this is the normal distribution (())(28:20) normal distribution is basically if you see that this is like a bell curve never touch a 0 but this will be this is how it looks this is at x equal to mu you have a peak at x equal to mu you have exponential function is 0 that is 1 and this is what you get that peak is at 1 divide by when x equal to mu this line this is y line this is my x.

So you get this this point is actually 1 divided by root 2 pi r sigma square this is the peak you get at x equal to mu, mu is the mean and sigma is standard deviation, okay standard deviation of that is normal distribution. So if you have like this this approximate width you ignore this error this length this length between this approximately is actually 2 alpha 2 sigma, okay 2 sigma the length of this is 2 sigma.

So by using this one so approximately width of this bell curve bell curve its peak is this peak of the bell curve is this and width of the bell curve is so peak the height of height of the bell curve you can call this height of bell curve is 1 by square root of 2 pi sigma square and width of the bell width of bell curve if you see this is actually approximately is approximately 2 sigma. So make use of these things or apply here so this is also looks like the such a bell curve, okay.

So if you do that what is that height, height of this height of S of x, t is at 1 divided by square root of 4 alpha square pi t, okay that is clear so that is true and what is the width approximate width width of S of x, t, okay approximately approximate width of this one, okay curve width approximate curve width positive curve width of S of x, t what you see is the two times and what you get is a 2 alpha root t, okay. So 2 times sigma is like root 2 here alpha is here root t so root t so that is what you have.

Now what happens as t tends to 0? As t tends to 0 so we are only going from positive side t cannot be so time is always positive so you are taking this limit from the positive. So what happens S at x, 0, okay so how does this look as a function so this geometrically this what happens the bell curve looks very very steep at x equal to 0 so mu is 0 here so here on this case x minus 0 square.

So at x equal to 0 you have infinite peak this is actually peak will be infinite, okay. So you see that this go into infinite and this width of this curve is becoming 0 this width width is width goes to 0 and height goes to infinity so let us say height goes to infinity is what you are seeing, okay. So where do you see such curve if you know delta a function is such thing so delta x if I define it as 0 if x is not equal to 0 and not defined, so infinity at x equal to 0, okay such is this is not a function this is actually a limit of certain normal functions.

So if you think of delta x as a limit of usual functions, okay that is why it is called generalized function limit of so this is called generalized function so which you see it as a limit of usual function so something like limit of all these usual peaks, okay so usual like this every time every time you increase but its integral value, okay its area is same we maintained the area as 1, okay. So if you maintain the functions f n of x dx equal to 1 from minus infinity infinity.

So if you do like that so what happens initially suppose you have f1 is like this f2 is you come inside and you increase the peak so that its value is whose integral has to be n is from 1, 2, 3 onwards this is my 1, this is my 2 and 3 and so on we will go on finally you end up finally some infinite thing, okay is where is steep peak finally this is how it goes. So this limit of is usual functions as n goes to infinity is my delta function so that is the generalized function.

So you see that this is also behaving the same fashion so S of x, 0 is actually by delta function so delta function, okay. So what is that delta function? So S of x, 0 so that is the delta function so that is why your solution x, y x, t u of x, t which is from minus infinity infinity S of x minus y f of y dy now what happens if x of x minus y comma t x minus y, t f of y dy this is the solution, now you take this limit t goes to 0, okay that is u at x, 0 so you take this limit t limit so that is y, 0.

So this is exactly minus infinity infinity this is now delta of x minus y f of y dy. So a delta function when you multiply with a function and integrate what you get is earlier so you have this is like a delta function scaled delta function at x translated delta function from 0 to y, so you get f at y at x, okay this is the delta function of y translated to kind of y minus x you can write, okay you want you can write y minus x or x minus y all are same, okay.

So x minus y you can write so because S of x minus y is this replacing x minus y of y, so this is symmetric this is the limit of symmetric function so does not matter if it is x minus y or y minus x so which is a function of my delta is a function of y if you view y equal to x this is infinity that will give you exactly fx, okay this is exactly my initial condition so you solution is actually satisfying the initial data if you think.

So what you get is even that is like delta function. So S is actually kind of delta function, so you have a bell curve as t goes to 0 your solution that involves S of x, t it looks like a bell curve as t goes to 0 it behaves like those kind of functions those curves finally eventually that converges to the delta function, okay. So we make use of this to so we make use of this interpretation and to finally to solve certain boundary value problems for example we will just we will do some more problems for example heat with source, okay let us do this one so what we need we will try to use this where do we use this let me put it so (())(37:32).

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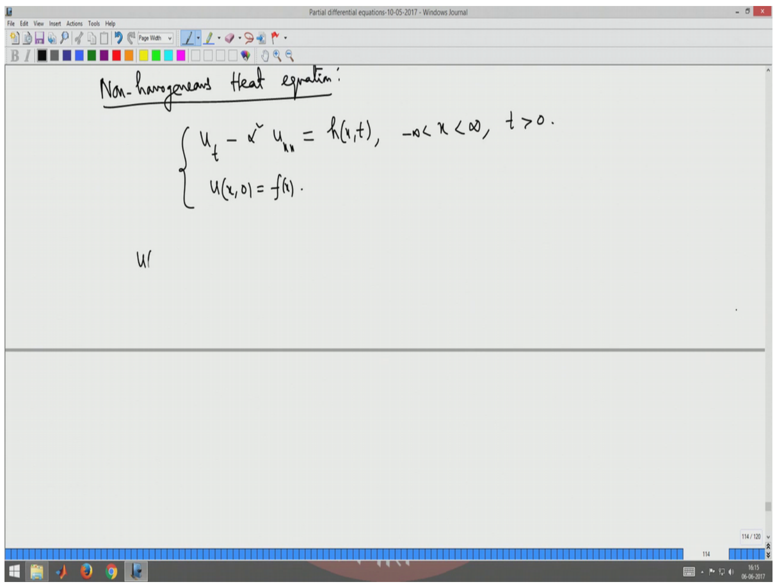

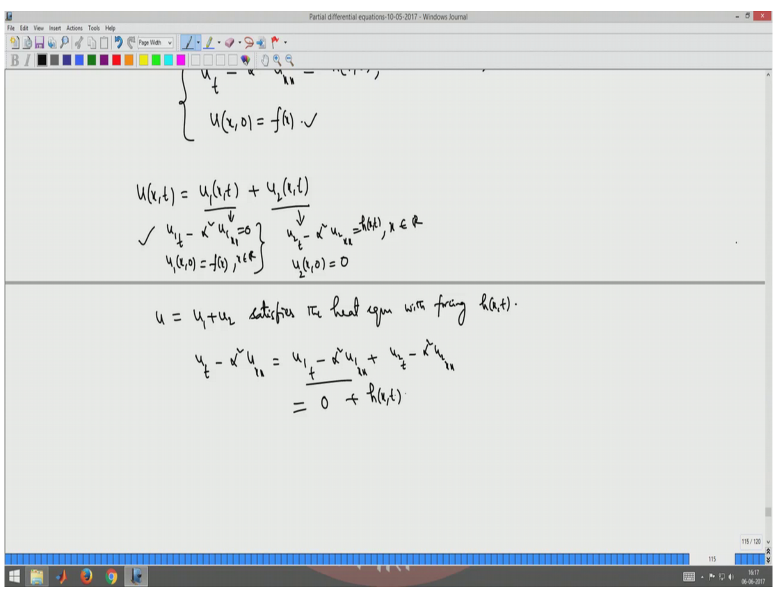

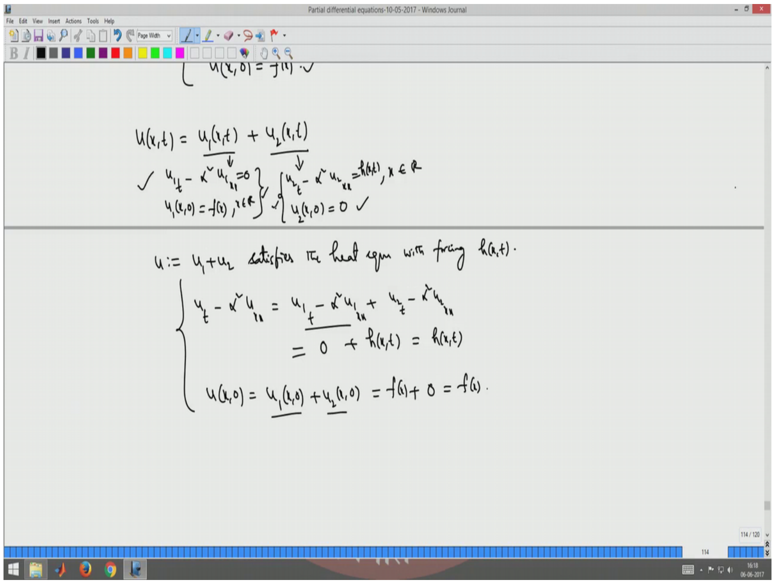

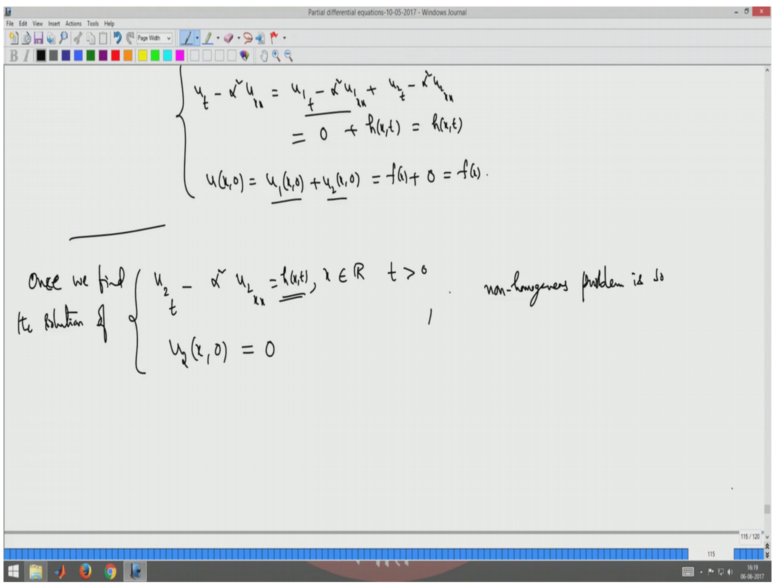

So we use this interpretation just to show you that we will try to solve the non-homogeneous heat equations, okay. So use the above interpretation so to solve a non-homogeneous heat equation so that is ut minus alpha square uxx now this is u is a function of x, t so the right hand side you have a force in you call this some h of x, t, okay.

So you have something like you have heating so outside you have a source, okay so you are heating the rod from outside some energy is coming inside the rod for all times, okay. So if you do this now between minus infinity to infinity when t is positive how do I solve this non-homogeneous equation with the initial data u at x, 0 equal to fx, okay. So to solve this non-homogeneous equation so what we do is we try to write u of x, t as u1 of x, t plus u2 of x, t like earlier.

So this is this will solve the homogeneous equation with the initial data and this is the this solves the non-homogeneous equation with 0 initial data so that it will solve so we will see this one so u1 satisfies u1 t minus alpha square u1 xx equal to 0 and then u1 at x, 0 equal to fx, okay. So this we know how to solve this is the initial value problem that we solved earlier last video, okay. So now I know this u1 of x, t so what happens to this u2, u2 actually satisfies the heat equation alpha square u2 xx equal to 0 for x belongs to R even here x belongs to R, okay.

So and what happens here u2 at x, 0 initial data is now 0, sorry now here this is actually satisfying non-homogeneous equation. So h of x, t x belongs to R so u2 of this so that u1 plus u2 that is u satisfies the heat equation satisfies the heat equation with so the non-homogeneous heat equation, okay with forcing h of x, t so what it means is ut now we just calculate ut minus alpha square uxx is actually equal to u1t minus alpha square u1 xx now plus u2 t minus alpha square u2 xx. So this is equal to 0 and this becomes h of x, t I just split it into two problems.

So finally is actually h of x, t so it solves u satisfies the non-homogeneous equation and u at x, 0 is actually same as if I define this way u1 at x, 0 plus u2 at x, 0 so what is u1 x, 0 that is fx and this becomes 0 this is 0 because that is how I defined the this problem for u2 so this is actually fx so that what you have is this u sum of these u1 and u2 actually satisfies the non-homogeneous equations. So if you solve these two problems this one and this one that means you solve the non-homogeneous equation this we already know and this we will try to calculate.

So we will just try to see this problem so u2 t minus alpha square u2 xx equal to 0 for x belongs to R and t positive and sorry this is non-homogeneous equation, so this one and u2 at x, 0 is actually equal to 0. So with 0 initial so initially you have a rod infinite rod with heat is 0 but there is a heat coming from outside so that is what is this forcing so this forcing term so then what happens to the rod so this is what is the this is what we need to find, okay find the solution of this that means you find the solution of the non-homogeneous equation, okay.

Once we find this once we find the solution of this problem then non-homogeneous is non-homogeneous problem then non-homogeneous problem is solved because I take this u2 and I add it with solution of this problem which we already know so that will give you the solution. So this is what we will see in the next video so we will try to solve this initial value problem in the next video along with using the method called operator method, okay.

So in the next video we will try to solve this initial value problem for this non-homogeneous equation with 0 initial condition using the operator method so we start with the simple ODE with something similar to this equation ut minus some a, a is a simply constant into uxx, okay equal to 0, then what happens we will just by looking at the solution we also we identify that the solution here we try to construct this called the operator method so we will just see this in the next video along with other possible problems that can be solved, okay thank you very much.